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Existence of solutions for a class of nonlinear integral equations on time scales

Svetlin G. Georgiev[a], Youssef N. Raffoul[b,*]

[a]*Department of Mathematics, Sorbonne University, Paris, France*
[b]*Department of Mathematics, University of Dayton, Dayton Ohio 45469-2316, U.S.A.*

Abstract. In this paper we investigate a class of nonlinear integral equations for existence of global classical solutions. We give conditions under which the considered equations have at least one, at least two and at least three solutions. To prove our main results we propose a new approach based upon recent theoretical results.

1. Introduction

In this paper, we investigate the following class of nonlinear integral equations on arbitrary time scales

$$x(t) = a(t) - \int_{t_0}^{t} C(t,s)(x(s) + H(s,x(s)))\Delta s, \quad t \in I, \tag{1.1}$$

where

(A1) $\mathbb{T}$ is a time scale with forward jump operator and delta differentiation operator σ and Δ, respectively, $t_0 \in \mathbb{T}$, $\sup \mathbb{T} = \infty$, $I = [t_0, \infty)_{\mathbb{T}}$, $a \in \mathcal{C}_{rd}(I)$, $0 \le a(t_0) \le B$ for some nonnegative constant B, $C \in \mathcal{C}_{rd}(I \times I)$, $H \in \mathcal{C}_{rd}(I \times \mathbb{R})$,

$$|H(s,y)| \le a_1(s) + a_2(s)|y|^p, \quad s \in I, \quad y \in \mathbb{R},$$

$a_1, a_2 \in \mathcal{C}_{rd}(I)$, $0 \le a_1, a_2 \le B$ on I, $p \ge 0$.

In [1], the equation (1.1) is investigated when $H(t,0) = 0$, $t \in I$, and there is a constant $k > 0$ such that

$$|H(t,y(t)) - H(t,z(t))| \le k|y(t) - z(t)|, \quad t \in I, \quad y,z \in \mathcal{M}, \tag{1.2}$$

where $\mathcal{M}$ is the Banach space of all bounded functions on I. Note that there are cases when the function H satisfies $(A1)$ and does not satisfy (1.2). For instance, the function $H(t,y(t)) = (y(t))^{\frac{1}{2}}$, $t \in I$, $y \in \mathcal{M}$, satisfies $(A1)$ but does not satisfy (1.2). Thus, the results in this paper can be considered as new and complimentary results to the results in [1].

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* Corresponding author: Youssef N. Raffoul
Email addresses: svetlingeorgiev1@gmail.com (Svetlin G. Georgiev), yraffoul1@udayton.edu (Youssef N. Raffoul)

In this paper, under the conditions $(A1)$ we will investigate the equation (1.1) for existence of at least one solution, at least two nonnegative and at least three nonnegative solutions. For this aim, firstly it is given a new integral representation of the solutions inspired by the papers [3], [7], [11] and reference therein. Then they are constructed two operators so that any fixed point of their sum is a solution to the considered problem.

The paper is organized as follows. In the next section, we give some auxiliary results. In Section 3 we prove existence of at least one classical solution for the equation (1.1). In Section 4, we prove existence of at least two nonnegative classical solutions. In Section 5, we prove existence of at least three nonnegative classical solutions. In Section 6, we give an example to illustrate our main results.

Throughout this work we assume a working knowledge of time scales calculus and the notation of time scales calculus.

2. Preliminary Results

Below, assume that X is a real Banach space. Now, we recall the definition for a completely continuous operator in a Banach space.

Definition 2.1. *Let $K : M \subset X \to X$ be a map. We say that K is compact if $K(M)$ is contained in a compact subset of X. K is called a completely continuous map if it is continuous and it maps any bounded set into a relatively compact set.*

The concept for k-set contraction is related to that of the Kuratowski measure of noncompactness which we recall for completeness.

Definition 2.2. *Let Ω_X be the class of all bounded sets of X. The Kuratowski measure of noncompactness $\alpha : \Omega_X \to [0, \infty)$ is defined by*

$$\alpha(Y) = \inf\left\{\delta > 0 : Y = \bigcup_{j=1}^{m} Y_j \quad and \quad diam(Y_j) \le \delta, \quad j \in \{1, \dots, m\}\right\},$$

where $diam(Y_j) = \sup\{\|x - y\|_X : x, y \in Y_j\}$ is the diameter of Y_j, $j \in \{1, \dots, m\}$.

Let α be a measure of noncompactness and $A, B \subset X$. Then some of the main properties of the measure of noncompactness are the following.

1. A is bounded if and only if $\alpha(A) < \infty$.
2. $\alpha(A) = \alpha(\overline{A})$, where $\overline{A}$ is the closure of A.
3. If A is compact, then $\alpha(A) = 0$. Conversely, if $\alpha(A) = 0$ and A is complete, then A is compact.
4. $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.
5. α is continuous with respect to the Hausdorff distance of sets.
6. $\alpha(aA) = |a|\alpha(A)$ for any $a \in \mathbb{R}$.

