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Hyperbolic Ricci solitons on trans-Sasakian manifolds

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Abstract. We investigate hyperbolic Ricci solitons on the trans-Sasakian manifolds. We prove that the trans-Sasakian manifolds under some conditions admit in hyperbolic Ricci soliton. Also, we provide two examples of hyperbolic Ricci solitons on the trans-Sasakian manifolds in dimensional 3 to prove our results.

1. Introduction

In 1985, Oubina [38] presented the trans-Sasakian manifold (TSM) as a class of almost contact metric manifolds (ACMM). Then, Blair and Oubina [6] obtained certain properties of these manifolds. We usually determine the TSM of (σ, θ)- type as $(M, \phi, \xi, \eta, g, \sigma, \theta)$, such that (ϕ, ξ, η, g) is an ACMM and both σ and θ are smooth maps on M. A TSM of kind $(0, 0)$, $(0, \theta)$ and $(\sigma, 0)$ is cosymplectic, θ-Kenmotsu [3, 29, 37, 47] and σ-Sasakian [31], respectively. In [9, 22, 42, 43, 48], some results on the structure of the TSMs are obtained. In addition, in [15, 17, 19, 21–23, 28, 35, 36, 48], the authors investigated compact TSMs with certain conditions on the smooth functions σ, θ as well as the vector field ξ that appears in their definition. Through these researches, restrictions under which a TSM can be considered homothetic to a Sasakian manifold have been discovered.
Essentially, a geometric flow is the evolution of these structures under a differential equation that takes into account a functional on a manifold. By utilizing this concept, we can determine canonical metrics on the underlying Riemannian manifolds. This is why geometric flows are such significant topics in differential geometry.
One of essential flows is the Ricci flow. Hamilton presented the notion of the Ricci soliton as an generalization of Einstein metrics and the particular solution to Ricci flow on a Riemannian structure in 1982 [27]. Assume that M is a pseudo-Riemannian Manifold (SRM). A triplet (g, χ, λ) is Ricci soliton [7], if

$$\mathcal{L}_\chi g + 2S + 2\lambda g = 0, \tag{1}$$

where λ is a real constant, S is the Ricci tensor, χ is potential vector field and $\mathcal{L}_\chi$ represent the Lie derivative in order χ.

The Ricci soliton can be classified into three types: expanding, steady, and shrinking, depending on the sign of a parameter λ. An expanding soliton corresponds to a positive λ, a steady soliton corresponds to a

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zero λ, and a shrinking soliton corresponds to a negative λ. In addition, a gradient Ricci soliton is a specific type of soliton where the soliton vector field χ is the gradient of some function ψ.

Hyperbolic geometric flow (HGF) is another geometric flow that is a set of nonlinear evolution PDEs of second order, and is defines by

$$\frac{\partial^2}{\partial t^2} g = -2Ric, \qquad g(0) = g_0, \qquad \frac{\partial g}{\partial t}(0) = k_0, \tag{2}$$

where k_0 is a symmetric 2-tensor field on M. The uniqueness and existences of (2) studied in [13] on closed Riemannian manifolds.
Let $(M^n, g(t))$ is a solution for the HGF on $(M, g(0))$. The self-similar solution of the HGF [1, 24] is as follows

$$S(g_0) + \lambda \mathcal{L}_X g_0 + (\mathcal{L}_X \circ \mathcal{L}_X) g_0 = \mu g_0, \tag{3}$$

for some constants λ and μ. With these assumptions, we state that g_0 is a hyperbolic Ricci soliton (HRS) and, we show it by $(M, g_0, X, \lambda, \mu)$ (briefly (g_0, X, λ, μ)). A HRS is an Einstein metric when X vanishes identically, Therefore, Einstein metric is a particular case of HRS. If $\lambda = \frac{1}{2}$ and X is a 2-Killing vector filed (KVF) [12, 34], i.e., $(\mathcal{L}_X \circ \mathcal{L}_X) g_0 = 0$ then a HRS is just a Ricci soliton. We state that $(M, g_0, \nabla f, \lambda, \mu)$ is a gradient HRS whenever $X = \nabla f$ for a map $f : M \to \mathbb{R}$.

In the presented paper, we investigate the HRSs on the TSMs. Also, we present two examples of HRSs on 3-TSMs. In this study, we will follow the outline described below. Firstly, in Section 2, we will review some important formulas and concepts corresponding to TSMs. Next, in Section 3, we will present the main findings of the study along with their respective proofs. In the final section, i.e., Section 4, we will explore two examples of 3-TSMs that are applicable in hyperbolic Ricci soliton.

2. Preliminaries

Assume that M is a $(2m+1)$-manifold and $\varpi_1, \varpi_2, \varpi_3$, and ϖ_4 are arbitrary vector fields on M. Also, we assume that g is a compatible Riemannian metric on M, η be a 1-form, ξ is a smooth vector field and ϕ is a $(1,1)$-tensor field so that

$$\phi^2(\varpi_1) = -\varpi_1 + \eta(\varpi_1)\xi, \eta(\xi) = 1, \tag{4}$$
$$g(\phi\varpi_1, \phi\varpi_2) = g(\varpi_1, \varpi_2) - \eta(\varpi_1)\eta(\varpi_2). \tag{5}$$

With the above assumptions, we call (ϕ, ξ, η, g) an almost contact structure (ACS) and the manifold (M, g) is called an ACMM [4, 5]. In this case, we get $g(\varpi_1, \phi\varpi_2) = -g(\phi\varpi_1, \varpi_2)$, $\eta \circ \phi = 0$, $\phi\xi = 0$ and $\eta(\varpi_1) = g(\varpi_1, \xi)$. The 2-form Φ is specified by

$$\Phi(\varpi_1, \varpi_2) = g(\varpi_1, \phi\varpi_2).$$

Let J be the ACS on $M \times \mathbb{R}$ that express by

$$J(\varpi_1, h\frac{d}{dt}) = (\phi\varpi_1 - h\xi, \eta(\varpi_1)\frac{d}{dt}),$$

for any smooth map h on $M \times \mathbb{R}$ and vector field ϖ_1. Let (M, ϕ, ξ, η, g) be an ACMM. Also, we set on $M \times \mathbb{R}$ the product metric G. According to [26], if $(M \times \mathbb{R}, J, G)$ belong to the type W_4, then M is called the TSM [38]. This can be represented by the equation [6]

$$(\nabla_{\varpi_1}\phi)\varpi_2 = \sigma\left(g(\varpi_1, \varpi_2)\xi - \eta(\varpi_2)\varpi_1\right) + \theta\left(g(\phi\varpi_1, \varpi_2)\xi - \eta(\varpi_2)\phi\varpi_1\right), \tag{6}$$

where σ and θ are smooth functions. Also, the TSM is said to be of type (σ, θ). By virtue of (6), we conclude

$$\nabla_{\varpi_1}\xi = -\sigma\phi\varpi_1 + \theta(\varpi_1 - \eta(\varpi_1)\xi), \tag{7}$$
$$(\nabla_{\varpi_1}\eta)\varpi_2 = -\sigma g(\phi\varpi_1, \varpi_2) + \theta g(\phi\varpi_1, \phi\varpi_2). \tag{8}$$

Using (6) and (7), we deduce

$$2\sigma\theta + \xi(\sigma) = 0. \tag{9}$$

Let R be the tensor of Riemannian curvature of M. Further, we get the following relations [18]

$$\begin{aligned} R(\varpi_1,\varpi_2)\xi &= (\sigma^2-\theta^2)(\eta(\varpi_2)\varpi_1-\eta(\varpi_1)\varpi_2)+2\sigma\theta(\eta(\varpi_2)\phi\varpi_1 \\ &\quad -\eta(\varpi_1)\phi\varpi_2)(\varpi_2(\sigma))\phi\varpi_1-(\varpi_1(\sigma))\phi\varpi_2+(\varpi_2(\theta))\phi^2\varpi_1 \\ &\quad -(\varpi_1(\theta))\phi^2\varpi_2, \end{aligned} \tag{10}$$

$$R(\varpi_1,\xi)\xi = (\sigma^2-\theta^2-\xi(\theta))\{\varpi_1-\eta(\varpi_1)\xi\}, \tag{11}$$

$$\begin{aligned} R(\xi,\varpi_1)\varpi_2 &= (\sigma^2-\theta^2)\{g(\varpi_1,\varpi_2)\xi-\eta(\varpi_2)\varpi_1\}+(\varpi_2(\sigma))\phi\varpi_1 \\ &\quad +2\sigma\theta\{g(\phi\varpi_2,\varpi_1)\xi+\eta(\varpi_2)\phi\varpi_1\}+(\varpi_2(\theta))\{\varpi_1-\eta(\varpi_1)\xi\} \\ &\quad +g(\phi\varpi_2,\varpi_1)\nabla\sigma-g(\phi\varpi_1,\phi\varpi_2)\nabla\theta. \end{aligned} \tag{12}$$

Also, the Ricci tensor S of a TSM (M^n,g) is determined by $S(\varpi_1,\varpi_2)=\sum_{i=1}^n g(R(e_i,\varpi_1)\varpi_2,e_i)$. Using (10), we arrive at

$$S(\varpi_1,\xi) = \Big((\sigma^2-\theta^2)(n-1)-\xi(\theta)\Big)\eta(\varpi_1)+\varpi_1(\theta)(2-n)-(\phi\varpi_1)\sigma, \tag{13}$$

where the orthogonal basis $\{e_i\}_{i=1}^n$ is a frame at any point of the manifold for the tangent space.

