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Pseudo-Ricci-Yamabe solitons on real hypersurfaces in the complex projective space

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Abstract. In this paper, we give a complete classification of Hopf *pseudo-Ricci-Yamabe solitons* on real hypersurfaces in the complex projective space $\mathbb{C}P^n$. As its applications, first we give a complete classification of *gradient pseudo-Ricci-Yamabe solitons* on real hypersurfaces with isometric Reeb flow in the complex projective space $\mathbb{C}P^n$. Next we prove that a contact real hypersurface in $\mathbb{C}P^n$ which admits the *gradient pseudo-Ricci-Yamabe soliton* is pseudo-Einstein.

1. Introduction

Among the class of Hermitian symmetric spaces with rank 1 of compact type, we have complex projective space $\mathbb{C}P^n = SU_{n+1}/S(U_1 \cdot U_n)$, which is geometrically quite different from the case of rank 2. It has a Kähler structure and Fubini-Study metric g of constant holomorphic sectional curvature 4 (see Romero [34], [35], and Smyth [38]). The complex projective space $\mathbb{C}P^n$ is considered as a kind of real Grassmann manifolds of compact type with rank 1 (see Kobayashi and Nomizu [26]).

Recently, Yamabe solitons and Ricci solitons on almost co-Kähler manifolds and three dimensional $N(k)$-contact manifolds have been investigated by Chaubey-De-Suh [10], and [18]. Moreover, the study of the Yamabe flow was initiated in the work of Hamilton [23], Morgan and Tian [27] and Perelman [30] as a geometric method to construct Yamabe metrics on Riemannian manifolds.

A time dependent metric $g(t)$ on a Riemannian manifold M is said to be evolved by the Yamabe flow if the metric g satisfies

$$\frac{\partial}{\partial t} g(t) = -\gamma g(t), \quad g(0) = g_0$$

on M, where γ denotes the scalar curvature on M. Then in this paper let us consider a *pseudo-Ricci-Yamabe soliton* $(M, V, \eta, \Omega, \delta, \rho, \gamma, g)$ as follows:

$$\frac{1}{2}(\mathcal{L}_V g)(X, Y) + \delta \mathrm{Ric}(X, Y) + \psi \eta(X)\eta(Y) = (\Omega - \frac{1}{2}\rho\gamma)g(X, Y) \tag{1.1}$$

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for any tangent vector fields X and Y on M, where Ω is said to be a *pseudo-Ricci-Yamabe soliton* constant, the functions δ, ρ and ψ are any constants and γ the scalar curvature on M, and $\mathcal{L}_V$ denotes the Lie derivative along the direction of the vector field V.

When the function ψ identically vanishes, the *pseudo-Ricci-Yamabe soliton* is said to be a *Ricci-Yamabe soliton*. When the function δ vanishes, it is said to be a *quasi-Yamabe soliton* due to Chaubey-Lee-Suh [15]. If the functions δ and ψ both vanish, the soliton (1.1) is said to be *Yamabe soliton*. We also say that the pseudo-Ricci-Yamabe soliton is shrinking, steady, and expanding according to the *pseudo-Ricci-Yamabe soliton* constant function $\Omega > 0$, $\Omega = 0$, and $\Omega < 0$ respectively.

On the other hand, it is well known that there exist two focal submanifolds of real hypersurfaces in Hermitian symmetric spaces of compact type and only one focal submanifold in Hermitian symmetric spaces of non-compact type (see Cecil and Ryan [8] and Helgason [22]). Since the complex projective space $\mathbb{C}P^n$ is a Hermitian symmetric space of compact type, any real hypersurface has two focal submanifolds (see Djorić and Okumura [19], Pérez [32]). Among them we consider two kinds of real hypersurfaces in $\mathbb{C}P^n$ with isometric Reeb flow or contact hypersurfaces. In $\mathbb{C}P^n$, Cecil and Ryan [8], and Okumura [28] gave a classification of real hypersurfaces with isometric Reeb flow as follows:

Theorem A. Let M be a real hypersurface of the complex projective space $\mathbb{C}P^n$, $n \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube of radius $0 < r < \frac{\pi}{2}$ around a totally geodesic $\mathbb{C}P^k \subset \mathbb{C}P^n$ for some $k \in \{0, \cdots, n-1\}$ or a tube of radius $\frac{\pi}{2} - r$ over $\mathbb{C}P^\ell$, where $k + \ell = n - 1$.

When a real hypersurface M in the complex projective space $\mathbb{C}P^n$ satisfies the formula $A\phi + \phi A = k\phi$, $k \neq 0$ and constant, we say that M is a *contact* real hypersurface in $\mathbb{C}P^n$. In the papers due to Blair [3], Okumura [28] and Yano and Kon [47], they introduce the classification of contact real hypersurfaces in $\mathbb{C}P^n$ as follows:

Theorem B. Let M be a connected orientable real hypersurface in the complex projective space $\mathbb{C}P^n$, $n \geq 3$. Then M is a contact real hypersurface if and only if M is congruent to an open part of a tube of radius $0 < r < \frac{\pi}{4}$ around an n-dimensional real projective space $\mathbb{R}P^n$ or a tube of radius $\frac{\pi}{4} - r$ over Q^{n-1}, where $0 < r < \frac{\pi}{4}$.

Motivated by these results, in this paper we give some characterizations of real hypersurfaces in the complex projective space $\mathbb{C}P^n$ regarding a family of geometric flows. Indeed, we know that a solution of the Ricci flow equation $\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}(g(t))$ is given by

$$\frac{1}{2}(\mathcal{L}_V g)(X, Y) + \mathrm{Ric}(X, Y) = \Omega g(X, Y),$$

where Ω is a constant and $\mathcal{L}_V$ denotes the Lie derivative along the direction of the vector field V (see Chaubey-Suh-De [11], Morgan-Tian [27], Perelman [30], Wang [45] and [46]). Then this solution (M, V, Ω, g) is said to be a *Ricci soliton* with potential vector field V and Ricci soliton constant Ω.

As a generalization of the notion of Ricci flow, the Ricci-Bourguignon flow (see Bourguignon [4] and [5], Catino-Cremaschi-Djadli-Mantegazza-Mazzieri [6]) is given by

$$\frac{\partial}{\partial t} g(t) = -2(\mathrm{Ric}(g(t)) - \frac{1}{2}\rho\gamma g(t)), \quad g(0) = g_0.$$

This family of geometric flows with $\rho = 0$ reduces to the Ricci flow $\frac{\partial}{\partial t} g(t) = -2\mathbb{R}\mathrm{ic}(g(t))$, $g(0) = g_0$. If the constant $\rho = 1$, it is said to be *Einstein flow*. The critical point of the following Einstein flow

$$\frac{\partial}{\partial t} g(t) = -2(\mathrm{Ric}(g(t)) - \frac{1}{2}\gamma g(t)), \quad g(0) = g_0,$$

implies that the Einstein gravitational tensor $\mathrm{Ric}(g(t)) - \frac{1}{2}\gamma g(t)$ vanishes. For a four-dimensional space time M^4, this is equivalent to the vanishing Ricci tensor by virtue of $d\gamma = 2\mathrm{div}(\mathrm{Ric})$. In this case M^4 becomes vacuum. That is, $g(t) = g(0)$, the metric is constant along the time (see O'Neill [29]). For $\rho = \frac{2}{n}$, the tensor $\mathrm{Ric} - \frac{\gamma}{n} g$ is said to be traceless Ricci tensor, and for $\rho = \frac{1}{n-1}$, it is said to be the Schouten tensor.

As another generalization of the notion of Ricci flow, recently, Güler and Crasmareanu [20] introduced a new geometric flow which is a scalar combination of Ricci and Yamabe flow by using Ricci-Yamabe map. This flow is also said to be a Ricci-Yamabe flow of type (δ, ρ). A solution to such a Ricci-Yamabe flow is called Ricci-Yamabe soliton if it moves only by one parameter group of diffeomorphism and scaling. In this case, the Riemannian manifold (M, g), $n > 2$ is said to be (δ, ρ)-Ricci-Yamabe soliton or simply Ricci-Yamabe soliton if it satisfies the equation

$$\frac{1}{2}\mathcal{L}_V g + \delta \mathrm{Ric}_g = (\Omega - \frac{1}{2}\rho\gamma) g,$$

where $\mathcal{L}_V g$ denotes the Lie derivative of the metric g along the vector field, γ the scalar curvature, and Ω, δ, ρ real scalars.

If the Ricci operator Ric of a real hypersurface M in $\mathbb{C}P^n$ satisfies

$$\mathrm{Ric}(X) = aX + b\eta(X)\xi \tag{1.2}$$

for smooth functions a, b on M, then M is said to be *pseudo-Einstein*. Then we introduce a complete classification for pseudo-Einstein Hopf real hypersurfaces in the complex projective space $\mathbb{C}P^n$ due to Cecil and Ryan [8] as follows:

Theorem C. Let M be a pseudo-Einstein real hypersurface in the complex projective space $\mathbb{C}P^n$, $m \geq 3$. Then M is locally congruent to one of the following:

(i) a geodesic hypersphere,
(ii) a tube of radius r around a totally geodesic $\mathbb{C}P^k$, $0 < k < n-1$, where $0 < r < \frac{\pi}{2}$ and $\cot^2 r = \frac{k}{n-k-1}$,
(iii) a tube of radius r around a complex quadric Q^{n-1} where $0 < r < \frac{\pi}{4}$ and $\cot^2 2r = n-2$.

