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Metallic shaped contact hypersurfaces of Kaehler manifolds

Cihan Özgür[a]

[a]*Department of Mathematics, İzmir Democracy University, 35140, Karabağlar, İzmir, Türkiye*

Abstract. We study on metallic shaped contact hypersurfaces of Kaehler manifolds. We show that a metallic shaped (κ, μ)-contact metric hypersurface of a Kaehler manifold has constant mean curvature. As a special case, we also consider product shaped Sasakian hypersurfaces of Kaehler manifolds.

1. Introduction

Let $F(0) = a$, $F(1) = b$ and $a, b, \mathfrak{p}$ and $\mathfrak{q}$ be real numbers. Then the generalized secondary Fibonacci sequence is given by the relation

$$F(k+1) = \mathfrak{p}F(k) + \mathfrak{q}F(k-1), \quad k \geq 1,$$

(see [7]) . When $\mathfrak{p} = \mathfrak{q} = 1$, we have the Fibonacci sequence. If the limit $x = \lim_{k\to\infty} \frac{F(k+1)}{F(k)}$ exists, then it is a root of the equation

$$x^2 - \mathfrak{p}x - \mathfrak{q} = 0, \tag{1}$$

(see [6]).

Let $\mathfrak{p}$ and $\mathfrak{q}$ be two integers. The positive solution $\sigma_{\mathfrak{p},\mathfrak{q}} = \frac{\mathfrak{p}+\sqrt{\mathfrak{p}^2+4\mathfrak{q}}}{2}$ of the equation (1) is called *member of the metallic means family* (briefly MMF) [6]. The numbers $\sigma_{\mathfrak{p},\mathfrak{q}}$ are called $(\mathfrak{p}, \mathfrak{q})$-*metallic numbers* [6]. For some special values of $\mathfrak{p}$ and $\mathfrak{q}$, we have some known metallic means. For example: If $\mathfrak{p} = \mathfrak{q} = 1$, then $\sigma_G = \frac{1+\sqrt{5}}{2}$ is the *golden mean*. If $\mathfrak{p} = 2$ and $\mathfrak{q} = 1$, then $\sigma_{\mathcal{A}g} = 1 + \sqrt{2}$ is the *silver mean*. If $\mathfrak{p} = 3$ and $\mathfrak{q} = 1$, then $\sigma_{Br} = \frac{3+\sqrt{13}}{2}$ is the *bronze mean*. If $\mathfrak{p} = 1$ and $\mathfrak{q} = 2$, then $\sigma_{Cu} = 2$ is the *copper mean*. If $\mathfrak{p} = 1$ and $\mathfrak{q} = 3$, then $\sigma_{Ni} = \frac{1+\sqrt{13}}{2}$ is the *nickel mean*.

So MMF is a generalization of the golden mean. The golden mean has many applications in biological growth, constructions of buildings, musics, paintings. The MMF are used in describing fractal geometry, quasiperiodic dynamics (see [7] and [8]).

The notion of a metallic shaped hypersurface was defined by the present author and N. Y. Özgür in [11]. M is called a *metallic shaped hypersurface* [11], if the shape operator $\mathcal{A}$ of M satisfies

$$\mathcal{A}^2 = \mathfrak{p}\mathcal{A} + \mathfrak{q}I,$$

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Email address: cihan.ozgur@idu.edu.tr (Cihan Özgür)

where I is the identity on the tangent bundle of M and $\mathfrak{p}$ and $\mathfrak{q}$ are positive integers. The full classification of the metallic shaped hypersurfaces in real space forms and Lorentzian space forms were given in [11] and [12], respectively. If $\mathfrak{p} = \mathfrak{q} = 1$, then the hypersurface is *golden shaped* [4]. The full classification of golden shaped hypersurfaces in real space forms were given by Crâşmăreanu, Hreţcanu and Munteanu in [4]. If $\mathfrak{p} = 2$ and $\mathfrak{q} = 1$, if $\mathfrak{p} = 3$ and $\mathfrak{q} = 1$, if $\mathfrak{p} = 1$ and $\mathfrak{q} = 2$, or if $\mathfrak{p} = 1$ and $\mathfrak{q} = 3$, then the hypersurface is called *silver shaped*, *bronze shaped*, *copper shaped* or *nickel shaped*, respectively [11].