Suppose that g is a pseudo-Riemannian metric on manifold M and S is the associated Ricci tensor. Also, let χ be a smooth vector field on M such that

$$S+\lambda\mathcal{L}_\chi g+(\mathcal{L}_\chi\circ\mathcal{L}_\chi)g=\mu g, \tag{14}$$

for some real constants μ and λ. We say that 4-tuple (g,χ,λ,μ) is a HRS.

3. Main results and their proofs

We consider the manifold M satisfies the HRS (14) and $\chi = f\xi$ for some smooth function f on M. Applying (7) we obtain

$$\begin{aligned} \mathcal{L}_{f\xi}g(\varpi_1,\varpi_2) &= g(\nabla_{\varpi_1}f\xi,\varpi_2)+g(\nabla_{\varpi_2}f\xi,\varpi_1) \\ &= (\varpi_1 f)\eta(\varpi_2)+fg(-\sigma\phi(\varpi_1)+\theta(\varpi_1-\eta(\varpi_1)\xi),\varpi_2)+(\varpi_2 f)\eta(\varpi_1) \\ &\quad +fg(\varpi_1,-\sigma\phi(\varpi_2)+\theta(\varpi_2-\eta(\varpi_2)\xi)) \\ &= (\varpi_1 f)\eta(\varpi_2)+(\varpi_2 f)\eta(\varpi_1)+2\theta f(g(\varpi_1,\varpi_2)-\eta(\varpi_1)\eta(\varpi_2)), \end{aligned} \tag{15}$$

thus

$$\begin{aligned} &(\mathcal{L}_{f\xi}(\mathcal{L}_{f\xi}g))(\varpi_1,\varpi_2) \\ &= \xi(\mathcal{L}_{f\xi}g(\varpi_1,\varpi_2))-\mathcal{L}_{f\xi}g(\mathcal{L}_{f\xi}\varpi_1,\varpi_2)-\mathcal{L}_{f\xi}g(\varpi_1,\mathcal{L}_{f\xi}\varpi_2) \\ &= \xi\left((\varpi_1 f)\eta(\varpi_2)+(\varpi_2 f)\eta(\varpi_1)+2\theta f(g(\varpi_1,\varpi_2)-\eta(\varpi_1)\eta(\varpi_2))\right) \\ &\quad -\Big(((\mathcal{L}_{f\xi}\varpi_1)f)\eta(\varpi_2)+(\varpi_2 f)\eta(\mathcal{L}_{f\xi}\varpi_1)+2\theta f(g(\mathcal{L}_{f\xi}\varpi_1,\varpi_2)-\eta(\mathcal{L}_{f\xi}\varpi_1)\eta(\varpi_2))\Big) \\ &\quad -\Big((\varpi_1 f)\eta(\mathcal{L}_{f\xi}\varpi_2)+((\mathcal{L}_{f\xi}\varpi_2)f)\eta(\varpi_1)+2\theta f(g(\varpi_1,\mathcal{L}_{f\xi}\varpi_2)-\eta(\varpi_1)\eta(\mathcal{L}_{f\xi}\varpi_2))\Big). \end{aligned}$$

Applying $\chi = f\xi$ and the above equation in (14) we conclude

$$\begin{aligned} &S(\varpi_1,\varpi_2)+\lambda\left((\varpi_1 f)\eta(\varpi_2)+(\varpi_2 f)\eta(\varpi_1)+2\theta f(g(\varpi_1,\varpi_2)-\eta(\varpi_1)\eta(\varpi_2))\right) \\ &+\xi\left((\varpi_1 f)\eta(\varpi_2)+(\varpi_2 f)\eta(\varpi_1)+2\theta f(g(\varpi_1,\varpi_2)-\eta(\varpi_1)\eta(\varpi_2))\right) \\ &-\Big(((\mathcal{L}_{f\xi}\varpi_1)f)\eta(\varpi_2)+(\varpi_2 f)\eta(\mathcal{L}_{f\xi}\varpi_1)+2\theta f(g(\mathcal{L}_{f\xi}\varpi_1,\varpi_2)-\eta(\mathcal{L}_{f\xi}\varpi_1)\eta(\varpi_2))\Big) \\ &-\Big((\varpi_1 f)\eta(\mathcal{L}_{f\xi}\varpi_2)+((\mathcal{L}_{f\xi}\varpi_2)f)\eta(\varpi_1)+2\theta f(g(\varpi_1,\mathcal{L}_{f\xi}\varpi_2)-\eta(\varpi_1)\eta(\mathcal{L}_{f\xi}\varpi_2))\Big) \\ &-\mu g(\varpi_1,\varpi_2)=0, \end{aligned} \tag{16}$$

We plug $\varpi_1 = \varpi_2 = \xi$ in the relation (16) and using (4) and (13) to obtain

$$(n-1)(\sigma^2 - \theta^2) - (n-1)\xi(\theta) + 2\lambda\xi(f) + 2\xi(\xi(f)) + 4(\xi(f))^2 - \mu = 0. \tag{17}$$

Thus, we get the following result:

Theorem 3.1. *Assume that $(M^n, g, \phi, \xi, \eta)$ is a TSM. If M satisfies in a HRS (g, χ, λ, μ) such that $\chi = f\xi$ then equation (17) holds.*

Suppose that M is a TSM and a, b are smooth maps on it. If we can write the Ricci tensor S as

$$S = ag + b\eta \otimes \eta,$$

then M is called η-Einstein. If M is an η-Einstein TS, then for some a and b we get $S = ag + b\eta \otimes \eta$. Equation (15) implies that

$$\mathcal{L}_\xi g(\varpi_1, \varpi_2) = 2\theta(g(\varpi_1, \varpi_2) - \eta(\varpi_1)\eta(\varpi_2)), \tag{18}$$

and

$$\begin{aligned}(\mathcal{L}_\xi(\mathcal{L}_\xi g))(\varpi_1, \varpi_2) = \; & 2\theta\left(\xi(g(\varpi_1, \varpi_2) - \eta(\varpi_1)\eta(\varpi_2))\right) \\ & +2\xi(\theta)\left(g(\varpi_1, \varpi_2) - \eta(\varpi_1)\eta(\varpi_2)\right) \\ & -2\theta\left(g(\mathcal{L}_\xi\varpi_1, \varpi_2) - \eta(\mathcal{L}_\xi\varpi_1)\eta(\varpi_2)\right) \\ & -2\theta\left(g(\varpi_1, \mathcal{L}_\xi\varpi_2) - \eta(\varpi_1)\eta(\mathcal{L}_\xi\varpi_2)\right).\end{aligned} \tag{19}$$

We have

$$\begin{aligned}g(\mathcal{L}_\xi\varpi_1, \varpi_2) &= g([\xi, \varpi_1], \varpi_2) \\ &= g(\nabla_\xi\varpi_1 - \nabla_{\varpi_1}\xi, \varpi_2) \\ &= g(\nabla_\xi\varpi_1 + \sigma\phi(\varpi_1) - \theta(\varpi_1 - \eta(\varpi_1)\xi), \varpi_2) \\ &= g(\nabla_\xi\varpi_1, \varpi_2) + \sigma g(\phi(\varpi_1), \varpi_2) - \theta g(\varpi_1, \varpi_2) + \theta\eta(\varpi_1)\eta(\varpi_2).\end{aligned} \tag{20}$$

Similarly

$$g(\varpi_1, \mathcal{L}_\xi\varpi_2) = g(\varpi_1, \nabla_\xi\varpi_2) + \sigma g(\varpi_1, \phi(\varpi_2)) - \theta g(\varpi_1, \varpi_2) + \theta(\varpi_1)\eta(\varpi_2).$$

Then

$$g(\mathcal{L}_\xi\varpi_1, \varpi_2) + g(\varpi_1, \mathcal{L}_\xi\varpi_2) = \xi(g(\varpi_1, \varpi_2)) - 2\theta(g(\varpi_1, \varpi_2) - \eta(\varpi_1)\eta(\varpi_2)). \tag{21}$$