Let M be a Hopf hypersurface in the complex projective space $\mathbb{C}P^n$. Then we have

$$A\xi = \alpha\xi$$

for the shape operator A with the Reeb function $\alpha = g(A\xi, \xi)$ on M in $G_2(\mathbb{C}^{n+2})$. When we consider a tensor field J for any vector field X on M , which is a Kähler structure on the tangent space T_zM, $z \in M$, then JX is given by

$$JX = \phi X + \eta(X)N,$$

where $\phi X = (JX)^T$ is the tangential component of the vector field JX, $\eta(X) = g(\xi, X)$, $\xi = -JN$, and N denotes a unit normal vector field on M.

In this paper we introduce a new notion named *generalized pseudo-anti commuting property* for the Ricci tensor of a real hypersurface M in the complex projective space $\mathbb{C}P^n$ as follows:

$$\mathrm{Ric}\,\phi + \phi\,\mathrm{Ric} = f\phi \tag{1.3}$$

for a smooth function f on M in $\mathbb{C}P^n$ (see Ki and Suh [24], and Yano and Kon [47]).

It is known that Einstein and pseudo-Einstein real hypersurfaces M in the complex projective space $\mathbb{C}P^m$ (see Besse [1], Cecil-Ryan [8]) satisfy the condition of *generalized pseudo-anti commuting* Ricci tensor, that is, $\mathrm{Ric}\phi + \phi\mathrm{Ric} = f\phi$, where f denotes a smooth function on M in $\mathbb{C}P^m$. Real hypersurfaces of type (B) in the complex projective space $\mathbb{C}P^m$, which are characterized by $A\phi + \phi A = k\phi$, $k \neq 0$ also satisfy the formula of *generalized pseudo-anti commuting* Ricci tensor.

Now in this paper, by using the notion of generalized pseudo-anti commuting Ricci tensor (1.2), we give a theorem as follows:

Main Theorem 1. Let M be a Hopf pseudo-Ricci-Yamabe soliton $(M, \xi, \delta, \Omega, \rho, \gamma, g)$ of the type (δ, ρ) in the complex projective space $\mathbb{C}P^n$, $n \geq 3$. Then M is locally congruent to one of the following:

(i) a geodesic hypersphere, $\Omega - \frac{1}{2}\rho\gamma = 2\{(n-1)\cot^2(r) + n\}\delta$, and $\psi = 2n\delta$,

(ii) a tube of radius r around a totally geodesic CP^k, $0 < k < n-1$, where $0 < r < \frac{\pi}{2}$, $\cot^2(r) = \frac{k}{n-k-1}$, $\Omega - \frac{1}{2}\rho\gamma = 2n\delta$, and $\psi = 2\delta$,

(iii) a trivial Yamabe soliton with $\Omega = \frac{1}{2}\rho\gamma$, and $\psi = 0$,

Let us denote by Df the gradient vector field of the function f on a real hypersurface M in the complex projective space $\mathbb{C}P^n$ defined by $g(Df, X) = g(\mathrm{grad} f, X) = X(f)$ for any tangent vector field X on M.

Now let us consider the *gradient pseudo-Ricci-Yamabe soliton* $(M, Df, \xi, \delta, \Omega, \rho, \gamma, g)$. It is a generalization of gradient η-Einstein soliton derived from a generalized Ricci potential for a Riemannian manifold (M, g) (see Catino-Mazzieri [7], Cernea-Guan [9]). It is defined by

$$\mathrm{Hess}(f) + \delta \mathrm{Ric} + \psi \eta \otimes \eta = (\Omega - \frac{1}{2}\rho\gamma)g,$$

where Hess(f) is defined by $\mathrm{Hess}(f) = \nabla Df$ and for any tangent vector fields X and Y on M

$$\mathrm{Hess}(f)(X, Y) = XY(f) - (\nabla_X Y)f.$$

Then a gradient pseudo-Ricci-Bourguignon soliton in $\mathbb{C}P^n$ can be defined by

$$\nabla_X Df + \delta \mathrm{Ric}(X) + \psi \eta(X)\xi = (\Omega - \frac{1}{2}\rho\gamma)X$$

for any vector field X tangent to M in $\mathbb{C}P^n$. Then first by Theorem A we can assert a classification theorem for gradient pseudo-Ricci-Bourguignon solitons in $\mathbb{C}P^n$ as follows:

Main Theorem 2. Let M be a real hypersurface in $\mathbb{C}P^n$ with isometric Reeb flow, $n \geq 3$. If it admits a gradient pseudo-Ricci-Yamabe soliton $(M, Df, \xi, \delta, \Omega, \rho, \gamma, g)$ of the type (δ, ρ), then M is locally congruent to one of the following

(i) a geodesic hypersphere, $\Omega - \frac{1}{2}\rho\gamma = 2\{(n-1)\cot^2(r) + n\}\delta$, and $\psi = 2n\delta$,

(ii) a tube of radius r around a totally geodesic $\mathbb{C}P^k$, $0 < k < n-1$, where $0 < r < \frac{\pi}{2}$, $\cot^2(r) = \frac{k}{n-k-1}$, $\Omega - \frac{1}{2}\rho\gamma = 2n\delta$, and $\psi = 2\delta$,

(iii) a trivial gradient Yamabe soliton with $\Omega = \frac{1}{2}\rho\gamma$, and $\psi = 0$,

Next by virtue of Theorem 1 let us consider a contact real hypersurface in the complex projective space $\mathbb{C}P^n$. Then we can assert a classification of gradient pseudo-Ricci-Yamabe soliton in $\mathbb{C}P^n$ as follows:

Main Theorem 3. Let M be a contact real hypersurface in the complex projective space $\mathbb{C}P^n$, $n \geq 3$. If it admits the gradient pseudo-Ricci-Yamabe soliton $(M, Df, \xi, \delta, \Omega, \rho, \gamma, g)$ of the type (δ, ρ), then M is pseudo-Einstein and locally congruent to a tube of radius r around a complex quadric Q^{n-1} where $0 < r < \frac{\pi}{4}$ and $\cot^2(2r) = n-2$. Moreover, the soliton constants are given by $\Omega - \frac{1}{2}\rho\gamma = 2n\delta$, and $\psi = 2(2n-1)\delta$, or otherwise it becomes a trivial gradient Yamabe soliton with $\Omega = \frac{1}{2}\rho\gamma$ and $\psi = 0$.

2. Some general equations

Let M be a real hypersurface in the complex projective space $\mathbb{C}P^n$ and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M. Then the vector field ξ is said to be the *Reeb* vector field on M in $\mathbb{C}P^n$. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

We now assume that M is a Hopf hypersurface. Then we have

$$A\xi = \alpha\xi,$$

where A denotes the shape operator of M in $\mathbb{C}P^n$ and the smooth function α is defined by $\alpha = g(A\xi, \xi)$ on M. When we consider the transformed JX by the Kähler structure J on $\mathbb{C}P^n$ for any vector field X on M in $\mathbb{C}P^n$, we may write

$$JX = \phi X + \eta(X)N.$$

Then by using Kähler structure $\bar{\nabla}J = 0$, we get the following

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi \text{ and } \nabla_X\xi = \phi AX,$$

where $\bar{\nabla}$ is the Levi-Civita connection of $\mathbb{C}P^n$ and ∇ the corresponding one on M.

Now we consider the equation of Codazzi

$$g((\nabla_X A)Y - (\nabla_Y A)X, Z) = \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y).$$

By the equation of Gauss, the curvature tensor $R(X, Y)Z$ for a real hypersurface M in $\mathbb{C}P^n$ induced from the curvature tensor $\bar{R}$ of $\mathbb{C}P^n$ can be described in terms of the almost contact structure tensor ϕ and the shape operator A of M in $\mathbb{C}P^n$ as follows:

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y \\ & + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ & + g(AY, Z)AX - g(AX, Z)AY \end{aligned} \tag{2.1}$$

for any vector fields $X, Y, Z \in T_zM$, $z \in M$. From this, contracting Y and Z on M in $\mathbb{C}P^n$, we get the Ricci tensor of a real hypersurface M in $\mathbb{C}P^n$ as follows:

$$\mathrm{Ric}(X) = (2n + 1)X - 3\eta(X)\xi + hAX - A^2X, \tag{2.2}$$

where h denotes the trace of the shape operator A of M in $\mathbb{C}P^n$. Then by contracting the Ricci operator in (2.2) the scalar curvature γ of M in $\mathbb{C}P^n$ is given by

$$\gamma = \sum_{i=1}^{2n-1} g(\mathrm{Ric}(e_i), e_i) = 4(n^2 - 1) + h^2 - \mathrm{Tr}A^2, \tag{2.3}$$

where the function h denotes the trace of the shape operator A of M in $\mathbb{C}P^n$.

Putting $Z = \xi$ in the Codazzi equation, we get

$$g((\nabla_X A)Y - (\nabla_Y A)X, \xi) = -2g(\phi X, Y).$$

Since we have assumed that M is Hopf in $\mathbb{C}P^n$, differentiating $A\xi = \alpha\xi$ gives

$$(\nabla_X A)\xi = (X\alpha)\xi + \alpha\phi AX - A\phi AX.$$

From this, the left side of the above equation becomes

$$\begin{aligned} & g((\nabla_X A)Y - (\nabla_Y A)X, \xi) \\ & = g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X) \\ & = (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((A\phi + \phi A)X, Y) - 2g(A\phi AX, Y). \end{aligned}$$

Putting $X = \xi$ in above two equations and using the almost contact structure of (M, g), we have

$$Y\alpha = (\xi\alpha)\eta(Y).$$

Inserting this formula into two previous equation implies

$$0 = 2g(A\phi AX, Y) - \alpha g((\phi A + A\phi)X, Y) - 2g(\phi X, Y).$$

By virtue of this equation, we can assert the following

Lemma 2.1. *Let M be a Hopf real hypersurface in* $\mathbb{C}P^n$, $n \geq 3$. *Then we obtain*

$$2A\phi AX = \alpha(A\phi + \phi A)X + 2\phi X$$

for any tangent vector field X on M.