M is called a *product-shaped hypersurface* [4], if the shape operator $\mathcal{A}$ satisfies

$$\mathcal{A}^2 = I.$$

Based on these observations, in the present paper, we consider metallic shaped contact hypersurfaces of Kaehler manifolds. It is shown that a metallic shaped (κ, μ)-contact metric hypersurface of a Kaehler manifold has constant mean curvature. As a special case, we also consider product shaped Sasakian hypersurfaces of Kaehler manifolds.

2. Contact Metric Manifolds

Let $M = (M^{2n+1}, \varphi, \xi, \eta, g)$ be a contact metric manifold with an almost contact metric structure (φ, ξ, η, g). A contact metric manifold $M = (M^{2n+1}, \varphi, \xi, \eta, g)$ is called a *Sasakian manifold*, if it is normal (see [1], [5]).

The (κ, μ)-nullity distribution of a contact metric manifold $M = (M^{2n+1}, \varphi, \xi, \eta, g)$ for the pair $(\kappa, \mu) \in \mathbb{R}^2$ is a distribution

$$N(\kappa, \mu) : p \to N_p(\kappa, \mu) = \left\{W \in T_pM : R(U, V)W = \kappa\left(g(V, W)U - g(U, W)V\right) + \mu(g(V, W)hU - g(U, W)hV)\right\},$$

where R is the curvature tensor of the contact metric manifold M (see [2]) and h is a $(1, 1)$-tensor field defined by $hU = \frac{1}{2}\left(\mathcal{L}_\xi\varphi\right)U$ where $\mathcal{L}_\xi$ denotes Lie differentiation in the direction of ξ, where $U, V, W \in TM$. For a contact metric manifold, it is clear that $trh = trh\varphi = 0$.

If the characteristic vector field ξ belongs to the (κ, μ)-nullity distribution, then

$$R(U, V)\xi = \kappa\left(\eta(V)U - \eta(U)V\right) + \mu(\eta(V)hU - \eta(U)hV),$$

where $\kappa \leq 1$. If $\kappa = 1$, then $M = (M^{2n+1}, \varphi, \xi, \eta, g)$ is Sasakian. If the characteristic vector field ξ belongs to the (κ, μ)-nullity distribution, then the contact metric manifold is called a (κ, μ)-*contact metric manifold* [2].

3. Hypersurfaces of Kaehler Manifolds

Let M be a $(2n + 1)$-dimensional orientable hypersurface isometrically embedded into a $(2n + 2)$-dimensional Kaehler manifold $\widetilde{M} = (\widetilde{M}^{2n+2}, g, J)$ with almost complex structure J and Kaehlerian metric g. Let υ be the unit normal vector field to M. Then the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_U V = D_U V + \sigma(U, V)\upsilon, \tag{2}$$

$$\widetilde{\nabla}_U \upsilon = -\mathcal{A}U, \tag{3}$$

where $\mathcal{A}$ is the shape operator of M and $\mathcal{A}$ and σ are related with $g(\mathcal{A}U, V) = \sigma(U, V)$, $U, V \in TM$ [3].

Since υ is the unit normal vector field, $J\upsilon$ is tangent to M. Setting

$$J\upsilon = \xi, \tag{4}$$

$$JX = \varphi X - \eta(X)\upsilon, \tag{5}$$

where φ is a $(1, 1)$-tensor field, η is a 1-form and $X \in TM$.