Since $\nabla_\xi\xi = -\sigma\phi(\xi) + \theta(\xi - \eta(\xi)\xi) = 0$, using (20) we have

$$\eta(\mathcal{L}_\xi\varpi_1) = g(\mathcal{L}_\xi\varpi_1, \xi) = g(\nabla_\xi\varpi_1, \xi) = \xi(g(\varpi_1, \xi)) = \xi(\eta(\varpi_1)),$$

similarly

$$\eta(\mathcal{L}_\xi\varpi_2) = \xi(\eta(\varpi_2)).$$

Thus

$$\eta(\mathcal{L}_\xi\varpi_1)\eta(\varpi_2) + \eta(\varpi_1)\eta(\mathcal{L}_\xi\varpi_2) = \xi(\eta(\varpi_1))\eta(\varpi_2) + \eta(\varpi_1)\xi(\eta(\varpi_2)) = \xi(\eta(\varpi_1)\eta(\varpi_2)). \tag{22}$$

Therefore, applying (21) and (22) in (19) we conclude

$$(\mathcal{L}_\xi(\mathcal{L}_\xi g))(\varpi_1, \varpi_2) = (4\theta^2 + 2\xi(\theta))(g(\varpi_1, \varpi_2) - \eta(\varpi_1)\eta(\varpi_2)). \tag{23}$$

Using (18) and (23) we infer

$$\begin{aligned}& S + \lambda\mathcal{L}_\xi g + (\mathcal{L}_\xi \circ \mathcal{L}_\xi)g - \mu g \\ &= ag + b\eta \otimes \eta + 2\lambda\theta g - 2\lambda\theta\eta \otimes \eta + (4\theta^2 + 2\xi(\theta))g - (4\theta^2 + 2\xi(\theta))\eta \otimes \eta - \mu g \\ &= (a + 2\lambda\theta + 4\theta^2 + 2\xi(\theta) - \mu)g + (b - 2\lambda\theta - 4\theta^2 - 2\xi(\theta))\eta \otimes \eta.\end{aligned}$$

From the above equation M accepts a HRS (g, ξ, λ, μ) if $\theta\lambda = \frac{b - 4\theta^2 - 2\xi(\theta)}{2}$ and $\mu = a + b$.

Hence, the next theorem is the result:

Theorem 3.2. *Suppose that M is an η-Einstein TSM of kind (σ, θ), that is, for some maps a and b on M, $S = ag + b\eta \otimes \eta$. Let $\frac{b-4\theta^2-2\xi(\theta)}{2\theta}$ and $a + b$ be two constants. Then manifold M satisfies a HRS $(g, \xi, \frac{b-4\theta^2-2\xi(\theta)}{2\theta}, a + b)$ provided $\theta \neq 0$. Also, if $b = 0$ then M accepts a HRS $(g, \xi, -2\theta, a)$.*

Let T and A be $(0, k)$ and symmetric $(0, 2)$ tensors, respectively, where $k \geq 1$. we define endomorphism $\varpi_1 \wedge_A \varpi_2$ by

$$(\varpi_1 \wedge_A \varpi_2)\chi = A(\varpi_2, \chi)\varpi_1 - A(\varpi_1, \chi)\varpi_2,$$

and

$$\begin{aligned}((\varpi_1 \wedge_A \varpi_2).T)(\zeta_1, \cdots, \zeta_k) &= -T((\varpi_1 \wedge_A \varpi_2)\zeta_1, \zeta_2, \cdots, \zeta_k)\\ &\quad -T(\zeta_1, (\varpi_1 \wedge_A \varpi_2)\zeta_2, \cdots, \zeta_k) - \cdots\\ &\quad -T(\zeta_1, \zeta_2, \cdots, (\varpi_1 \wedge_A \varpi_2)\zeta_k),\end{aligned}$$

where $\chi, \zeta_1, \cdots, \zeta_k$ are vector fields on M. Also, we express $R.T$ and $\Gamma(A, T)$ as follow

$$\begin{aligned}(R(\varpi_1, \varpi_2).T)(\zeta_1, \cdots, \zeta_k) &= -T(R(\varpi_1, \varpi_2)\zeta_1, \zeta_2, \cdots, \zeta_k)\\ &\quad -T(\zeta_1, R(\varpi_1, \varpi_2)\zeta_2, \cdots, \zeta_k) - \cdots\\ &\quad -T(\zeta_1, \zeta_2, \cdots, R(\varpi_1, \varpi_2)\zeta_k),\end{aligned}$$

and

$$\Gamma(A, T)(\zeta_1, \cdots, \zeta_k; \varpi_1, \varpi_2) = ((\varpi_1 \wedge_A \varpi_2).T)(\zeta_1, \cdots, \zeta_k).$$

Now assume that (M^n, g) is a (σ, θ)-type TSM justifying $R.S = f(q)\Gamma(g, S)$, where $f \in C^\infty(M)$, $q \in \{x \in M : \Gamma(g, S) \neq 0 \text{ at } x\}$ and $f(q) \neq \sigma^2 - \theta^2$, that is manifold M is partially Ricci pseudosymmetric. Then from [39] we have

$$S(\varpi_1, \varpi_2) = (1 - n)(\theta^2 - \sigma^2)g(\varpi_1, \varpi_2).$$

Thus, Theorem 3.2 implies that:

Corollary 3.3. *Suppose that M is a TSM of type (σ, θ) satisfy the condition $R.S = f(q)\Gamma(g, S)$ for some map f such that $f(q) \neq \sigma^2 - \theta^2$ for some $q \in \{x \in M : \Gamma(g, S) \neq 0 \text{ at } x\}$. Let $\frac{2\theta^2+\xi(\theta)}{\theta}$ and $\sigma^2 - \theta^2$ be two constants. Then manifold M satisfies a HRS $(g, \xi, -\frac{2\theta^2+\xi(\theta)}{\theta}, (n-1)(\sigma^2 - \theta^2))$.*

Let M be an ACMM in dimension n and r is the scalar curvature. The Weyl conformal curvature tensor $\mathcal{W}$ is defined by

$$\begin{aligned}&\mathcal{W}(\varpi_1, \varpi_2)\varpi_3 =\\ &\quad R(\varpi_1, \varpi_2)\varpi_3 - \frac{1}{n-2}\{S(\varpi_2, \varpi_3)\varpi_1 - S(\varpi_1, \varpi_3)\varpi_2 + g(\varpi_2, \varpi_3)Q\varpi_1\\ &\quad -g(\varpi_1, \varpi_3)Q\varpi_2\} + \frac{r}{(n-1)(n-2)}\{g(\varpi_2, \varpi_3)\varpi_1 - g(\varpi_1, \varpi_3)\varpi_2\},\end{aligned}$$

where Q is Ricci operator.
For a TSM, if condition $R.\mathcal{W} = 0$ is satisfied, we say that this manifold is Weyl-semisymmetric. Now, suppose that (M^n, g) is a TSM, that is, $(R(\varpi_1, \varpi_2).\mathcal{W})(\varpi_3, \varpi_4)\varpi_5 = 0$. From [39] we have

$$S(\varpi_1, \varpi_2) = (\frac{r}{n-1} - \sigma^2 + \theta^2)g(\varpi_1, \varpi_2) - (\frac{r}{n-1} - n(\sigma^2 - \theta^2))\eta(\varpi_1)\eta(\varpi_2).$$

Also, if $\mathcal{W} = 0$ then the above equation is true [2]. Therefore Theorem 3.2 yields the next result.

Corollary 3.4. *Let (M^n, g) be a Weyl-semisymmetric (or Weyl-conformally flat) TSM of type (σ, θ). Then manifold M satisfies a HRS $(g, \xi, \frac{b-4\theta^2-2\xi(\theta)}{2\theta}, a+b)$ if $\theta \neq 0$, $\frac{b-4\theta^2-2\xi(\theta)}{2\theta}$ and $\sigma^2 - \theta^2$ are two constants, where $a = \frac{r}{n-1} - \sigma^2 + \theta^2$ and $b = -(\frac{r}{n-1} - n(\sigma^2 - \theta^2))$.*

Now, suppose that (M^n, g) be a TSM (σ, θ)-type and also be Einstein-semisymmetric, that is, satisfying the condition $R.\varpi = 0$, or equivalently,

$$(R(\varpi_1, \varpi_2).\varpi)(\varpi_3, \varpi_4) = 0,$$

where $\varpi = S - \frac{r}{n}g$ is Einstein tensor. Then from [39] we have

$$S(\varpi_1, \varpi_2) = (1-n)(\theta^2 - \sigma^2)g(\varpi_1, \varpi_2).$$

Therefore, applying Theorem 3.2 we deduce,

Corollary 3.5. *Assume that M is an Einstein-semisymmetric TSM of type (σ, θ) and $\sigma^2 - \theta^2$ is an nonzero constant. Then manifold M satisfies a HRS $(g, \xi, -\frac{2\theta^2+\xi(\theta)}{\theta}, (n-1)(\sigma^2 - \theta^2))$ if $\frac{2\theta^2+\xi(\theta)}{\theta}$ is a constant.*

Definition 3.6. *A Weyl-pseudosymmetric TSM is a TSM M provided that*

$$R.\mathcal{W} = f_{\mathcal{W}}\Gamma(g, \mathcal{W}),$$

where $f_{\mathcal{W}}$ is a function on $\{x \in M : \mathcal{W} \neq 0 \text{ at } x\}$ [22, 23].