In the proof of our Theorems 1 and 2, we want to provide further details about Hopf real hypersurfaces in the complex projective space. By using the formulas given in section 3 we want to introduce an important lemma due to Okumura [28] and Yano and Kon [47] as follows:

Lemma 2.2. *Let M be a Hopf real hypersurface in* $\mathbb{C}P^n$. *Then the Reeb function* α *is constant. Moreover, if* $X \in \mathcal{C}$ *is a principal curvature vector of M with principal curvature* λ, *then* $2\lambda \neq \alpha$ *and* ϕX *is a principal curvature vector of M with principal curvature* $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.

3. Fundamental Theorem and some Propositions

By using Takagi's classification of homogeneous real hypersurfaces in the complex projective space $\mathbb{C}P^n$ (see [42], [43], and [44]), Kimura [25] gave the following

Theorem 3.1. *Let M be a Hopf real hypersurface in the complex projective space* $\mathbb{C}P^n$, $n \geq 2$, *with constant principal curvatures. Then, M is holomorphic congruent to an open part of the following hypersurfaces in* $\mathbb{C}P^n$.

- (A_1) *geodesic hyperspheres, that is, a tube of radius r around a totally geodesic* $\mathbb{C}P^{n-1}$ *with* $0 < r < \frac{\pi}{2}$,
- (A_2) *a tube of radius r around a totally geodesic* $\mathbb{C}P^k$ $(1 \leq k \leq n-2)$ *with* $0 < r < \frac{\pi}{2}$,
- (B) *a tube of radius r around a totally geodesic real projective space* $\mathbb{R}P^n$ *in* $\mathbb{C}P^n$ *with* $0 < r < \frac{\pi}{4}$,
- (C) *a tube of radius r around the Segre embedding of* $\mathbb{C}P^1 \times \mathbb{C}P^k$, $n = 2k+1$ *and* $n \geq 5$ *with* $0 < r < \frac{\pi}{4}$,
- (D) *a tube of radius r around a complex Grassmannian* $G_2(\mathbb{C}^3) = SU(5)/S(U(3)U(2))$ *and* $n = 9$ *with* $0 < r < \frac{\pi}{4}$,
- (E) *a tube of radius r around a Hermitian symmetric space* $SO(10)/U(5)$ *and* $n = 15$ *with* $0 < r < \frac{\pi}{4}$.

In literature, a real hypersurface is said to be of type (A) if it is of type (A_1) or (A_2) in $\mathbb{C}P^n$. By using (2.2) and (2.3), we introduce an important proposition due to Cecil-Ryan [8], Djorić and Okumura [19] as follows:

Proposition 3.2. *Let M be the tube of radius* $0 < r < \frac{\pi}{2}$ *around the totally geodesic* $\mathbb{C}P^k$, $k \in \{1, \cdots, n-2\}$ *in* $\mathbb{C}P^n$, *which is called a real hypersurface of type* (A_2). *Then the following statements hold:*

(1) *M is a Hopf hypersurface with constant principal curvatures.*

(2) *The principal curvatures and corresponding principal curvature spaces of M are given by*

principal curvature	eigenspace	multiplicity
$\lambda = \cot(r)$	$T_\lambda = \{X \in TM \,\vert\, X \perp \xi, AX = \lambda X\}$	2ℓ
$\mu = -\tan(r)$	$T_\mu = \{X \in TM \,\vert\, X \perp \xi, AX = \mu X\}$	$2k$
$\alpha = 2\cot(2r)$	$T_\alpha = \mathbb{R}JN = \text{span}\{\xi\}$	1

where $\ell = n - k - 1$.

(3) *The shape operator A commutes with the structure tensor field* ϕ *as*

$$A\phi = \phi A.$$

That is, such a real hypersurface has isometric Reeb flow.

(4) *The trace h of the shape operator A and its square* h^2 *becomes the following respectively*

$$h = (2\ell+1)\cot(r) - (2k+1)\tan(r),$$

$$h^2 = (2\ell+1)^2\cot^2(r) + (2k+1)^2\tan^2(r) - 2(2\ell+1)(2k+1).$$

(5) *The trace of the matrix A^2 is given by*

$$\mathrm{Tr}A^2 = (2\ell + 1)\cot^2(r) + (2k + 1)\tan^2(r) - 2.$$

(6) *The scalar curvature γ of the tube M is given by*

$$\gamma = 4(n-1)n - 8k\ell + 2(2\ell + 1)\ell\cot^2(r) + 2(2k + 1)k\tan^2(r).$$

According to Theorem 3.1, there exist another tubes of radius $0 < r < \frac{\pi}{2}$ around the totally geodesic $\mathbb{C}P^{n-1}$ in $\mathbb{C}P^n$, which is said to be *of type* (A_1). Such hypersurfaces are tubes over complex projective hyperplanes and also *geodesic spheres*. For example, the geodesic sphere centered at πe_0 with radius $\frac{\pi}{2} - r$ coincides with the tube of radius r over the totally geodesic $\mathbb{C}P^{n-1} = \pi\{z\,|\,z_0 = 0\}$. Here, π is the canonical projection from $S^{2n+1}(1) \to \mathbb{C}P^n$ with fiber S^1. The principal curvatures of one are related to those of the other by replacing the parameter r by $\frac{\pi}{2} - r$. Thus, these tubes satisfy the following statements:

Proposition 3.3. *Let M be the tube of radius $0 < r < \frac{\pi}{2}$ around the totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^n$, either $k = 0$ or $k = n - 1$. Then the following statements hold:*

(1) *M is a Hopf hypersurface with constant principal curvatures.*
(2) *The principal curvatures and corresponding principal curvature spaces of M are given by*

- $k = 0$

principal curvature	eigenspace	multiplicity
$\lambda = \cot(r)$	T_λ	$2n-2$
$\alpha = 2\cot(2r)$	$T_\alpha = \mathbb{R}JN$	1

- $k = n - 1$

principal curvature	eigenspace	multiplicity
$\mu = -\tan(r)$	T_μ	$2n-2$
$\alpha = 2\cot(2r)$	$T_\alpha = \mathbb{R}JN$	1

(3) *The shape operator A commutes with the structure tensor field ϕ as*

$$A\phi = \phi A,$$

that is, the Reeb flow of M is isometric.
(4) *The scalar curvatures γ of the tube M of type (A_1) and (A_2) are, respectively, given as follows.*

- $k = 0$

$$\begin{aligned}\gamma &= 4(n^2-1) + h^2 - \mathrm{Tr}A^2\\ &= 4n(n-1) + 2(2n-1)\cot^2 r,\end{aligned}$$

where $\mathrm{Tr}A = h = (2n-1)\cot r - \tan r$, $h^2 = (2n-1)^2\cot^2 r + \tan^2 r - 2(2n-1)$ *and* $\mathrm{Tr}A^2 = (2n-1)\cot^2 r + \tan^2 r - 2$.

- $k = n - 1$

$$\begin{aligned}\gamma &= 4(n^2-1) + h^2 - \mathrm{Tr}A^2\\ &= 4n(n-1) + 2(2n-1)\tan^2 r,\end{aligned}$$

where $h = \cot r - (2n-1)\tan r$, $h^2 = (2n-1)^2\tan^2 r + \cot^2 r - 2(2n-1)$ *and* $\mathrm{Tr}A^2 = (2n-1)\tan^2 r + \cot^2 r - 2$.

Now, let M be a tube of radius r, $0 < r < \frac{\pi}{4}$, over the real projective space $\mathbb{R}P^n$, which is said to be of type (B) and a contact real hypersurface in the complex projective space $\mathbb{C}P^n$. It also can be regarded as a tube of radius $\frac{\pi}{4} - r$ over a totally geodesic complex quadric Q^{n-1}. Then by (2.2) and (2.3), we want to give an important proposition due to Cecil and Ryan [8] as follows:

Proposition 3.4. *Let M be the tube of radius $0 < r < \frac{\pi}{4}$ around the complex quadric Q^{n-1} in $\mathbb{C}P^n$. Then the following statements hold:*

(1) *M is a Hopf hypersurface.*
(2) *The principal curvatures and corresponding principal curvature spaces of M are*

principal curvature	*eigenspace*	*multiplicity*
$\lambda = -\cot(\frac{\pi}{4} - r)$	T_λ	$n-1$
$\mu = \tan(\frac{\pi}{4} - r)$	T_μ	$n-1$
$\alpha = 2\cot(2r)$	$\mathbb{R}JN$	1

(3) *The shape operator A and the structure tensor field ϕ satisfy*

$$A\phi + \phi A = k\phi, \quad k \neq 0 : const.$$

(4) *The trace h of the shape operator A and its square h^2 becomes the following respectively*

$$h = \mathrm{Tr}A = 2\cot(2r) - 2(n-1)\tan(2r),$$
$$h^2 = 4\cot^2(r) + 4(n-1)^2\tan^2(2r) - 8(n-1).$$