From (4) by differentiation along M, and by the use of (2) and (3), we have

$$D_X\xi = -\varphi\mathcal{A}X.$$

Hence because of (4) and (5), (η, ξ, φ, g) defines an almost contact metric structure on M. From (5), by differentiation along M, and by using of (2) and (3), we find

$$(D_X\varphi)Y = \sigma(X, Y)\xi - \eta(Y)\mathcal{A}X,$$

(see [13]).

We suppose that the almost contact metric structure induced on M is a contact metric structure. Such a hypersurface is called a *contact hypersurface of the Kaehler manifold* $\widetilde{M}$ [13]. By easy calculations, we have the following formulas (see [13]):

$$\mathcal{A}\xi = (tr(\mathcal{A}) - 2n)\xi \tag{6}$$

and

$$\mathcal{A}X = X + hX + (tr(\mathcal{A}) - 2n - 1)\eta(X)\xi. \tag{7}$$

From [13], we know that a contact hypersurface of a Kaehler manifold is a Hopf hypersurface. Because of this reason, to study on a contact hypersurface of a Kaehler manifold is more interesting than other real hypersurfaces of a Kaehler manifold (see [13]). For more details about hypersurfaces of Kaehler manifolds, we refer to [9], [10] and [14].

4. Main Results

Let M be a contact hypersurface of a Kaehler manifold $\widetilde{M}$. Similar to the definition of a metallic shaped hypersurface in a real space form given in [11], we can define the metallic shaped contact hypersurface of a Kaehler manifold as follows:

Definition 4.1. *Let M be a contact hypersurface in a Kaehler manifold $\widetilde{M}$. Then M is called metallic shaped, if the shape operator $\mathcal{A}$ of M satisfies the condition*

$$\mathcal{A}^2X = \mathfrak{p}\mathcal{A}X + \mathfrak{q}X \tag{8}$$

for any vector field X in TM, which is not parallel to the characteristic vector field ξ, where $\mathfrak{p}$ and $\mathfrak{q}$ are positive integers.

From (7), we have

$$\mathcal{A}^2X = \mathcal{A}(\mathcal{A}X) = \mathcal{A}X + \mathcal{A}hX + (tr(\mathcal{A}) - 2n - 1)\eta(X)\mathcal{A}\xi.$$

Then using (6) and (7), we get

$$\mathcal{A}^2X = X + 2hX + h^2X + (tr(\mathcal{A}) - 2n - 1)(tr(\mathcal{A}) - 2n + 1)\eta(X)\xi. \tag{9}$$

Since M is a metallic shaped contact hypersurface, from (7), (8) and (9), it follows that

$$(1 - \mathfrak{p} - \mathfrak{q})\, X + (2 - \mathfrak{p})\, hX + h^2X + (tr(\mathcal{A}) - 2n - 1)(tr(\mathcal{A}) - 2n + 1 - \mathfrak{p})\eta(X)\xi = 0.$$

Contracting the last equation, we find

$$(1 - \mathfrak{p} - \mathfrak{q})\,(2n + 1) + trh^2 + (tr(\mathcal{A}) - 2n - 1)(tr(\mathcal{A}) - 2n + 1 - \mathfrak{p}) = 0. \tag{10}$$

Then we can state the following Lemma:

Lemma 4.2. *Let M be a metallic shaped contact hypersurface in a Kaehler manifold $\widetilde{M}^{2n+2}$. Then the condition* (10) *is satisfied on M.*

Now assume that M is a non-Sasakian (κ,μ)-contact metric hypersurface of a Kaehler manifold $\widetilde{M}^{2n+2}$. From [2], we know that for an $(2n+1)$-dimensional (κ,μ)-contact metric manifold, we have $h^2=(\kappa-1)\varphi^2$, which implies $trh^2=2n(1-\kappa)$. So substituting $trh^2=2n(1-\kappa)$ in (10), we get

$$(1-\mathfrak{p}-\mathfrak{q})(2n+1)+2n(1-\kappa)+(tr(\mathcal{A})-2n-1)(tr(\mathcal{A})-2n+1-\mathfrak{p})=0$$

or equivalently

$$(tr\mathcal{A})^2-(4n+\mathfrak{p})\,tr\mathcal{A}+4n^2+2n(2-\kappa-\mathfrak{q})-\mathfrak{q}=0. \tag{11}$$