Assume that M is a Weyl-pseudosymmetric (σ, θ)-type TSM, $\theta \neq 0$, and $f_{\mathcal{W}} \neq \sigma^2 - \theta^2$. Then from [39] we obtain

$$S(\varpi_1, \varpi_2) = (\frac{r}{n-1} - \sigma^2 + \theta^2)g(\varpi_1, \varpi_2) - (\frac{r}{n-1} - n(\sigma^2 - \theta^2))\eta(\varpi_1)\eta(\varpi_2).$$

Hence from Theorem 3.2 we deduce,

Corollary 3.7. *Suppose that M is a Weyl-pseudosymmetric (σ, θ)-type TSM with $f_{\mathcal{W}} \neq \sigma^2 - \theta^2$. Then manifold M satisfies a HRS $(g, \xi, \frac{b-4\theta^2-2\xi(\theta)}{2\theta}, a+b)$ provided $\theta \neq 0$, $\frac{b-4\theta^2-2\xi(\theta)}{2\theta}$ and $\sigma^2-\theta^2$ are two constants, where $a = \frac{r}{n-1}-\sigma^2+\theta^2$ and $b = -(\frac{r}{n-1} - n(\sigma^2 - \theta^2))$.*

Definition 3.8. *A Weyl Ricci pseudosymmetric TSM is a TSM M provided that*

$$\mathcal{W}.S = f_S\Gamma(g, S),$$

where f_S is a function on $\{x \in M : \mathcal{W} \neq 0 \text{ at } x\}$ [22, 23].

Assume that M is a Weyl Ricci pseudosymmetric (σ, θ)-type TSM and $\theta \neq 0$. Then from [39] we have

$$S(\varpi_1, \varpi_2) = (\frac{r}{n-1} - \sigma^2 + \theta^2)g(\varpi_1, \varpi_2) - (\frac{r}{n-1} - n(\sigma^2 - \theta^2))\eta(\varpi_1)\eta(\varpi_2).$$

Hence from Theorem 3.2 we deduce,

Corollary 3.9. *Suppose that M is a Weyl-pseudosymmetric TSM of type (σ, θ). Then manifold M satisfies a HRS $(g, \xi, \frac{b-4\theta^2-2\xi(\theta)}{2\theta}, a+b)$ provided $\theta \neq 0$, $\frac{b-4\theta^2-2\xi(\theta)}{2\theta}$ and $\sigma^2 - \theta^2$ are two constants, where $a = \frac{r}{n-1} - \sigma^2 + \theta^2$ and $b = -(\frac{r}{n-1} - n(\sigma^2 - \theta^2))$.*

Definition 3.10. *A partially Ricci pseudosymmetric TSM is a TSM M provided that*

$$R.S = f(q)\Gamma(g, S),$$

for $f \in C^\infty(M)$ and $q \in \{x \in M : \Gamma(g, S) \neq 0 \text{ at } x\}$ [3].

Suppose that M is a partially Ricci pseudosymmetric (σ, θ)-type TSM and $\theta \neq 0$. Then from [39] we have

$$S(\varpi_1, \varpi_2) = (n-1)(\sigma^2 - \theta^2)g(\varpi_1, \varpi_2).$$

Hence from Theorem 3.2 we deduce,

Corollary 3.11. *Suppose that M is a Weyl Ricci pseudosymmetric TSM of type (σ, θ), $f(q) \neq \sigma^2 - \theta^2$, and $\theta \neq 0$. Let $\frac{2\theta^2+\xi(\theta)}{\theta}$ and $\sigma^2 - \theta^2$ be two constants. Then manifold M satisfies a HRS $(g, \xi, -\frac{2\theta^2+\xi(\theta)}{\theta}, (n-1)(\sigma^2 - \theta^2))$.*

Definition 3.12. *A connected TSM (M^n, g) is anointed projectively flat if the limitation*

$$P(\varpi_1, \varpi_2)\varpi_3 = 0,$$

be maintained, where projectively tensor P is represented by

$$P(\varpi_1, \varpi_2)\varpi_3 = R(\varpi_1, \varpi_2)\varpi_3 - \frac{1}{n-1}\{S(\varpi_2, \varpi_3)\varpi_1 - S(\varpi_1, \varpi_3)\varpi_2\}.$$

Let (M^n, g) be a projectively flat TSM (σ, θ)-type. So, from [2] we have

$$S(\varpi_1, \varpi_2) = (n-1)(\sigma^2 - \theta^2)g(\varpi_1, \varpi_2).$$

Hence from Theorem 3.2 we deduce,

Corollary 3.13. *Suppose that (M^n, g) is a projectively flat TSM of type (σ, θ). Let $\frac{2\theta^2+\xi(\theta)}{\theta}$ and $\sigma^2 - \theta^2$ be two constants. Then manifold M satisfies a HRS $(g, \xi, -\frac{2\theta^2+\xi(\theta)}{\theta}, (n-1)(\sigma^2 - \theta^2))$.*

Let M be a TSM (σ, θ)-type satisfies the condition $R.P = 0$. Then from [2] we have

$$S(\varpi_1, \varpi_2) = (n-1)(\sigma^2 - \theta^2)g(\varpi_1, \varpi_2) + (\frac{r}{1-n} + n(\sigma^2 - \theta^2))\eta(\varpi_1)\eta(\varpi_2).$$

Hence from Theorem 3.2 the following result is obtained.

Corollary 3.14. *Suppose that (M^n, g) is a TSM (σ, θ)-type which satisfies $R.P = 0$ and $\theta \neq 0$. Then M satisfies a HRS $(g, \xi, \frac{b-4\theta^2-2\xi(\theta)}{2\theta}, a+b)$ if $\frac{b-4\theta^2-2\xi(\theta)}{2\theta}$ and $a+b$ are two constant where $a = (n-1)(\sigma^2 - \theta^2)$ and $b = (\frac{r}{1-n} + n(\sigma^2 - \theta^2))$.*

Definition 3.15. *A ϕ-projectively flat TSM is a TSM which*

$$\phi^2 P(\phi\varpi_1, \phi\varpi_2)\phi\varpi_3 = 0.$$

Assume that M is a 3-dimensional ϕ-projectively TSM. Then from Proposition 4.2 of [14] we have

$$S(\varpi_1, \varpi_2) = \frac{r}{3}g(\varpi_1, \varpi_2).$$

Then, applying Theorem 3.2 we deduce,

Corollary 3.16. *Let M be a ϕ-projectively 3-TSM of type (σ, θ). Then manifold M satisfies a HRS $(g, \xi, -\frac{2\theta^2+\xi(\theta)}{\theta}, \frac{r}{3})$ if $\frac{2\theta^2+\xi(\theta)}{\theta}$ and r are two constants.*

Definition 3.17. *Suppose that (M^n, g) is a TSM. The concircular curvature tensor C on M is specified by*

$$C(\varpi_1, \varpi_2)\varpi_3 = R(\varpi_1, \varpi_2)\varpi_3 - \frac{r}{n(n-1)}\left(g(\varpi_2, \varpi_3)\varpi_1 - g(\varpi_1, \varpi_3)\varpi_2\right). \tag{24}$$

Now, suppose that a TSM (σ,θ)-type M^n is concircularly flat, that means,

$$C(\varpi_1,\varpi_2)\varpi_3 = 0.$$

Therefore, we can conclude from [2] that:

$$S(\varpi_1,\varpi_2) = (n-1)(\sigma^2-\theta^2)g(\varpi_1,\varpi_2) - (\frac{r}{n-1} - (\sigma^2-\theta^2))\eta(\varpi_1)\eta(\varpi_2).$$

Hence from Theorem 3.2 we deduce,

Corollary 3.18. *Suppose that (M^n, g) is a concircularly flat TSM (σ,θ)-type. Then manifold M satisfies a HRS $(g, \xi, \frac{b-4\theta^2-2\xi(\theta)}{2\theta}, a+b)$ if $\frac{b-4\theta^2-2\xi(\theta)}{2\theta}$ and $a+b$ are two constants where $a = (n-1)(\sigma^2-\theta^2)$ and $b = -(\frac{r}{n-1} - (\sigma^2-\theta^2))$.*