(5) *The trace of the matrix A^2 is given by*

$$\mathrm{Tr}A^2 = 4\cot^2(2r) + 4(n-1)\tan^2(2r).$$

(6) *The scalar curvature γ of the tube M is given by*

$$\gamma = 4(n-1)^2 + 4(n-1)(n-2)\tan^2(2r).$$

(7) *For $\cot^2(r) = n-2$, M is pseudo-Einstein such that*

$$\mathrm{Ric(X)} = 2\mathrm{n}\mathrm{X} - 2(2\mathrm{n}-1)\eta(\mathrm{X})\xi.$$

4. Hopf Pseudo-Ricci-Yamabe soliton in $\mathbb{C}P^n$

Now let us introduce pseudo-Ricci-Yamabe soliton $(M, \xi, \delta, \Omega, \rho, \gamma, g)$ of type (δ, ρ) which is a solution of the pseudo-Ricci-Yamabe flow as follows:

$$\frac{1}{2}(\mathcal{L}_V g)(X,Y) + \delta \mathrm{Ric}(X,Y) + \psi\eta(X)\eta(Y) = (\Omega - \frac{1}{2}\rho\gamma)g(X,Y),$$

for any tangent vector fields X and Y on M, where Ω is a Ricci-Yamabe soliton constant, δ, ψ, and ρ any constants and γ the constant scalar curvature on M, and $\mathcal{L}_V$ denotes the Lie derivative along the direction of the vector field V (see Chaubey-Siddiqi-Prakasha [14], and Morgan-Tian [27]). Then let us consider the Reeb vector field ξ as the pseudo-Ricci-Yamabe soliton vector field V as follows:

$$\frac{1}{2}(\mathcal{L}_\xi g)(X,Y) + \delta \mathrm{Ric}(X,Y) + \psi\eta(X)\eta(Y) = (\Omega - \frac{1}{2}\rho\gamma)g(X,Y), \tag{4.1}$$

Then by virtue of the Lie derivative $(\mathcal{L}_\xi g)(X,Y) = g(\nabla_X\xi, Y) + g(\nabla_Y\xi, X)$, the formula (4.1) can be given by

$$\delta \mathrm{Ric}(X) = \frac{1}{2}(A\phi - \phi A)X - \psi\eta(X)\xi + (\Omega - \frac{1}{2}\rho\gamma)X. \tag{4.2}$$

From this, by applying the structure tensor ϕ to both sides, we get the following two formulas

$$\delta \mathrm{Ric}(\phi X) = \frac{1}{2}(A\phi^2 - \phi A\phi)X - \psi\eta(\phi X)\xi + (\Omega - \frac{1}{2}\rho\gamma)\phi X,$$

and

$$\delta\phi\mathrm{Ric}(X) = \frac{1}{2}(\phi A\phi - \phi^2 A)X - \psi\eta(X)\phi\xi + (\Omega - \frac{1}{2}\rho\gamma)\phi X.$$

By using the almost contact structure (ϕ, ξ, η, g) in the right side above, we know that the *generalized pseudo-anti commuting property* holds as follows:

$$\delta(\mathrm{Ric}(\phi X) + \phi\mathrm{Ric}(X)) = 2(\Omega - \frac{1}{2}\rho\gamma)\phi X. \tag{4.3}$$

Now let us consider (4.3) into the following two cases.
Case 1. $\delta\neq 0$.

Then by Lemmas 2.1 and 2.2, if $X\in T_\lambda$, then $\phi X\in T_\mu$, where $\mu = \frac{\alpha\lambda-2}{2\lambda-\alpha}$. Then by substituting (2.2) into (4.3), we have the following for $X\in T_\lambda$

$$\lambda^2 + \mu^2 - h(\lambda+\mu) = k, \tag{4.4}$$

where the function k is given by $k = -\frac{2}{\delta}(\Omega - \frac{1}{2}\rho\gamma) + 2(2n+1)$. Then substituting $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$ into (4.4), it gives the following

$$\begin{aligned} &4\lambda^4 - 4(\alpha+h)\lambda^3 + 2(\alpha^2 + \alpha h - 2k)\lambda^2 \\ &\quad + 4(\alpha - h + \alpha k)\lambda + 4 + 2\alpha h - k\alpha^2 = 0. \end{aligned} \tag{4.5}$$

Let λ_1, λ_2, λ_3 and λ_4 are the roots of the above biquadric equation. Then from the relations of the roots and coefficient of the equation (4.4) it follows that

$$\begin{aligned} &\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \alpha + h \\ &\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = (\alpha^2 + \alpha h - 2k)/2 \\ &\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_4 = \alpha - h + \alpha k \\ &\lambda_1\lambda_2\lambda_3\lambda_4 = (4 + 2\alpha h - k\alpha^2)/4. \end{aligned} \tag{4.6}$$

Here we consider the trace h of the shape operator A of M in complex projective space $\mathbb{C}P^n$. Then it is defined by

$$h = \alpha + m_1\lambda_1 + m_2\lambda_2 + m_1\frac{\alpha\lambda_1 + 2}{2\lambda_1 - \alpha} + m_2\frac{\alpha\lambda_2 + 2}{2\lambda_2 - \alpha}.$$

From this, together with (4.6) and note that the principal curvature α is constant, and the scalar curvature γ in (2.3) is

$$\gamma = -4(n^2 - 1) + h^2 - \mathrm{Tr}A^2, \tag{4.7}$$

where $\mathrm{Tr}A^2$ is given by

$$\mathrm{Tr}A^2 = \alpha^2 + m_1\lambda_1^2 + m_2\lambda_2^2 + m_1\Big(\frac{\alpha\lambda_1 - 2}{2\lambda_1 - \alpha}\Big)^2 + m_2\Big(\frac{\alpha\lambda_2 - 2}{2\lambda_2 - \alpha}\Big)^2. \tag{4.8}$$

So substituting h and k into (4.6), and using (4.7) and (4.8), we can see that (4.6) consists of four linearly independent equations with constant multiplicities m_1, m_1, m_2 and m_2 of the principal curvatures λ_1, μ_1, λ_2, and μ_2 respectively. Consequently, it can be asserted that M has at most 5 distinct constant principal curvatures in $\mathbb{C}P^n$. Then we have introduced Theorem 3.1, which is an important result due to Kimura [25], mentioned in section 3

Now a geodesic hypersphere ($k = 0, k = n-1$) and a tube of radius r over $\mathbb{C}P^k$ in $\mathbb{C}P^n$ for some $k\in\{1, \cdots, n-2\}$, belongs to the class of the tube of the first in Theorem 3.1 respectively. So by Theorem A, they are characterized by the commuting shape operator. That is, $A\phi = \phi A$. Accordingly, from the notion of pseudo-Ricci-Bourguignon soliton $(M, \xi, \Omega, \rho, \gamma, g)$ of M, (4.1) becomes

$$\delta\text{Ric} = (\Omega - \frac{1}{2}\rho\gamma)g - \psi\eta\otimes\eta.$$

Moreover, by Proposition 3.3 and (2.2), we get the following

$$\delta\text{Ric}(X) = 2\delta\{(n-1)\cot^2(r) + n\}X - 2n\delta\eta(X)\xi.$$

Then comparing these two formulas implies $\Omega - \frac{1}{2}\rho\gamma = 2\delta\{(n-1)\cot^2(r) + n\}$ and $\psi = 2n\delta$.

Now let us consider of type (A_2). Then it means that the gradient pseudo-Ricci-Yamabe soliton (4.1) becomes pseudo-Einstein, $\delta\text{Ric}(X) = (\Omega - \frac{1}{2}\rho\gamma)X - \psi\eta(X)\xi$. From this, we assert that $\Omega - \frac{1}{2}\rho\gamma = 2n\delta$ and $\psi = 2\delta$ for a non-vanishing constant $\delta \neq 0$.

In order to show this, let us put $\text{Ric}(X) = aX + b\eta(X)\xi$, where $a = \frac{1}{\delta}(\Omega - \frac{1}{2}\rho\gamma)$ and $b = -\frac{\psi}{\delta}$. Then $a + b = g(\text{Ric}(\xi), \xi) = 2n - 2$ by virtue of $\cot^2(r) = \frac{k}{n-k-1}$, which is used to be Einstein for real hypersurfaces of type (A_2). Moreover, by using Proposition 3.2, for real hypersurfaces of type (A_2) we get easily $\text{Ric}(X) = 2nX$ for any $X \in T_\lambda$, and $\text{Ric}(Y) = 2nY$ for any $Y \in T_\mu$. So it becomes $a = 2n$ and $b = -2$ respectively. This gives the above soliton constants.