Measures of noncompactness are however useful in the study of infinite-dimensional Banach spaces.

Definition 2.3. *A mapping $K : X \to X$ is said to be k-set contraction if there exists a constant $k \ge 0$ such that*

$$\alpha(K(Y)) \le k\alpha(Y)$$

for any bounded set $Y \subset X$.

Obviously, if $K : X \to X$ is a completely continuous mapping, then K is 0-set contraction(see [6]).

Proposition 2.4. *(Leray-Schauder nonlinear alternative [2]) Let $C \subset E$ be a convex, closed subset in a Banach space E, $0 \in U \subset C$ where U is an open set. Let $f : \overline{U} \to C$ be a continuous, compact map. Then*

(a) *either f has a fixed point in $\overline{U}$,*

(b) *or there exist $x \in \partial U$, and $\lambda \in (0, 1)$ such that $x = \lambda f(x)$.*

To prove our existence result we will use the following fixed point theorem. Its proof can be found in [4].

Theorem 2.5. *Let E be a Banach space, Y a closed, convex subset of E, $0 \in Y$,*

$$U = \{x \in Y : \|x\| < R\},$$

with $R > 0$. Consider two operators T and S, where

$$Tx = \varepsilon x, \ x \in \overline{U},$$

for $\varepsilon > 0$ and $S : \overline{U} \to E$ be such that

(i) *$I - S : \overline{U} \to Y$ continuous, compact and*

(ii) *$\{x \in \overline{U} : x = \lambda(I - S)x, \quad \|x\| = R\} = \emptyset$, for any $\lambda \in \left(0, \frac{1}{\varepsilon}\right)$.*

Then there exists $x^* \in \overline{U}$ such that

$$Tx^* + Sx^* = x^*.$$

Definition 2.6. *Let X and Y be real Banach spaces. A map $K : X \to Y$ is called expansive if there exists a constant $h > 1$ for which one has the following inequality*

$$\|Kx - Ky\|_Y \geq h\|x - y\|_X$$

for any $x, y \in X$.

Now, we will recall the definition for a cone in a Banach space.

Definition 2.7. *A closed, convex set $\mathcal{P}$ in X is said to be cone if*

1. *$\alpha x \in \mathcal{P}$ for any $\alpha \geq 0$ and for any $x \in \mathcal{P}$,*
2. *$x, -x \in \mathcal{P}$ implies $x = 0$.*

Denote $\mathcal{P}^* = \mathcal{P} \backslash \{0\}$. The next result is a fixed point theorem which we will use to prove existence of at least two nonnegative global classical solutions of the IVP (1.1). For its proof, we refer the reader to [5], [8] and [9].

Theorem 2.8. *Let $\mathcal{P}$ be a cone of a Banach space E; Ω a subset of $\mathcal{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to \mathcal{P}$ is an expansive mapping, $S : \overline{U}_3 \to E$ is a completely continuous map and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $u_0 \in \mathcal{P}^*$ such that the following conditions hold:*

(i) *$Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda u_0)$,*

(ii) *there exists $\varepsilon \geq 0$ such that $Sx \neq (I - T)(\lambda x)$, for all $\lambda \geq 1 + \varepsilon$, $x \in \partial U_2$ and $\lambda x \in \Omega$,*

(iii) *$Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.*

Then $T + S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that

$$x_1 \in \partial U_2 \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

The following result will be used to prove the existence of three nonnegative solutions of our problem. For the proof, we use the same arguments used in [5] and [10].

Theorem 2.9. *Let $\mathcal{P}$ be a cone of a Banach space E; Ω a subset of $\mathcal{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to E$ is an expansive mapping, $S : \overline{U}_3 \to E$ is a completely continuous one and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exist $w_0 \in \mathcal{P}^*$ and $\varepsilon > 0$ small enough such that the following conditions hold:*

(i) *$Sx \neq (I - T)(\lambda x)$, for all $\lambda \geq 1 + \varepsilon$, $x \in \partial U_1$ and $\lambda x \in \Omega$,*

(ii) *$Sx \neq (I - T)(x - \lambda w_0)$, for all $\lambda \geq 0$ and $x \in \partial U_2 \cap (\Omega + \lambda w_0)$,*

(iii) *$Sx \neq (I - T)(\lambda x)$, for all $\lambda \geq 1 + \varepsilon$, $x \in \partial U_3$ and $\lambda x \in \Omega$.*

Then $T + S$ has at least three non trivial fixed points $x_1, x_2, x_3 \in \mathcal{P}$ such that

$$x_1 \in \overline{U}_1 \cap \Omega \text{ and } x_2 \in (U_2 \setminus \overline{U}_1) \cap \Omega \text{ and } x_3 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

In $X = \mathcal{C}_{rd}(I)$ we introduce the norm

$$\|u\| = \sup_{t \in I} |u(t)|,$$

provided it exists.