This gives us

$$tr(\mathcal{A})=\frac{(4n+\mathfrak{p})\mp\sqrt{\mathfrak{p}^2+8n(\mathfrak{p}+\mathfrak{q}+\kappa-2)+4\mathfrak{q}}}{2},$$

which is a constant, since $\mathfrak{p}$ and $\mathfrak{q}$ are positive integers.

Then we can state the following result:

Theorem 4.3. *Let M be a metallic shaped (κ,μ)-contact metric hypersurface in a Kaehler manifold $\widetilde{M}^{2n+2}$. Then M has constant mean curvature* $H=\frac{(4n+\mathfrak{p})+\sqrt{\mathfrak{p}^2+8n(\mathfrak{p}+\mathfrak{q}+\kappa-2)+4\mathfrak{q}}}{2(2n+1)}$ *or* $H=\frac{(4n+\mathfrak{p})-\sqrt{\mathfrak{p}^2+8n(\mathfrak{p}+\mathfrak{q}+\kappa-2)+4\mathfrak{q}}}{2(2n+1)}$.

If M is a Sasakian hypersurface in a Kaehler manifold $\widetilde{M}^{2n+2}$, then $\kappa=1$.

So we have the following corollary:

Corollary 4.4. *Let M be a metallic shaped Sasakian hypersurface in a Kaehler manifold $\widetilde{M}^{2n+2}$. Then M has constant mean curvature* $H=\frac{(4n+\mathfrak{p})+\sqrt{\mathfrak{p}^2+8n(\mathfrak{p}+\mathfrak{q}-1)+4\mathfrak{q}}}{2(2n+1)}$ *or* $H=\frac{(4n+\mathfrak{p})-\sqrt{\mathfrak{p}^2+8n(\mathfrak{p}+\mathfrak{q}-1)+4\mathfrak{q}}}{2(2n+1)}$.

If M is a Sasakian hypersurface with unit mean curvature in a Kaehler manifold $\widetilde{M}^{2n+2}$, then from (11), we have

$$(1-\mathfrak{p}-\mathfrak{q})(2n+1)=0,$$

which is impossible, since $\mathfrak{p}$ and $\mathfrak{q}$ are positive integers.

Hence we obtain the following result:

Theorem 4.5. *There does not exist a metallic shaped Sasakian hypersurface with unit mean curvature in a Kaehler manifold $\widetilde{M}^{2n+2}$.*

Now assume that M is a metallic shaped (κ,μ)-contact metric hypersurface in a Kaehler manifold $\widetilde{M}^{2n+2}$. From the Gauss equation, we have

$$\widetilde{Ric}(X,Y)-\widetilde{g}(\widetilde{R}(\upsilon,X)Y,\upsilon)=Ric(X,Y)+g(\mathcal{A}X,\mathcal{A}Y)-tr(\mathcal{A})g(\mathcal{A}X,Y),$$

where Ric and $\widetilde{Ric}$ denote the Ricci tensors of M and $\widetilde{M}^{2n+2}$, respectively. By a contraction from the last equation, we have

$$\widetilde{scal}-2\widetilde{Ric}(\upsilon,\upsilon)=scal+\|\mathcal{A}\|^2-(tr(\mathcal{A}))^2, \tag{12}$$

where $scal$ and $\widetilde{scal}$ denote the scalar curvatures of M and $\widetilde{M}^{2n+2}$, respectively. Now assume that $\widetilde{M}^{2n+2}$ is an Kaehler Einstein manifold. So we can write $\widetilde{Ric}(X,Y)=\lambda\widetilde{g}(X,Y)$ for some constant λ. By a contraction, the scalar curvature of $\widetilde{M}^{2n+2}$ is $\widetilde{scal}=(2n+2)\lambda$. On the other hand by equation (8), we have $\|\mathcal{A}\|^2=\mathfrak{p}tr(\mathcal{A})+(2n+1)\mathfrak{q}$. Hence the equation (12) turns into

$$2n\lambda=scal+\mathfrak{p}tr(\mathcal{A})+(2n+1)\mathfrak{q}-(tr(\mathcal{A}))^2. \tag{13}$$