In fact, in detail let us consider a real hypersurface of type (A_2) in Proposition 3.2. First along the Reeb direction with $\cot^2(r) = \frac{k}{n-k-1}$ and $\ell = n - k - 1$ we get

$$\begin{aligned}\text{Ric}(\xi) =& \{2(n-1) + h\alpha - \alpha^2\}\xi \\ =& \Big[2(n-1) + \{(2\ell+1)\cot(r) - (2k+1)\tan(r)\}2\cot(2r) - (2\cot(2r))^2\Big]\xi \\ =& \Big[2(n-1) + (2\ell+1)\cot^2(r) + (2k+1)\tan^2(r) \\ & - (2\ell+1) - (2k+1) - \{\cot^2(r) + \tan^2(r) - 2\}\Big]\xi \\ =& \Big[2n - 2k - 2\ell - 2 + 2\ell\cot^2(r) + 2k\tan^2(r)\Big]\xi \\ =& \{4(n-1) - 2k - 2\ell\}\xi \\ =& (2n-2)\xi.\end{aligned}$$

Then it becomes $a+b = 2n-2$. Next we consider for any $X \in T_\lambda$, $\lambda = \cot(r)$ with multiplicities 2ℓ in Proposition 3.2. Then it becomes

$$\begin{aligned}\text{Ric}(X) =& \{(2n+1) + h\lambda - \lambda^2\}X \\ =& \Big[(2n+1) + 2\ell\cot^2(r) - (2k+1)\Big]X \\ =& 2nX\end{aligned}$$

and for any $Y \in T_\mu$, $\mu = -\tan(r)$ with multiplicities $2k$ in Proposition 3.2 we get another formula

$$\begin{aligned}\text{Ric}(Y) =& \{(2n+1) + h\mu - \mu^2\}Y \\ =& \Big[(2n+1) + 2k\tan^2(r) - (2\ell+1)\Big]Y \\ =& 2nY,\end{aligned}$$

where we have used $\cot^2(r) = \frac{k}{n-k-1}$ and $\ell = n - k - 1$ in above two equations. Then both two equations above means that $a = 2n$, which gives $b = (a+b) - a = -2$. From this, we give a detailed assertion mentioned above.

Consequently, by Theorem C there exists such a hypersurface in the complex projective space $\mathbb{C}P^n$ in our Main theorem 1 in the introduction.

Case 2. $\delta = 0$.

For the constant $\delta = 0$, by (4.1) it becomes

$$\frac{1}{2}(\mathcal{L}_\xi g)(X, Y) + \psi\eta(X)\eta(Y) = (\Omega - \frac{1}{2}\rho\gamma)g(X, Y),$$

From this, by putting $X = \xi$, (4.1) reduces to the following

$$\psi = \Omega - \frac{1}{2}\rho\gamma.$$

Then, together with (4.2), it follows the following for the vanishing constant $\delta = 0$

$$\frac{1}{2}(A\phi - \phi A)X - \psi(\eta(X)\xi - X) = 0.$$

Then from contracting this equation we get

$$0 = \psi\sum_{i=1}^{2n-1}(\eta(e_i)g(\xi, e_i) - g(e_i, e_i)) = -2(n-1)\psi.$$

So the constant $\psi = 0$ and $\Omega - \frac{1}{2}\rho\gamma = 0$. So it becomes a trivial Yamabe soliton and $A\phi = \phi A$.

Next, let us check real hypersurfaces of type (B), that is, the third case (iii) in Theorem 3.1. It is characterized by $A\phi + \phi A = \ell\phi$, where $\ell \neq 0$: constant. Moreover, by Proposition 4.4, the principal curvature are given by $\lambda = -\cot(\frac{\pi}{4} - r)$, $\mu = \tan(\frac{\pi}{4} - r)$ and $\alpha = 2\cot(2r)$. So $\ell = \lambda + \mu = -\frac{4}{\alpha}$. For any $X \in T_\lambda$ the vector field $\phi X \in T_\mu$. So (4.2) gives the following for $X \in T_\lambda$

$$\begin{aligned}\delta\mathrm{Ric}(X) =& \frac{1}{2}(\mu - \lambda)\phi X - \psi\eta(X)\xi + (\Omega - \frac{1}{2}\rho\gamma)X \\ =& \frac{1}{2}(\mu - \lambda)\phi X + (\Omega - \frac{1}{2}\rho\gamma)X,\end{aligned} \tag{4.9}$$

where we have used $\eta(X) = 0$ for any $X \in T_\lambda$.

On the other hand, by (2.2) the left side of (4.9) becomes the following for any $X \in T_\lambda$

$$\delta\mathrm{Ric}(X) = \delta\{(2n+1) + (h - \lambda)\lambda\}X, \tag{4.10}$$

where the function h denotes the trace of the shape operator A of M in $\mathbb{C}P^n$. By virtue of (4.10), the first term in the right side of (4.9) is skew-symmetric and the other terms are symmetric. Accordingly, if we take the inner product of (4.9) with any $\phi X \in T_\mu$ for $X \in T_\lambda$ and use (4.10), naturally we get $\lambda = \mu$. That is, $-\cot(\frac{\pi}{4} - r) = \tan(\frac{\pi}{4} - r)$, which gives a contradiction. Consequently, it means that there does not exist a contact hypersurface which admits a pseudo-Ricci-Yamabe soliton in complex projective space $\mathbb{C}P^n$.

Finally, in the remained case, we check of type (C), (D) and (E) in Theorem 3.1 due to Kimura [25].
From (4.4) the principal curvatures (λ_1, μ_1), and (λ_2, μ_2) satisfies the following respectively

$$\lambda_1^2 + \mu_1^2 - h(\lambda_1 + \mu_1) = k, \tag{4.11}$$

and

$$\lambda_2^2 + \mu_2^2 - h(\lambda_2 + \mu_2) = k. \tag{4.12}$$

Then by viture above two fomulas we can assert the following lemma

Lemma 4.1. *The mean curvature h is given by*

$$h = \alpha - \frac{4}{\alpha}.$$

Proof. In Theorem 3.1, let us put 5 distinct constant principal curvatures by $\alpha = 2\cot(2r)$, $\lambda_1 = \cot(r)$, $\mu_1 = -\tan(r)$, $\lambda_2 = \cot(r - \frac{\pi}{4})$ and $\mu_2 = -\tan(r - \frac{\pi}{4})$ with multiplicities 1, m_1, m_1, m_2 and m_2 respectively. Then from (4.11) and (4.12) we get the following formulas respectively

$$\alpha^2 + 2 - h\alpha = k,$$

and

$$\frac{16}{\alpha^2} + 2 + \frac{4h}{\alpha} = k.$$

Then these two equations imply $\alpha^4 - h\alpha^3 - 4\alpha h - 16 = 0$. From this, it follows that $\alpha h = \alpha^2 - 4$. This gives a complete proof of above lemma. □

For a type (*C*) in Theorem 3.1 the principal curvatures are given by $2\cot(2r)$, $\cot(r)$, $-\tan(r)$, $\cot(r - \frac{\pi}{4})$ and $-\tan(r - \frac{\pi}{4})$ with its multiplicities 1, $n-3$, $n-3$, 2 and 2 respectively. Its dimension is $\dim M = 4p - 3$. So the trace of the shape operator h becomes

$$h = \alpha + (n-3)\alpha - \frac{8}{\alpha} = \alpha - \frac{4}{\alpha},$$

where we have used Lemma 4.1 in the second equality. From this, it becomes a tube of radius r such that $\cot^2(2r) = \frac{1}{n-3}$ around the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^k$, $n = 2k+1$ and $n \geq 5$.

For a type (*D*) in Theorem 3.1 the principal curvatures are given by $2\cot(2r)$, $\cot(r)$, $-\tan(r)$, $\cot(r - \frac{\pi}{4})$ and $-\tan(r - \frac{\pi}{4})$ with its multiplicities 1, 4, 4, 4 and 4 respectively. Its dimension is $\dim M = 17$. So the trace of the shape operator h becomes

$$h = 5\alpha + 4(-\frac{4}{\alpha}).$$

From this, together with Lemma 4.1, we get the following

$$\alpha^2 = 4\cot^2(2r) = 3.$$

It means a tube of radius r with $\cot(2r) = \frac{\sqrt{3}}{2}$ around a complex Grassmannian $G_2(\mathbb{C}^3) = SU(5)/S(U(3)U(2))$ and $n = 9$ with $0 < r < \frac{\pi}{4}$.

For a type (*E*) in Theorem 4.1 the principal curvatures are given by $2\cot(2r)$, $\cot(r)$, $-\tan(r)$, $\cot(r - \frac{\pi}{4})$ and $-\tan(r - \frac{\pi}{4})$ with its multiplicities 1, 8, 8, 6 and 6 respectively. Its dimension is $\dim M = 29$. So the trace of the shape operator h becomes

$$h = \alpha + 8\alpha + 6(-\frac{4}{\alpha}).$$

From this, also together with Lemma 4.2, it follows that

$$8\alpha^2 = 20.$$

It means a tube of radius r with $\cot^2(2r) = \frac{\sqrt{5}}{2\sqrt{2}}$ around a Hermitian symmetric space $SO(10)/U(5)$ and $n = 15$ with $0 < r < \frac{\pi}{4}$.

Summing up all of types (*C*), (*D*) and (*E*) mentioned above, we are now going to make a further process as follows:

For $X \in T_{\lambda_1}$, we get $\phi X \in T_{\mu_1}$. Then the following holds from the assumption of pseudo-Ricci-Yamabe for the non-vanishing soliton constant $\delta \neq 0$ and (2.2)

$$\begin{aligned}\delta \mathrm{Ric}(X) =& \frac{1}{2}(\mu_1 - \lambda_1)\phi X + (\Omega - \frac{1}{2}\rho\gamma)X \\ =& \delta\{(2n+1) + h\lambda_1 - \lambda_1^2\}X.\end{aligned} \tag{4.13}$$

For $X \in T_{\mu_1}$, we get $\phi X \in T_{\lambda_1}$. Then the following also holds from the assumption of pseudo-Ricci-Yamabe and (2.2)

$$\begin{aligned}\delta \mathrm{Ric}(X) =& \frac{1}{2}(\lambda_1 - \mu_1)\phi X + (\Omega - \frac{1}{2}\rho\gamma)X \\ =& \delta\{(2n+1) + h\mu_1 - \mu_1^2\}X.\end{aligned} \tag{4.14}$$

Then from (4.13) and (4.14) it follows that

$$0 = (\lambda_1 - \mu_1)(h - (\lambda_1 + \mu_1)).$$

From this, together with Lemma 4.2, we get

$$\alpha - \frac{4}{\alpha} = h = \lambda_1 + \mu_1 = 2\cot(2r) = \alpha.$$

This gives a contradiction.