The contact holomorphic Riemannian curvature tensor or briefly $(CHR)_3$-curvature tensor [32, 33] in an AC Riemannain manifold is defined by

$$\begin{aligned}
&16(CHR)_3(\varpi_1,\varpi_2,\varpi_3,\varpi_4)\\
&= 3\{R(\varpi_1,\varpi_2,\varpi_3,\varpi_4) + R(\phi\varpi_1,\phi\varpi_2,\varpi_3,\varpi_4) + R(\varpi_1,\varpi_2,\phi\varpi_3,\phi\varpi_4)\\
&+R(\phi\varpi_1,\phi\varpi_2,\phi\varpi_3,\phi\varpi_4)\} - R(\varpi_1,\varpi_3,\phi\varpi_4,\phi\varpi_2) - R(\phi\varpi_1,\phi\varpi_3,\varpi_4,\varpi_2)\\
&-R(\varpi_1,\varpi_4,\phi\varpi_2,\phi\varpi_3) - R(\phi\varpi_1,\phi\varpi_4,\varpi_2,\varpi_3) + R(\phi\varpi_1,\varpi_3,\phi\varpi_4,\varpi_2)\\
&+R(\varpi_1,\phi\varpi_3,\varpi_4,\phi\varpi_2) + R(\phi\varpi_1,\varpi_4,\varpi_2,\phi\varpi_3) + R(\varpi_1,\phi\varpi_4,\phi\varpi_2,\varpi_3)\\
&+\eta(\varpi_1)\mathcal{P}(\varpi_3,\varpi_4,\varpi_2) - \eta(\varpi_2)\mathcal{P}(\varpi_3,\varpi_4,\varpi_1) + \eta(\varpi_3)\mathcal{P}(\varpi_1,\varpi_2,\varpi_4)\\
&-\eta(\varpi_4)\mathcal{P}(\varpi_1,\varpi_2,\varpi_3) + \eta(\varpi_1)\eta(\varpi_4)\mathcal{Q}(\varpi_2,\varpi_3) - \eta(\varpi_1)\eta(\varpi_3)\mathcal{Q}(\varpi_2,\varpi_4)\\
&+\eta(\varpi_2)\eta(\varpi_3)\mathcal{Q}(\varpi_4,\varpi_1) - \eta(\varpi_2)\eta(\varpi_4)\mathcal{Q}(\varpi_3,\varpi_1),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{P}(\varpi_1,\varpi_2,\varpi_3) \quad &= \quad 3\{R(\varpi_1,\varpi_2,\varpi_3,\xi) + R(\phi\varpi_1,\phi\varpi_2,\varpi_3,\xi)\} + R(\phi\varpi_1,\phi\varpi_3,\varpi_2,\xi)\\
&\quad +R(\phi\varpi_3,\phi\varpi_2,\varpi_1,\xi) - R(\varpi_1,\phi\varpi_3,\phi\varpi_2,\xi) - R(\phi\varpi_3,\varpi_2,\phi\varpi_1,\xi),
\end{aligned}$$

and

$$\mathcal{Q}(\varpi_1,\varpi_2) \quad = \quad 3R(\xi,\varpi_1,\varpi_2,\xi) - R(\xi,\phi\varpi_1,\phi\varpi_2,\xi).$$

Now assume that a TSM (σ,θ)-type (M^n, g) is $(CHR)_3$-flat and θ is constant, that is, $(CHR)_3(\varpi_1,\varpi_2,\varpi_3,\varpi_2) = 0$. Then from [32] we have

$$S(\varpi_1,\varpi_2) = \frac{3n-5}{4}(\sigma^2-\theta^2)g(\varpi_1,\varpi_2) + \frac{n+1}{4}(\sigma^2-\theta^2)\eta(\varpi_1)\eta(\varpi_2).$$

Hence from Theorem 3.2 we deduce,

Corollary 3.19. *Suppose that (M^n, g) is a $(CHR)_3$-flat TSM (σ,θ)-type and σ,θ be two constants. Then manifold M satisfies a HRS $(g, \xi, \frac{b-4\theta^2}{2\theta}, (n-1)(\sigma^2-\theta^2))$ if $\theta \neq 0$ where $b = \frac{n+1}{4}(\sigma^2-\theta^2)$.*

In 2011, Mantica and Molinari [30] stated the concept of generalized $\mathcal{Z}$ tensor on a SRM M as follows

$$\mathcal{Z} = S + fg,$$

where f is some smooth map on M. A three-dimensional TSM M is anointed weakly $\mathcal{Z}$ symmetric [41] if the nonvanishing $\mathcal{Z}$ tensor justified

$$(\nabla_{\varpi_1}\mathcal{Z})(\varpi_2,\varpi_3) = A(\varpi_1)\mathcal{Z}(\varpi_2,\varpi_3) + B(\varpi_2)\mathcal{Z}(\varpi_1,\varpi_3) + D(\varpi_3)\mathcal{Z}(\varpi_1,\varpi_2),$$

where A, B and D are 1-forms on M. From [8], a three-dimensional weakly $\mathcal{Z}$ symmetric TSM (σ,θ)-type with $B \neq D$ and $grad\theta \neq \phi grad\sigma$ is η-Einstein manifold and

$$S(\varpi_1,\varpi_2) = -fg(\varpi_1,\varpi_2) + \frac{[2(\sigma^2-\theta^2) - \xi(\theta) + f]^2}{f + 2(\sigma^2-\theta^2-\xi(\theta))}\eta(\varpi_1)\eta(\varpi_2).$$

Hence from Theorem 3.2 we deduce,

Corollary 3.20. *Suppose that (M^n, g) is a weakly $\mathcal{Z}$ symmetric three-dimensional TSM (σ,θ)-type with $B \neq D$, $\theta \neq 0$, and $grad\theta \neq \phi grad\sigma$. Then manifold M satisfies a HRS $(g, \xi, \frac{b-4\theta^2-2\xi(\theta)}{2\theta}, -f+b)$ if $\frac{b-4\theta^2-2\xi(\theta)}{2\theta}$ and $-f+b$ are constants where $b = \frac{[2(\sigma^2-\theta^2)-\xi(\theta)+f]^2}{f+2(\sigma^2-\theta^2-\xi(\theta))}$.*

From [21] we have the following theorem.

Theorem 3.21. *Suppose that $(M, \phi, \xi, \eta, g, \sigma, \theta)$ be a compact and connected 3-dimensional TSM (σ,θ)-type. If the Ricci tensor curvature $Ric(\xi,\xi)$ satisfies $0 < Ric(\xi,\xi) < 2(\sigma^2-\theta^2)$, then M is homothetic to a Sasakian manifold.*

Hence from Theorem 3.2 the following result is obtained.

Corollary 3.22. *Assume that M is a 3-dimensional connected and compact TSM (σ,θ)-type and admits in HRS (g,ξ,λ,μ) such that $0 < \mu < 2(\sigma^2-\theta^2)$. Then M is homothetic to a Sasakian manifold.*

Definition 3.23. *A conformal KVF on SRM (M, g) is a vector field χ that*

$$(\mathcal{L}_\chi g)(\varpi_1, \varpi_2) = 2hg(\varpi_1, \varpi_2), \tag{25}$$

where h is some function on M. Let χ be a conformal KVF on M. Then, χ reduces to

- *proper vector field if h is not constant,*
- *KVF when $h = 0$,*
- *homothetic vector field when h is a constant.*

Let χ be a conformal KVF on TSM (σ,θ)-type and satisfies in (25). Then

$$\begin{aligned}((\mathcal{L}_\chi \circ \mathcal{L}_\chi)g)(\varpi_1,\varpi_2) &= \chi(\mathcal{L}_\chi g(\varpi_1,\varpi_2)) - \mathcal{L}_\chi g(\mathcal{L}_\chi \varpi_1, \varpi_2) - \mathcal{L}_\chi g(\varpi_1, \mathcal{L}_\chi \varpi_2)\\ &= \chi(2hg(\varpi_1,\varpi_2)) - 2hg(\mathcal{L}_\chi\varpi_1,\varpi_2) - 2hg(\varpi_1,\mathcal{L}_\chi\varpi_2)\\ &= 2\chi(h)g(\varpi_1,\varpi_2) + 2h\mathcal{L}_\chi g(\varpi_1,\varpi_2)\\ &= (2\chi(h)+4h^2)g(\varpi_1,\varpi_2).\end{aligned} \tag{26}$$

By inserting (26) in the equation (14) we obtain

$$S(\varpi_1,\varpi_2) + 2h\lambda g(\varpi_1,\varpi_2) + (2\chi(h)+4h^2)g(\varpi_1,\varpi_2) - \mu g(\varpi_1,\varpi_2) = 0. \tag{27}$$

Replacing ϖ_1 and ϖ_2 by ξ in the above relation we get

$$((n-1)(\sigma^2-\theta^2) - (n-1)\xi(\theta) + 2h\lambda + 2\chi(h) + 4h^2 - \mu)\eta(\varpi_1) = 0. \tag{28}$$

The following theorem is concluded considering that ϖ_1 is arbitrary in the above relation.