So from Theorem 4.3, since $tr(\mathcal{A})$ is a constant for a metallic shaped (κ,μ)-contact metric hypersurface in a Kaehler manifold $\widetilde{M}^{2n+2}$, the equation (13) gives us the scalar curvature $scal$ of M is a constant.

Then we can state the following theorem:

Theorem 4.6. *Let M be a metallic shaped (κ, μ)-contact metric hypersurface in an Kaehler Einstein manifold $\widetilde{M}^{2n+2}$. Then M has constant scalar curvature.*

Complex manifolds with a Ricci-flat Kaehler metric are called *Calabi-Yau manifolds* [15]. A Calabi-Yau manifold is not a complex space-form, if it is not flat [15].

Now assume that M is a metallic shaped (κ, μ)-contact metric hypersurface in a Calabi-Yau manifold $\widetilde{M}^{2n+2}$. Then from (12), we have

$$scal + \|\mathcal{A}\|^2 - (tr(\mathcal{A}))^2 = 0. \tag{14}$$

Since for a metallic shaped (κ, μ)-contact metric hypersurface of a Kaehler manifold, $tr(\mathcal{A})$ and $\|A\|^2$ are constants, from (14), the scalar curvature $scal$ of M is also a constant.

So we have the following result:

Corollary 4.7. *Let M be a metallic shaped (κ, μ)-contact metric hypersurface in a Calabi-Yau manifold $\widetilde{M}^{2n+2}$. Then M has constant scalar curvature.*

If the second fundamental form of a contact metric hypersurface M of a Kaehler manifold is a linear combination of the metric tensor and $\eta \otimes \eta$, then M is called a *C-umbilical hypersurface* [13].

Now assume that M is a product-shaped hypersurface in a Kaehler manifold $\widetilde{M}$. Then its shape operator satisfies the condition $\mathcal{A}^2 = I$. Hence from (10), we have

$$trh^2 + (tr(\mathcal{A}) - 2n - 1)(tr(\mathcal{A}) - 2n + 1) = 0.$$

If M is Sasakian, then $(tr(\mathcal{A}) - 2n - 1)(tr(\mathcal{A}) - 2n + 1) = 0$. Since $tr(\mathcal{A})$ is a constant, the last equation gives us either $tr(\mathcal{A}) = 2n + 1$ or $tr(\mathcal{A}) = 2n - 1$. If $tr(\mathcal{A}) = 2n + 1$, then it has unit mean curvature and moreover from (7), it is totally umbilical. If M is 3-dimensional and $\widetilde{M}$ is a 4-dimensional Calabi-Yau manifold, then $\widetilde{M}$ is flat at each point of M (see the proof of Theorem 2 in [13]). If $tr(\mathcal{A}) = 2n - 1$, then from (7), $\mathcal{A}X = X - 2\eta(X)\xi$. Hence M is C-umbilical.

Thus we obtain the following theorem:

Theorem 4.8. *Let M be a Sasakian hypersurface in a Kaehler manifold $\widetilde{M}$. Then M is product shaped if and only if either M is totally umbilical with unit mean curvature (if M is 3-dimensional and $\widetilde{M}$ is a 4-dimensional Calabi-Yau manifold, then $\widetilde{M}$ is flat at each point of M) or M is a C-umbilical hypersurface whose shape operator $\mathcal{A}$ is of the form $\mathcal{A} = I - 2\eta \otimes \xi$.*

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