On the other hand, for vanishing soliton constant $\delta = 0$, (4.13) or (4.14) implies $\lambda_1 = \mu_1$, that is, $\cot(r) = -\tan(r)$. This gives also a contradiction. Accordingly, real hypersurfaces of types (C), (D) and (E) can not satisfy the assumption of pseudo-Ricci-Yamabe soliton. Consequently, we give a complete proof of our Main Theorem 1 in the introduction.

5. Gradient Pseudo-Ricci-Yamabe soliton on isometric Reeb flow in $\mathbb{C}P^n$

In this section let M be a tube of radius r, $0 < r < \frac{\pi}{2}$, over a totally geodesic $\mathbb{C}P^k$, $k \in \{0, 1, \cdots, n-2, n-1\}$ in $\mathbb{C}P^n$, which is said to be of type (A_1) or of type (A_2). In Theorem A, we have mentioned that the Reeb flow on M in $\mathbb{C}P^n$ is isometric if and only if M is locally congruent to a totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^n$ for $k \in \{0, 1, \cdots, n-1\}$. Then for $k = 0$ or $k = n-1$ we say that M is a geodesic hypersphere which is said to be of type (A_1) and it has with two distinct principal curvatures. For $k \in \{1, \cdots, n-2\}$, M is locally congruent to a tube over $\mathbb{C}P^k$ in $\mathbb{C}P^n$. Moreover, it is said to be of type (A_2) and has with three distinct constant principal curvatures.

Then the shape operator of M in the complex projective space $\mathbb{C}P^n$ with isometric Reeb flow can be expressed as

$$A = \begin{bmatrix} \alpha & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cot(r) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \cot(r) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -\tan(r) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & -\tan(r) \end{bmatrix}$$

for three constant principal curvatures $\alpha = 2\cot(2r)$, $\cot(r)$ and $-\tan(r)$ with multiplicities 1, 2ℓ and $2k$ respectively, where $\ell = n - k - 1$.

Then, by putting $X = \xi$ in (2.2), and using $A\xi = \alpha\xi$, we have the following

$$\begin{aligned}\mathrm{Ric}(\xi) =& (2n+1)\xi - 3\xi + hA\xi - A^2\xi \\ =& 2(n-1)\xi + (h\alpha - \alpha^2)\xi \\ =& \kappa\xi,\end{aligned}$$

where we have put $\kappa = 2(n-1) + h\alpha - \alpha^2$. So by Propositions 3.2 and 3.3, the constant κ is given by

$$\begin{aligned}\kappa =&2(n-1) + (h\alpha - \alpha^2)\\ =&2(n-1) + \{(2\ell+1)\cot(r) - (2k+1)\tan(r)\}2\cot(2r) - (2\cot(2r))^2\\ =&2(n-1) + 2\{\ell\cot^2(r) + k\tan^2(r) - (k+\ell)\}\\ =&2\ell\cot^2(r) + 2k\tan^2(r).\end{aligned}$$

Then by taking the covariant derivative we get the following two formulas

$$(\nabla_X \mathrm{Ric})\xi = \kappa\phi AX - \mathrm{Ric}(\phi AX),$$

and

$$(\nabla_\xi \mathrm{Ric})X = h(\nabla_\xi A)X - (\nabla_\xi A^2)X.$$

Since M admits the gradient pseudo-Ricci-Yamabe soliton $(M, Df, \xi, \delta, \Omega, \rho, \gamma, g)$, we could consider the soliton vector field W as $W = Df$ for any smooth function on M. In the introduction we have noted that Hess(f) is defined by Hess$(f) = \nabla Df$ for any tangent vector fields X and Y on M in such a way that

$$\mathrm{Hess}(f)(X, Y) = g(\nabla_X Df, Y).$$

Then the gradient pseudo-Ricci-Yamabe soliton $(M, Df, \xi, \delta, \Omega, \rho, \gamma, g)$ can be given by

$$\nabla_X Df + \delta\mathrm{Ric}(X) + \psi\eta(X)\xi = (\Omega - \frac{1}{2}\rho\gamma)X \tag{5.1}$$

for any tangent vector field X on M. Then by covariant differentiation, it gives

$$\begin{aligned}&\nabla_X\nabla_Y Df + \delta(\nabla_X \mathrm{Ric})(Y) + \delta\mathrm{Ric}(\nabla_X Y)\\ &\quad + \psi(\nabla_X\eta)(Y)\xi + \psi\eta(\nabla_X Y)\xi + \psi\eta(Y)\phi AX\\ =&(\Omega - \frac{1}{2}\rho\gamma)\nabla_X Y\end{aligned}$$

for any vector field X and Y tangent to M in $\mathbb{C}P^n$. From this, together with the above two formulas for the derivative of Ricci operator and the constant scalar curvature γ for the isomeric Reeb flow, it follows that

$$\begin{aligned}R(\xi, Y)Df =&\nabla_\xi\nabla_Y Df - \nabla_Y\nabla_\xi Df - \nabla_{[\xi,Y]}Df\\ =&\delta(\nabla_Y \mathrm{Ric})\xi - \delta(\nabla_\xi \mathrm{Ric})Y + \psi\phi AY\\ =&(\delta\kappa + \psi)\phi AY - \delta\mathrm{Ric}(\phi AY) - \delta h(\nabla_\xi A)Y + \delta(\nabla_\xi A^2)Y.\end{aligned} \tag{5.2}$$

Then from the curvature tensir $R(X, Y)Z$ given in (2.1) we have the following for a real hypersurface M in $\mathbb{C}P^n$ with isometric Reeb flow

$$\begin{aligned}R(\xi, Y)Df =&g(Y, Df)\xi - g(\xi, Df)Y\\ &+ g(AY, Df)A\xi - g(A\xi, Df)AY.\end{aligned} \tag{5.3}$$

From this, let us take a vector field $Y \in T_\lambda$, $\lambda = \cot(r)$. Moreover, we can decompose the tangent space $T\mathbb{C}P^n$ as

$$T\mathbb{C}P^n = T_\lambda \oplus T_\mu \oplus T_\alpha \oplus \mathbb{R}N,$$

where $\lambda = \cot(r)$, $\mu = -\tan(r)$ and $\alpha = 2\cot(2r)$. Then for $Y \in T_\lambda$ (5.3) gives

$$\begin{aligned}&R(\xi, Y)Df\\ =&g(Y, Df)\xi - g(\xi, Df)Y + \alpha\lambda g(Y, Df)\xi - \alpha\lambda g(\xi, Df)Y\\ =&(1 + \alpha\lambda)\{g(Y, Df)\xi - g(\xi, Df)Y\}.\end{aligned} \tag{5.4}$$

Then by taking the inner product of (5.4) with the Reeb vector field ξ and using (5.2), it follows that $(1+\alpha\lambda)g(Y, Df) = \cot^2(r)g(Y, Df) = 0$. But $\cot^2(r)\neq 0$ for the radius $0 < r < \frac{\pi}{2}$ of isometric Reeb flow M in $\mathbb{C}P^n$. It means the following for any $Y \in T_\lambda$

$$g(Y, Df) = 0. \tag{5.5}$$

Now let us check (5.3) for $Y \in T_\mu$, $\mu = -\tan(r)$. Then (5.3) gives

$$R(\xi, Y)Df = g(Y, Df)\xi - g(\xi, Df)Y + \alpha\mu g(Y, Df)\xi - \alpha\mu g(\xi, Df)Y. \tag{5.6}$$

Then by taking the inner product (5.6) with the Reeb vector field ξ and $Y \in T_\mu$ respectively and using (5.2), we get

$$(1+\alpha\mu)g(Y, Df) = 0 \quad \text{and} \quad (1+\alpha\mu)g(\xi, Df) = 0, \tag{5.7}$$

where $g(R(\xi, Y)Df, \xi) = 0$ and the left side $g(R(\xi, Y)Df, Y) = 0$ is given by virtue of the following formulas

$$g(\phi AY, Y) = \mu g(\phi Y, Y) = 0,$$

$$\mathrm{Ric}(\phi AY) = \mu\{(2n+1) + \mu h - \mu^2\}\phi Y,$$

and

$$g((\nabla_\xi A)Y, Y) = -\mu g(\nabla_\xi Y, Y) = 0.$$

Since $1+\alpha\mu = 1 + (\cot(r) - \tan(r))(-\tan(r)) = \tan^2(r) \neq 0$ for $0 < r < \frac{\pi}{2}$ for isometric Reeb flow M in $\mathbb{C}P^n$, (5.7) implies that

$$g(Y, Df) = 0 \text{ and } g(\xi, Df) = 0 \tag{5.8}$$

for any $Y \in T_\mu$, $\mu = -\tan r$.

For a geodesic hypersphere of type (A_1) in $\mathbb{C}P^n$ it holds either $g(Y, Df) = 0$ for $Y\in T_\lambda = \mathcal{C}$ or for $Y\in T_\mu = \mathcal{C}$ from the above decomposition, where $\mathcal{C}$ denotes the orthogonal complement of the Reeb vector field ξ in the tangent space TM of M in $\mathbb{C}P^n$. Of course, it also holds $g(\xi, Df) = 0$ for a geodesic hypersphere in $\mathbb{C}P^n$. Moreover, a geodesic hypersphere in $\mathbb{C}P^n$, it can be decomposed as

$$T\mathbb{C}P^n = T_\lambda \oplus T_\alpha \oplus \mathbb{R}N,$$

or otherwise

$$T\mathbb{C}P^n = T_\mu \oplus T_\alpha \oplus \mathbb{R}N.$$

Then as in section 4, $\Omega - \frac{1}{2}\gamma\delta = 2\{(n-1)\cot^2(r) + n\}$ and $\psi = 2n\delta$.