Theorem 3.24. *If χ is a conformally KVF on M and g is a TSM (σ,θ)-type metric on M that satisfies the HRS (g,χ,λ,μ), that means $\mathcal{L}_\chi g = 2hg$, then M is η-Einstein and*

$$(n-1)(\sigma^2-\theta^2-\xi(\theta)) + 2h\lambda + 2\chi(h) + 4h^2 - \mu = 0. \tag{29}$$

Definition 3.25. *Assume that (M, g) is a SRM and χ is a nonzero vector field on it. We call the vector field χ torse-forming whenever [45],*

$$\nabla_\zeta \chi = h\zeta + \tau(\zeta)\chi, \tag{30}$$

where h is a smooth map, τ is a 1-form, ∇ is the Levi-Civita connection (LCC) of metric and ζ is a vector filed on M. the vector field χ becomes

- *concircular [11, 44] whenever the 1-form τ in (30) vanishes identically,*

- *concurrent [40, 46] if $h = 1$ and the 1-form τ in (30) vanishes identically,*
- *parallel vector field if in equation (30) $h = \tau = 0$,*
- *torqued vector field [10] if in equation (30) $\tau(\chi) = 0$.*

Suppose that (g, χ, λ, μ) is a HRS on a TSM (σ, θ)-type where χ is a torse-forming vector filed and fulfilled the (30). Then

$$\mathcal{L}_\chi g(\varpi_1, \varpi_2) = 2hg(\varpi_1, \varpi_2) + \tau(\varpi_1)g(\chi, \varpi_2) + \tau(\varpi_2)g(\chi, \varpi_1) \tag{31}$$

and

$$\begin{aligned}(\mathcal{L}_\chi(\mathcal{L}_\chi g))(\varpi_1, \varpi_2) &= \chi\left(2hg(\varpi_1, \varpi_2) + \tau(\varpi_1)g(\chi, \varpi_2) + \tau(\varpi_2)g(\chi, \varpi_1)\right)\\ &\quad -2hg(\mathcal{L}_\chi\varpi_1, \varpi_2) - \tau(\mathcal{L}_\chi\varpi_1)g(\chi, \varpi_2) - \tau(\varpi_2)g(\chi, \mathcal{L}_\chi\varpi_1)\\ &\quad -2hg(\varpi_1, \mathcal{L}_\chi\varpi_2) - \tau(\varpi_1)g(\chi, \mathcal{L}_\chi\varpi_2) - \tau(\mathcal{L}_\chi\varpi_2)g(\chi, \varpi_1).\end{aligned} \tag{32}$$

On the other hand,

$$g(\mathcal{L}_\chi\varpi_1, \varpi_2) = g(\nabla_\chi\varpi_1, \varpi_2) - hg(\varpi_1, \varpi_2) - \tau(\varpi_2)g(\chi, \varpi_1), \tag{33}$$

similarly

$$g(\varpi_1, \mathcal{L}_\chi\varpi_2) = g(\varpi_1, \nabla_\chi\varpi_2) - hg(\varpi_1, \varpi_2) - \tau(\varpi_1)g(\chi, \varpi_2). \tag{34}$$

Thus,

$$\begin{aligned}g(\mathcal{L}_\chi\varpi_1, \varpi_2) + g(\varpi_1, \mathcal{L}_\chi\varpi_2) &= \chi(g(\varpi_1, \varpi_2)) - 2hg(\varpi_1, \varpi_2) - \tau(\varpi_2)g(\chi, \varpi_1)\\ &\quad -\tau(\varpi_1)g(\chi, \varpi_2).\end{aligned} \tag{35}$$

Also, we have

$$\begin{aligned}\tau(\mathcal{L}_\chi\varpi_1) &= \tau(\nabla_\chi\varpi_1 - \nabla_{\varpi_1}\chi) = \tau(\nabla_\chi\varpi_1 - h\varpi_1 - \tau(\varpi_1)\chi)\\ &= \tau(\nabla_\chi\varpi_1) - h\tau(\varpi_1) - \tau(\varpi_1)\tau(\chi),\end{aligned}$$

similarly

$$\tau(\mathcal{L}_\chi\varpi_2) = \tau(\nabla_\chi\varpi_2) - h\tau(\varpi_2) - \tau(\varpi_2)\tau(\chi).$$

Therefore, applying the above equations in (32) we obtain

$$\begin{aligned}(\mathcal{L}_\chi(\mathcal{L}_\chi g))(\varpi_1, \varpi_2) &= (2\chi(h) + 4h^2)g(\varpi_1, \varpi_2) + 4h\tau(\varpi_2)g(\chi, \varpi_1)\\ &\quad +\chi\left(\tau(\varpi_1)g(\chi, \varpi_2) + \tau(\varpi_2)g(\chi, \varpi_1)\right) + 4h\tau(\varpi_1)g(\chi, \varpi_2)\\ &\quad -\tau(\nabla_\chi\varpi_1)g(\chi, \varpi_2) + 2\tau(\varpi_1)\tau(\chi)g(\chi, \varpi_2)\\ &\quad +2\tau(\varpi_2)\tau(\chi)g(\chi, \varpi_1) - \tau(\nabla_\chi\varpi_2)g(\chi, \varpi_1)\\ &\quad -\tau(\varpi_2)g(\nabla_\chi\varpi_1, \chi) - \tau(\varpi_1)g(\nabla_\chi\varpi_2, \chi).\end{aligned} \tag{36}$$

Putting $\varpi_1 = \varpi_2 = \xi$ in (31) and (36) we infer

$$\mathcal{L}_\chi g(\xi, \xi) = 2h + 2\tau(\xi)\eta(\chi) \tag{37}$$

and

$$\begin{aligned}(\mathcal{L}_\chi(\mathcal{L}_\chi g))(\xi, \xi) &= (2\chi(h) + 4h^2) + \chi\left(\tau(\xi)\eta(\chi) + \tau(\xi)\eta(\chi)\right)\\ &\quad +4h\tau(\xi)\eta(\chi) + 4h\tau(\xi)\eta(\chi)\\ &\quad -2\tau(\nabla_\chi\xi)\eta(\chi) + 4\tau(\xi)\tau(\chi)\eta(\chi) - 2\tau(\xi)g(\nabla_\chi\xi, \chi)\\ &= (2\chi(h) + 4h^2) + \chi\left(\tau(\xi)\eta(\chi) + \tau(\xi)\eta(\chi)\right)\\ &\quad +4h\tau(\xi)\eta(\chi) + 4h\tau(\xi)\eta(\chi) + 4\tau(\xi)\tau(\chi)\eta(\chi)\\ &\quad +2\tau(\xi)\eta(\chi) + 4(\eta(\chi))^2\tau(\xi) + 2\tau(\xi)|\chi|^2.\end{aligned} \tag{38}$$

Applying (37) and (38) into (14) we arrive at

$$(n-1)(\sigma^2-\theta^2)-(n-1)\xi(\theta)-\mu+\lambda(2h+2\tau(\xi)\eta(\chi))+(2\chi(h)+4h^2)$$
$$+\chi\left(\tau(\xi)\eta(\chi)+\tau(\xi)\eta(\chi)\right)+4h\tau(\xi)\eta(\chi)+4h\tau(\xi)\eta(\chi)$$
$$+4\tau(\xi)\tau(\chi)\eta(\chi)+2\tau(\xi)\eta(\chi)+4(\eta(\chi))^2\tau(\xi)+2\tau(\xi)|\chi|^2=0. \tag{39}$$

Therefore, the following theorem can be stated.

Theorem 3.26. *If χ is a torse-forming vector filed and satisfied in (30) and g is a metric of a TSM (σ,θ)-type which satisfies the HRS (g,χ,λ,μ), then the equation (39) holds.*

Corollary 3.27. *If χ is concircular vector filed and g is the metric of TSM M satisfies the HRS (g,χ,λ,μ), that is, $\nabla_{\varpi}\chi=h\varpi$ for all ϖ's as a vector field, then $(n-1)(\sigma^2-\theta^2-\xi(\theta))-\mu+2h\lambda+2\chi(h)+4h^2=0$.*

Garcia-Parrado and Senovilla [25] introduced bi-conformal vector fields, then De et al. in [16] utilizing the the Ricci tensor field S and the metric tensor field g specified Ricci bi-conformal vector fields as follows.

Definition 3.28. *Let (M,g) be a Riemannian manifold and χ is a vector field on M. If the following equations hold, then we say that χ is Ricci bi-conformal vector field.*

$$(\mathcal{L}_\chi g)(\varpi_1,\varpi_2)=\alpha g(\varpi_1,\varpi_2)+\beta S(\varpi_1,\varpi_2), \tag{40}$$

and

$$(\mathcal{L}_\chi S)(\varpi_1,\varpi_2)=\alpha S(\varpi_1,\varpi_2)+\beta g(\varpi_1,\varpi_2), \tag{41}$$

for non-zero smooth functions α and β.

Let χ is the Ricci bi-conformal vector field on TSM (σ,θ)-type (M^n,g), which satisfies the HRS (g,χ,λ,μ) and satisfies in (40) and (41). We get

$$\mathcal{L}_\chi(\mathcal{L}_\chi g)=(\alpha^2+\beta^2+\chi(\alpha))g+(2\alpha\beta+\chi(\beta))S. \tag{42}$$

Inserting (40) and (42) in (14) we have

$$(1+\lambda\beta+2\alpha\beta+\chi(\beta))S(\varpi_1,\varpi_2)+(\lambda\alpha-\mu+\alpha^2+\beta^2+\chi(\alpha))g(\varpi_1,\varpi_2)=0. \tag{43}$$

Substituting $\varpi_1=\varpi_2=\xi$ in the last relation we arrive at

$$(1+\chi(\beta)+2\alpha\beta+\lambda\beta)(n-1)(\sigma^2-\theta^2-\xi(\theta))+(\lambda\alpha-\mu+\alpha^2+\beta^2+\chi(\alpha))=0, \tag{44}$$

and

$$\Big(1+\chi(\beta)+2\alpha\beta+\lambda\beta\Big)\Big(S(\varpi_1,\varpi_2)-(n-1)(\sigma^2-\theta^2-\xi(\theta))g(\varpi_1,\varpi_2)\Big)=0. \tag{45}$$

Set $F=1+\lambda\beta+2\alpha\beta+\chi(\beta)$ and $G=\lambda\alpha-\mu+\alpha^2+\beta^2+\chi(\alpha)$.

Taking the Lie derivative of both sides of (43) and using (40) and (41), we deduce

$$(\alpha F+\beta G+\chi(F))S(\varpi_1,\varpi_2)+(\alpha G+\beta F+\chi(G))g(\varpi_1,\varpi_2)=0. \tag{46}$$

Putting $\varpi_1=\varpi_2=\xi$ in the last relation we conclude

$$(\alpha F+\beta G+\chi(F))(n-1)(\sigma^2-\theta^2-\xi(\theta))+\alpha G+\beta F+\chi(G)=0. \tag{47}$$

Using (44) and (47) we infer

$$F\Big(\beta-(n-1)^2\beta(\sigma^2-\theta^2-\xi(\theta))-(n-1)\chi(\sigma^2-\theta^2-\xi(\theta))\Big)=0.$$

If $F\neq 0$ then the equation (45) implies that manifold is a Einstein manifold and $r=n(n-1)(\sigma^2-\theta^2-\xi(\theta))$, where $\sigma^2-\theta^2-\xi(\theta)$ is constant. If $F=0$ then $G=0$. Therefore, the following theorem can be stated.

Theorem 3.29. *Suppose that χ is the Ricci bi-conformal vector field on a TSM (σ,θ)-type M with metric g and satisfies in (40) and (41). Also, let (M,g) satisfies the HRS (g,χ,λ,μ), then manifold M is a Einstein manifold, $\beta(1-(n-1)^2(\sigma^2-\theta^2-\xi(\theta))^2)=0$, and $r=n(n-1)(\sigma^2-\theta^2-\xi(\theta))$ or $\lambda=-\frac{1}{\beta}(1+2\alpha\beta+\chi(\beta))$ and $\mu=-\frac{\alpha}{\beta}(1+\chi(\beta))-\alpha^2+\beta^2+\chi(\alpha)$.*

4. Examples

We give two examples of TSMs admit in HRS in this section.

Example 4.1. *Suppose that* $M = \{(\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3) \in \mathbb{R}^3 | \mathfrak{e}_3 \neq 0\}$ *where* $(\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3)$ *is the canonic coordinates in* $\mathbb{R}^3$ *and vector fields*

$$\varpi_1 = e^{-2\mathfrak{e}_3}(\frac{\partial}{\partial \mathfrak{e}_1} + \frac{\partial}{\partial \mathfrak{e}_2}), \qquad \varpi_2 = -e^{-2\mathfrak{e}_3}(\frac{\partial}{\partial \mathfrak{e}_1} - \frac{\partial}{\partial \mathfrak{e}_2}), \qquad \varpi_3 = \frac{\partial}{\partial \mathfrak{e}_3},$$

are linearly independent on M. We determine the metric g by

$$g(\varpi_s, \varpi_t) = \begin{cases} 1, & \text{if } s = t, \\ 0, & \text{otherwise,} \end{cases}$$

where $s, t \in \{1, 2, 3\}$*. We state an ACS* (ϕ, ξ, η) *for M by*

$$\xi = \varpi_3, \quad \eta(\zeta) = g(\zeta, \varpi_3), \quad \phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where ζ *is a vector field. Note that the relations* $g(\phi\zeta, \phi W) = g(\zeta, W) - \eta(\zeta)\eta(W)$, $\eta(\xi) = 1$ *and* $\phi^2(\zeta) = \eta(\zeta)\xi - \zeta$ *hold. Thus, an AC structure* (M, ϕ, ξ, η, g) *defines on M. We get*

$[,]$	ϖ_1	ϖ_2	ϖ_3
ϖ_1	0	0	$2\varpi_1$
ϖ_2	0	0	$2\varpi_2$
ϖ_3	$-2\varpi_1$	$-2\varpi_2$	0

The LCC ∇ *of M is presented by*

$$\nabla_{\varpi_i}\varpi_j = \begin{pmatrix} -2\varpi_3 & 0 & 2\varpi_1 \\ 0 & -2\varpi_3 & 2\varpi_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The structure (ϕ, ξ, η) *confirms the relation* $(\nabla_\zeta\phi)W = g(\phi\zeta, W) - \eta(W)\phi\zeta$ *and* $\nabla_\zeta\xi = \zeta - \eta(\zeta)\xi$*, thus* (M, ϕ, ξ, η, g) *becomes a TSM* $(0, 1)$*-type. The curvature tensor nonvanishing terms are:*

$$R(\varpi_1, \varpi_2)\varpi_1 = 4\varpi_2, \ R(\varpi_1, \varpi_2)\varpi_2 = -4\varpi_1,$$
$$R(\varpi_1, \varpi_3)\varpi_1 = 4\varpi_3, \ R(\varpi_2, \varpi_3)\varpi_2 = 4\varpi_3,$$
$$R(\varpi_1, \varpi_3)\varpi_3 = -4\varpi_1, \ R(\varpi_2, \varpi_3)\varpi_3 = -4\varpi_2.$$

Therefore, we get

$$S = \begin{pmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{pmatrix} = -8g.$$

If we consider $\chi = \xi$ *then* $\mathcal{L}_\chi g = 2(g - \eta \otimes \eta)$ *and* $(\mathcal{L}_\chi \circ \mathcal{L}_\chi)g = -4^2(g - \eta \otimes \eta)$*. Hence,* $(g, \xi, \lambda = 2, \mu = -8)$ *is a HRS on manifold M.*

Example 4.2. *Suppose that* $M = \{(\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3) \in \mathbb{R}^3 | \mathfrak{e}_3 \neq 0\}$ *where* $(\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3)$ *is the canonic coordinates in* $\mathbb{R}^3$*. Also, let*

$$\varpi_1 = \mathfrak{e}_3\frac{\partial}{\partial \mathfrak{e}_1}, \qquad \varpi_2 = \mathfrak{e}_3\frac{\partial}{\partial \mathfrak{e}_2}, \qquad \varpi_3 = \mathfrak{e}_3\frac{\partial}{\partial \mathfrak{e}_3},$$

are linearly independent on M. We characterize

$$g(\varpi_s, \varpi_t) = \begin{cases} 1, & \text{if } s = t, \\ 0, & \text{otherwise,} \end{cases}$$

were $s, t \in \{1, 2, 3\}$ *as Riemannain metric* g *on* M. *Assume that an ACS* (ϕ, ξ, η) *on* M *given by*

$$\xi = \varpi_3, \quad \eta(\zeta) = g(\zeta, \varpi_3), \quad \phi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for all vector field ζ. *Then one can easily show that the relations* $\eta(\xi) = 1$, $g(\phi\zeta, \phi W) = g(\zeta, W) - \eta(\zeta)\eta(W)$ *and* $\phi^2(\zeta) = -\zeta + \eta(\zeta)\xi$ *are true. Thus* (M, ϕ, ξ, η, g) *represents an ACS on* M. *We get*

[,]	ϖ_1	ϖ_2	ϖ_3
ϖ_1	0	0	$-\varpi_1$
ϖ_2	0	0	-ϖ_2
ϖ_3	ϖ_1	ϖ_2	0

Applying the Koszul's formula we see that the LCC ∇ *of* M *is determined by*

$$\nabla_{\varpi_i}\varpi_j = \begin{pmatrix} \varpi_3 & 0 & -\varpi_1 \\ 0 & \varpi_3 & -\varpi_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, we find out that the structure (ϕ, ξ, η) *satisfies the formula* $(\nabla_\zeta\phi)W = g(\phi\zeta, W) - \eta(W)\phi\zeta$ *and* $\nabla_\zeta\xi = \zeta - \eta(\zeta)\xi$. *Hence* (M, ϕ, ξ, η, g) *becomes a TSM* $(0, 1)$*-type. The curvature tensor nonvanishing terms are:*

$$R(\varpi_1, \varpi_2)\varpi_1 = \varpi_2, \; R(\varpi_1, \varpi_2)\varpi_2 = -\varpi_1,$$
$$R(\varpi_1, \varpi_3)\varpi_1 = \varpi_3, \; R(\varpi_2, \varpi_3)\varpi_2 = \varpi_3,$$
$$R(\varpi_1, \varpi_3)\varpi_3 = -\varpi_1, \; R(\varpi_2, \varpi_3)\varpi_3 = -\varpi_2.$$

Thus, we obtain

$$S = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2g.$$

If we assume that $\chi = \xi$ *then* $\mathcal{L}_\chi g = 2(g - \eta \otimes \eta)$ *and* $(\mathcal{L}_\chi \circ \mathcal{L}_\chi)g = -4(g - \eta \otimes \eta)$. *Therefore* $(g, \xi, \lambda = 2, \mu = 2)$ *is a HRS on manifold* M.