For another isometric Reeb flow we could consider real hypersurfaces of type (A_2) in $\mathbb{C}P^n$. Then by Proposition 3.2, we may put $\mathrm{Ric}X = aX + b\eta(X)\xi$, where $a = \frac{1}{\delta}(\Omega - \frac{1}{2}\rho\gamma)$ and $b = -\frac{\psi}{\delta}$. Then $a+b = g(\mathrm{Ric}(\xi), \xi) = 2n-2$ by virtue of $\cot^2(r) = \frac{k}{n-k-1}$, which is used to be pseudo-Einstein for real hypersurfaces of type (A_2). Moreover, by using Proposition 3.2, for real hypersurfaces of type (A_2) we get easily $\mathrm{Ric}(X) = 2nX$ for any $X\in T_\lambda$, and $\mathrm{Ric}(Y) = 2nY$ for any $Y\in T_\mu$. So it becomes $a = 2n$ and $b = -2$ respectively. This gives the above soliton constants.

Summing up (5.5), (5.8) and the above documents, the gradient Df of the smooth function f is identically vanishing, that is, $g(Y, Df) = 0$ for any tangent vector field $Y \in T_zM$, $z \in M$. Consequently, we conclude that the gradient pseudo-Ricci-Yamabe soliton $(M, Df, \xi, \delta, \Omega, \rho, \gamma, g)$ is trivial. That is, $Df = 0$, the potential function f is constant on M. This means that the gradient pseudo-Ricci-Yamabe soliton (5.1) becomes pseudo-Einstein, that is, $\delta\mathrm{Ric}(X) = (\Omega - \frac{1}{2}\gamma)X - \psi\eta(X)\xi$, where $\Omega - \frac{1}{2}\rho\gamma = 2n\delta$ and $\psi = 2\delta$ for a non-vanishing constant $\delta\neq 0$.

When the constant function $\delta = 0$ vanishes, it implies that $Df = 0$ and $\Omega = \frac{1}{2}\rho\gamma$ and the constant $\psi = 0$. So it becomes a trivial gradient Yamabe soliton. Then by Theorem C, we get a complete proof of our Main Theorem 2 in the Introduction.

6. Gradient pseudo-Ricci-Yamabe soliton on contact real hypersurfaces in $\mathbb{C}P^n$

In this section, we want to give a property for gradient pseudo-Ricci-Yamabe soliton on a contact real hypersurface M in the complex projective space $\mathbb{C}P^n$. Then by Theorem B the scalar curvature γ is constant. The gradient pseudo-Ricci-Yamabe soliton $(M, Df, \xi, \delta, \Omega, \rho, \gamma, g)$ gives the following for any tangent vector field X on M in $\mathbb{C}P^n$

$$\nabla_X Df + \delta \mathrm{Ric}(X) + \psi\eta(X)\xi = (\Omega - \frac{1}{2}\rho\gamma)X. \tag{6.1}$$

Then by differentiating (6.1), the curvature tensor of $Df = \mathrm{grad} f$ is given by the following

$$\begin{aligned} R(X,Y)Df =& \delta\nabla_X\nabla_Y Df - \delta\nabla_Y\nabla_X Df - \nabla_{[X,Y]}Df \\ =& -(\nabla_X \mathrm{Ric})Y - \mathrm{Ric}(\nabla_X Y) - \psi(\nabla_X\eta)(Y)\xi - \psi\eta(\nabla_X Y)\xi \\ & -\psi\eta(Y)\nabla_X\xi + (\Omega - \frac{1}{2}\rho\gamma)\nabla_X Y \\ & + \delta(\nabla_Y \mathrm{Ric})X + \delta\mathrm{Ric}(\nabla_Y X) + \psi(\nabla_Y\eta)(X)\xi + \psi\eta(\nabla_Y X)\xi \\ & + \psi\eta(X)\nabla_Y\xi - (\Omega - \frac{1}{2}\rho\gamma)\nabla_X Y \\ & + \delta\mathrm{Ric}([X,Y]) - (\Omega - \frac{1}{2}\rho\gamma)[X,Y] + \psi\eta([X,Y])\xi \\ =& \delta(\nabla_Y\mathrm{Ric})X - \delta(\nabla_X\mathrm{Ric})Y - \psi(\nabla_X\eta)(Y)\xi + \psi(\nabla_Y\eta)(X)\xi \\ & - \psi\eta(Y)\nabla_X\xi + \psi\eta(X)\nabla_Y\xi \end{aligned} \tag{6.2}$$

where we have used the Ricci soliton constant θ and gradient pseudo-Ricci-Bourguignon soliton constant Ω, and the scalar curvature γ is constant on a contact real hypersurface M in $\mathbb{C}P^n$ in Proposition 3.4.

Now let us assume that M is a contact real hypersurface in $\mathbb{C}P^n$, which is characterized by

$$A\phi + \phi A = k\phi, \text{where } k{\neq}0 : \text{const.}$$

Then it is Hopf and the Ricci operator is given by

$$\mathrm{Ric}(X) = (2n+1)X - 3\eta(X)\xi + hAX - A^2X$$

for any tangent vector field X on M. From this, let us put $X = \xi$. Then M being Hopf and $A\xi = \alpha\xi$ implies

$$\mathrm{Ric}(\xi) = \ell\xi,$$

where $\ell = 2(n-1) + h\alpha - \alpha^2$ is constant, and the mean curvature $h = \mathrm{Tr}A$ is constant for a contact hypersurface M in $\mathbb{C}P^n$. Then by taking covariant derivative to the Ricci operator, we have

$$(\nabla_X\mathrm{Ric})\xi = \nabla_X(\mathrm{Ric}(\xi)) - \mathrm{Ric}(\nabla_X\xi) = \ell\phi AX - \mathrm{Ric}(\phi AX),$$

and

$$\begin{aligned} (\nabla_\xi\mathrm{Ric})(X) =& \nabla_\xi(\mathrm{Ric}X) - \mathrm{Ric}(\nabla_\xi X) \\ =& h(\nabla_\xi A)X - (\nabla_\xi A^2)X. \end{aligned}$$

From (6.2), together with above formula, by putting $X = \xi$ we have the following for a contact hypersurface M in $\mathbb{C}P^n$

$$\begin{aligned} R(\xi,Y)Df =& \delta(\nabla_Y\mathrm{Ric})\xi - \delta(\nabla_\xi\mathrm{Ric})Y \\ & - \psi(\nabla_\xi\eta)(Y)\xi + \psi(\nabla_Y\eta)(\xi)\xi - \psi\eta(Y)\nabla_\xi\xi + \psi\eta(\xi)\nabla_Y\xi \\ =& (\delta\ell + \psi)\phi AY - \delta\mathrm{Ric}(\phi AY) - h(\nabla_\xi A)Y + (\nabla_\xi A^2)Y. \end{aligned} \tag{6.3}$$

Then the diagonalization of the shape operator A of the contact real hypersurface in complex hyperbolic quadric $\mathbb{C}P^n$ is given by

$$A = \begin{bmatrix} 2\cot(2r) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & -\cot(\frac{\pi}{4}-r) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & -\cot(\frac{\pi}{4}-r) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \tan(\frac{\pi}{4}-r) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \tan(\frac{\pi}{4}-r) \end{bmatrix}.$$

Here by Proposition 3.4 the principal curvatures are given by $\alpha = 2\cot(2r)$, $\lambda = -\cot(\frac{\pi}{4}-r)$ and $\mu = \tan(\frac{\pi}{4}-r)$ with multiplicities 1, $n-1$ and $n-1$ respectively. All of these principal curvatures satisfy

$$\kappa = \lambda + \mu = -\cot(\frac{\pi}{4} - r) + \tan(\frac{\pi}{4} - r) = -2\tan(2r) = -\frac{4}{\alpha}.$$

On the other hand, the curvature tensor $R(X,Y)Z$ of M induced from the curvature tensor $\bar{R}(X,Y)Z$ of the complex projective space $\mathbb{C}P^n$ gives

$$\begin{aligned} R(\xi, Y)Df =& g(Y, Df)\xi - g(\xi, Df)Y \\ &+ g(AY, Df)A\xi - g(A\xi, Df)AY \\ =&(1+\alpha\lambda)\{g(Y, Df)\xi - g(\xi, Df)Y\} \end{aligned} \tag{6.4}$$

for any $Y \in T_\lambda$, $\lambda = -\cot(\frac{\pi}{4} - r)$ for a contact real hypersurface M in the complex projective space $\mathbb{C}P^n$. Consequently, (6.3) and (6.4) give

$$(\delta\ell + \psi)\phi AY - \delta\mathrm{Ric}(\phi AY) - h(\nabla_\xi A)Y + (\nabla_\xi A^2)Y = (1+\alpha\lambda)\{g(Y, Df)\xi - g(\xi, Df)Y\}.$$

From this, by taking the inner product with the Reeb vector field ξ, we have

$$(1+\alpha\lambda)g(Y, Df) = 0. \tag{6.5}$$

Then for any $Y \in T_\lambda$ in (6.5) it follows that

$$g(Y, Df) = 0, \tag{6.6}$$

where we have noted that $1+\alpha\lambda = 1+2\cot(2r)(-\cot(\frac{\pi}{4}-r)) \neq 0$. Because if we assume that $1 = 2\cot(2r)\cot(\frac{\pi}{4}-r)$, then $\tan(2r) = 2\cot(\frac{\pi}{4} - r)$. Then it follows that

$$(\cos(r) - \sin(r))\sin(r)\cos(r) = (\cos(r) + \sin(r))^2(\cos(r) - \sin(r)),$$

which gives $\sin(r)\cos(r) = -1$. This gives us a contradiction for $0 < r < \frac{\pi}{4}$. Accordingly, the gradient vector field Df is orthogonal to the eigenspace T_λ, that is, $g(Y, Df) = 0$ for any $Y \in T_\lambda$.

Next, we consider for $Y \in T_\mu$, $\mu = \tan(\frac{\pi}{4} - r)$ in Proposition 3.4. Then using these properties in (6.3) and (6.4) implies the following

$$(\delta\ell + \psi)\phi AY - \delta\mathrm{Ric}(\phi AY) - h(\nabla_\xi A)Y + (\nabla_\xi A^2)Y = (1+\alpha\mu)\{g(Y, Df)\xi - g(\xi, Df)Y\}.$$

From this, by taking the inner product with the Reeb vector field ξ, we get

$$g(Y, Df) = 0 \quad \text{for any} \quad Y \in T_\mu, \tag{6.7}$$

where $1+\alpha\mu\neq 0$. If we assume that $1+\alpha\mu=0$, then by Proposition 3.4, we get $1+2\cot(2r)\tan(\frac{\pi}{4}-r)=0$. Then it gives $-\tan(2r)=2\frac{\cos(r)-\sin(r)}{\cos(r)+\sin(r)}$. Since $\tan(2r)=\frac{\sin(2r)}{\cos(2r)}$, we get the following

$$\begin{aligned}(\cos(r)+\sin(r))\sin(r)\cos(r) &= -(\cos(r)-\sin(r))(\cos^2(r)-\sin^2(r))\\ &= -(\cos(r)-\sin(r))^2(\cos(r)+\sin(r)).\end{aligned}$$

From $\cos(r)+\sin(r)\neq 0$ we get $\sin(r)\cos(r)=1$, which gives also a contradiction for $0<r<\frac{\pi}{4}$.

Finally, let us take the inner product the above formula with $Y\in T_\mu$, and use $AY=\mu Y$, $A\phi Y=\lambda\phi Y$ for a contact hypersurface in $\mathbb{C}P^n$, we have

$$\begin{aligned}-(1+\alpha\mu)g(\xi,Df) =&(\delta\ell+\psi)g(\phi AY,Y)-\delta g(\mathrm{Ric}(\phi AY),Y)\\ &-hg((\nabla_\xi A)Y,Y)+g((\nabla_\xi A^2)Y,Y)\\ =&0,\end{aligned}$$

where in the second equality we have used the following formulas

$$\begin{aligned}\mathrm{Ric}(\phi AY) =&(2n+1)\phi AY+hA\phi AY-A^2\phi AY\\ =&\mu\{(2n+1)+\lambda h-\lambda^2\}\phi Y,\end{aligned}$$

$$\begin{aligned}g((\nabla_\xi A)Y,Y) =&g(\nabla_\xi(AY)-A\nabla_\xi Y,Y)\\ =&g(\mu\nabla_\xi Y-A\nabla_\xi Y,Y)=0.\end{aligned}$$

and

$$\begin{aligned}g((\nabla_\xi A^2)Y,Y) =&g(\nabla_\xi(A^2Y)-A^2\nabla_\xi Y,Y)\\ =&g(\mu^2\nabla_\xi Y-A^2\nabla_\xi Y,Y)\\ =&\mu^2 g(\nabla_\xi Y,Y)-\mu^2 g(\nabla_\xi Y,Y)=0.\end{aligned}$$

From this, together with $1+\alpha\mu\neq 0$, we can assert that

$$g(\xi,Df)=0. \tag{6.8}$$

Consequently, from (6.6), (6.7) and (6.8) it follows that the gradient vector field Df is identically vanishing on the tangent space $T_xM=T_\lambda\oplus T_\mu\oplus T_\alpha$, $x\in M$. Then $Df=0$ in (6.1) means that M is pseudo-Einstein, that is, $\delta\mathrm{Ric}(X)=(\Omega-\frac{1}{2}\rho\gamma)X-\psi\eta(X)\xi$, $x\in M$. Since $\lambda+\mu=-\frac{4}{\alpha}$ mentioned before, we get the following

Lemma 6.1. *Let M be a contact real hypersurface in $\mathbb{C}P^n$, $n\geq 3$. If M satisfies a gradient pseudo-Ricci-Yamabe soliton, then M is pseudo-Einstein and*

$$h=\lambda+\mu.$$

Proof. By the above discussions, we can assert that M is pseudo-Einstein. Then Theorem C gives $\cot^2(2r)=n-2$. From this and the Reeb function $\alpha=2\cot(2r)$, follows that

$$\begin{aligned}&h+\frac{4}{\alpha}=2\cot(2r)-2(n-1)\tan(2r)+2\tan(2r)\\ &=2\cot(2r)-2(n-2)\tan(2r)\\ &=2\frac{\cot^2(2r)-(n-2)}{\cot(2r)}\\ &=0.\end{aligned}$$

From this, together with $\lambda+\mu=-\frac{4}{\alpha}$, it becomes $h=\lambda+\mu$. This completes the proof of our lemma. □

Then if we put $\mathrm{Ric}(X) = aX + b\eta(X)\xi$, then the constants a and b can be calculated as follows:

Proposition 6.2. *Let M be a contact real hypersurface in $\mathbb{C}P^n$, $n \geq 3$. If M satisfies gradient pseudo-Ricci-Yamabe soliton, then M is pseudo-Einstein and the soliton constants are given by*

$$a = \Omega - \frac{1}{2}\rho\gamma = 2n, \quad \textit{and} \quad b = -\psi = -2(n-1)\delta.$$

Proof. Since M is pseudo-Einstein, we may put $\mathrm{Ric}(X) = aX + b\eta(X)\xi$. Then from (3.2) it follows that

$$\begin{aligned} a + b =& g(\mathrm{Ric}(\xi), \xi) = 2(n-1) + h\alpha - \alpha^2 \\ =& 2(n-1) + \{2\cot(2r) - 2(n-1)\tan(2r)\}(2\cot(2r)) - (2\cot(2r))^2 \\ =& 2(n-1) - 4(n-1) = -2(n-1). \end{aligned}$$

Next for any vector field $X \in T_\lambda$, (3.2) implies the following

$$\mathrm{Ric}(X) = (2n+1)X + hAX - A^2X = \{(2n+1) + h\lambda - \lambda^2\}X.$$

Then by using Lemma 6.1 it follows from $\lambda - \cot(\frac{\pi}{4}r)$ and $\mu = \tan(\frac{\pi}{4} - r)$

$$\begin{aligned} a =& g(\mathrm{Ric}(X), X) = (2n+1) + h\lambda - \lambda^2 \\ =& (2n+1) + (\lambda + \mu)\lambda - \lambda^2 \\ =& (2n+1) + \lambda\mu = (2n+1) - 1 = 2n. \end{aligned}$$

Then the other constant $b = (a+b) - a == -2(n-1) - 2n = -2(2n-1)$. So from the pseudo-Einstein property

$$\mathrm{Ric}(X) = aX + b\eta(X)\xi = (\Omega - \frac{1}{2}\gamma)X - \psi\eta(X)\xi$$

we can assert above proposition. □

Then summing up the above discussion, together with Lemma 6.1 and Proposition 6.2, we give a complete proof of our Main Theorem 3 in the introduction.

Remark 6.3. The metric g of a Riemannian manifold M of dimension $n \geq 3$ is said to be a *gradient η-Einstein soliton* [7] if there exists a smooth function f on M such that

$$\mathrm{Ric(X)} + \nabla^2\mathrm{f} + \psi\eta(\mathrm{X})\xi = (\Omega + \frac{1}{2}\gamma)\mathrm{X},$$

where γ denotes the scalar curvature of M and Ω and ψ are η-Einstein gradient soliton constants on M. Here $\nabla^2 f$ denotes the Hessian operator of g and f the Einstein potential function of the η-gradient Einstein soliton. So this soliton is an example of gradient pseudo-Ricci-Yamabe soliton of type $(\delta, \rho) = (1, -1)$.

Remark 6.4. Let M be a contact real hypersurface in $\mathbb{C}P^n$, $n \geq 3$, with gradient η-Einstein soliton. Then Lemma 6.1 implies that M is pseudo-Einstein. So by Theorem C it satisfies $\cot^2(2r) = n - 2$. From this and Proposition 3.4 implies that the scalar curvature is given by

$$\begin{aligned} \gamma =& 4(n-1)^2 + 4(n-1)(n-2)\tan^2(2r) \\ =& 4n(n-1). \end{aligned}$$

Moreover, by the definition of η-Einstein soliton, the soliton constant ρ is given by $\rho = -1$. Then by Proposition 6.2, it gives

$$\Omega = 2n + \frac{1}{2}\gamma = 2n + 2n(n-1) = 2n^2 > 0$$

for $n \geq 3$. This means that the gradient η-Einstein soliton becomes shrinking.

Data Availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declaration of Competing Interest

The present authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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