References

[1] S. Azami and G. Fasihi-Ramndi, *Hyperbolic Ricci soliton on warped product manifolds*, Filomat, **37**:20 (2023), 6843-6853.

[2] C. S. Bagewadi and Veakatesha, *Some curvature tensors on a trans-Sasakian manifold*, Tuk J. Math. **31**(2007), 111-121.

[3] T.Q. Binh, U. C. De and L. Tamàssy, *On partially pseudo symmetric K- contact Riemannian manifolds*, Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis **18** (2002), 19-25.

[4] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Mathematics, **509**. Springer-Verlag, Berlin-New York (1976).

[5] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, **203**, Birkhauser Boston Inc. (2002).

[6] D. E. Blair and J. A. Oubina, *Conformal and related changes of metric on the product of two almost contact metric manifolds*, Publ. Mathematiques, **34** (1990), 199-207.

[7] C. Calin and M. Crasmareanu, *From the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds*, Bull. Malays. Math. Soc. **33**(3) (2010) 361-368.

[8] S. K. Chaubey and U. C. De, *Three-dimensional trans-Sasakian manifolds and solitons*, Arab J. Math. Sci., **28**(1)(2022), 112-125.

[9] X. Chen, *Einstein-Weyl structures on trans-Sasakian manifolds*, Math. Slovaca, **69**(6)(2019), 1425-1436.

[10] B. Y. Chen, *Classification of torqued vector fields and its applications to Ricci solitons*, Kragujevac J. Math., **41**(2) (2017), 239-250.

[11] B. Y. Chen, *A simple characterization of generalized Robertson-Walker space-times*, Gen. Relativ. Gravitation, **46**(12) (2014), Article ID 1833.

[12] J. X. Cruz Neto, I. D. Melo and P. A. Sousa, *Non-existence of strictly monotone vector fields on certain Riemannian manifolds*, Acta Math. Hung. **146** (2015), 240-246.

[13] W. R. Dai, D. X. Kong, K. Liu, *Hyperbolic gometric flow (I): short-time existence and nonlinear stability*, Pure and applied mathematics quarterly, **6** (2010), 331-359.

[14] K. De and U. C. De, *Projective curvature tensor in 3-dimensional connected trans-Sasakian manifolds*, Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica **55**(2)(2016), 29-40.

[15] U.C. De and N. Guha, *On generalized recurrent manifolds*, Proc. Math. Soc. **7** (1991), 7-11.
[16] U. C. De, A. Sardar, and A. Sarkar, *Some conformal vector fields and conformal Ricci solitons on $N(k)$-contact metric manifolds*, AUT J. Math. Com., **2**(1) (2021), 61-71.
[17] U.C. De and A. Sarkar, *On three-Dimensional trans-Sasakian manifolds*. Extracta Math. **23**(2008), 265-277.
[18] U.C. De and M. M. Tripathi, *Ricci tensor in 3-dimensional trans-Sasakian manifolds*, Kyungpook Math. J., **2**(2003), 247-255.
[19] S. Deshmukh, *Trans-Sasakian manifolds homothetic to Sasakian manifolds*. Mediterr. J. Math. **13**(2016), 2951-2958.
[20] S. Deshmukh and F. Alsolamy, *A note on compact trans-Sasakian manifolds*. Mediterr. J. Math. **13**(2016), 2099-2104.
[21] S. Deshmukh and M. M. Tripathi, *A note on trans-Sasakian manifolds*, Math. Slovaca, **63**(6)(2013), 1361-1370.
[22] R. Deszcz and W. Grycak, *On some class of warped product manifolds*, Bull. Inst. Math. Acad. Sinica. **15** (1987), 311-322.
[23] R. Deszcz, F. Defever, L. Verstraelen and L. Vraneken, *On pseudosymmetric spacetimes*, J. Math. Phys. **35** (1994), 5908-5921.
[24] H. Faraji, S. Azami, G. Fasihi-Ramandi, *Three dimensional homogeneous hyperbolic Ricci soliton*, J. Nonlinear Math. Phys., (2022), 1-21.
[25] A. Garcia-Parrado and J. M. M. Senovilla, *Bi-conformal vector fields and their applications*, Classical Quantum Gravity, **21**(8) (2004), 2153-2177.
[26] A. Gray and L. M. Hervalla, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Math. Pura Appl., **123**(4)(1980), 35-58.
[27] R. S. Hamilton, *The Ricci flow on surfaces, Mathematics and general relativity*, Contemp. Math. Santa Cruz, CA, 1986, **71**, American Math. Soc. (1988), 237-262.
[28] R. Ma and D. Pei, *Reeb flow invariant ⋆-Ricci operators on trans-Sasakian three-Manifolds*. Math. Slovaca, **71**(2021), 749-756.
[29] V. Mangione, *Harmonic maps and stability on f-Kenmotsu manifolds*, Int. J. Math. Math. Sci., 2008(2008).
[30] C. A. Mantica and L. G. Molinari, *Weakly Z symmetric manifolds*, Acta Math. Hung., **135**(1-2)(2012), 80-96.
[31] J. C. Marrero, *The local structure of trans-Sasakian manifolds*, Ann. Mat. Pura Appl., **162**(1992), 77-86.
[32] K. Matsumoto, *A $(CHR)_3$-flat trans-Sasakian manifold*, Proc. Int. Geom. Cent. **12**(2)(2019), 11-25.
[33] K. Matsumoto, *A new curvaturlike tensor field in an almost contact Riemannian manifold II*, Publ. Inst. Math. (Beograd) (N.S), **103**(117)(2018), 113-128.
[34] S. Z. Németh, *Five kinds of monotone vector fields*, Pure Math. Appl., **9**(1998), 417-428.
[35] M. Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere.* J. Math. Soc. Jpn. **14**(1962), 333-340.
[36] M. Okumura, *Certain almost contact hypersurfaces in Kaehlerian manifolds of constant holomorphic sectional curvatures*. Tohoku Math. J. **16**(1964), 270-284.
[37] Z. Olszak and R. Rosca, *Normal locally conformal almost cosymplectic manifolds*, Publ. Math. Debrecen, **39**(1991), 315-323.
[38] J. A. Oubina, *New classes of almost contact metric structures*, Publ. Math. Debrecen, **32** (1985), 187-193.
[39] G. P. Pokhariyal, S. Yadav and S. K. Chaubey, *Ricci solitons on trans-Sasakian manifolds*, Differ. Geom.-Dyn. Syst., **20**(2018), 138-158.
[40] J. A. Schouten, *Ricci Calculus*, Springer-Verlag (1954), Berlin.
[41] R. Tamassy and T. Q. Binh, *On weak symmetries of Einstein and Sasakian manifolds*. Tensor, N.S., **53**(1993), 140-48.
[42] W. Wang and X. Liu, *Ricci tensors on trans-Sasakian 3-manifolds*. Filomat, **32** (2018), 4365-4374.
[43] Y. Wang and W. Wang, *A remark on trans-Sasakian 3-manifolds*. Rev. Union Mater. Argent. **60**(2019), 257-264.
[44] K. Yano, *Concircular geometry I. Concircular tranformations*, Proc. Imp. Acad. Tokyo, **16**(1940), 195-200.
[45] K. Yano, *On the torse-forming directions in Riemannian spaces*, Proc. Imp. Acad. Tokyo **20** (1944), 340-345.
[46] K. Yano and B. Y. Chen, *On the concurrent vector fields of immersed manifolds*, Kodai Math. Sem. Rep., **23** (1971), 343-350.
[47] A. Yildiz, U. C. De and M. Turan, *On 3-dimensional f-Kenmotsu manifolds and Ricci solitons*, Ukrainian Math. J., **65**(2013), 684-693.
[48] Y. Zhao, *Some characterizations of property of trans-Sasakian 3-Manifolds*. Math. Math. Slovaca, **71**(2021), 383-390.