3. Existence of at Least One Solution

In this section, we will prove that the equation (1.1) has at least one solution. For $u \in X$, define the operator

$$S_1(u)(t) = u(t) - a(t) + \int_{t_0}^{t} C(t,s)(u(s) + H(s,u(s)))\Delta s, \quad t \in I.$$

Note that if $u \in X$ satisfies the equation

$$S_1(u)(t) = 0, \quad t \in I,$$

then u is a solution to the equation (1.1).

Let

$$B_1 = \max\{1, B, B^p\}$$

and

$$f(t) = 1 + |a(t)| + \int_{t_0}^{t} |C(t,s)|(1 + a_1(s) + a_2(s))\Delta s, \quad t \in I.$$

Lemma 3.1. *Suppose $(A1)$. If $u \in X$, $\|u\| \leq B$, then*

$$|S_1(u)(t)| \leq B_1 f(t), \quad t \in I.$$

Proof. We have

$$\begin{aligned}|S_1(u)(t)| &\leq |u(t)| + |a(t)| + \int_{t_0}^{t} |C(t,s)|(|u(s)| + a_1(s) + a_2(s)|u(s)|^p)\Delta s \\ &\leq B + |a(t)| + \int_{t_0}^{t} |C(t,s)|(B + a_1(s) + a_2(s)B^p)\Delta s \\ &\leq B_1\left(1 + |a(t)| + \int_{t_0}^{t} C(t,s)(1 + a_1(s) + a_2(s))\Delta s\right) \\ &= B_1 f(t), \quad t \in I.\end{aligned}$$

This completes the proof. □

In addition, we suppose

(A2) there exist a positive function $g \in C_{rd}(I)$ and a constant A such that

$$\int_{0}^{t} g(s)f(s)\Delta s \leq A$$

for any $t \in I$.

In the last section, we will give an example for a function g and a constant A that satisfy $(A2)$. For $u \in X$, define the operator

$$S_2(u)(t) = \int_{t_0}^{t} g(s)f(s)\Delta s, \quad t \in I.$$

Lemma 3.2. *Suppose* $(A1)$ *and* $(A2)$*. If* $u \in X$ *and* $\|u\| \leq B$*, then*

$$\|S_2 u\| \leq AB_1.$$

Proof. We have

$$\begin{aligned}|S_2(u)(t)| &= \left|\int_{t_0}^{t} g(s)S_1(u)(s)\Delta s\right| \\ &\leq \int_{t_0}^{t} g(s)|S_1(u)(s)|\Delta s \\ &\leq B_1 \int_{t_0}^{t} g(s)f(s)\Delta s \\ &\leq AB_1, \quad t \in I,\end{aligned}$$

whereupon we get the desired result. This completes the proof. □

Lemma 3.3. *Suppose* $(A1)$ *and* $(A2)$. *If* $u \in X$ *satisfies the equation*

$$S_2(u)(t) = C, \quad t \in I, \tag{3.1}$$

for some constant C, *then* u *is a solution to the equation* (1.1).

Proof. We differentiate with respect to t the equation (3.1) and we find

$$g(t)S_1(u)(t) = 0, \quad t \in I,$$

whereupon

$$S_1(u)(t) = 0, \quad I.$$

Hence, we conclude that u is a solution to the equation (1.1). This completes the proof. □

Below, suppose

(A3) $\varepsilon \in (0, 1)$, A, B and B_1 satisfy the inequalities $\varepsilon B_1(1 + A) < 1$ and $AB_1 < B$.

In the last section, we will give an example for the constants ε, A, B and B_1. Our main result in this section is as follows.

Theorem 3.4. *Suppose* $(A1)$-$(A3)$. *Then the equation* (1.1) *has at least one solution in* X.

Proof. Let $\widetilde{\widetilde{Y}}$ denote the set of all equi-continuous families in X with respect to the norm $\|\cdot\|$. Let also,

$$\widetilde{Y} = \{u \in \widetilde{\widetilde{Y}} : u(t_0) \geq 0\},$$

$Y = \overline{\overline{Y}}$ be the closure of $\widetilde{Y}$,

$$U = \{u \in Y : \|u\| < B\}.$$

For $u \in \overline{U}$, define the operators

$$Tu(t) = \varepsilon u(t),$$

$$Su(t) = u(t) - \varepsilon u(t) + 3\varepsilon B - \varepsilon S_2(u)(t), \quad t \in I.$$

For $u \in \overline{U}$, we have

$$\begin{aligned}
\|(I - S)u\| &= \|\varepsilon u + \varepsilon S_2(u) - 3\varepsilon B\| \\
&\leq \varepsilon\|u\| + \varepsilon\|S_2(u)\| + 3\varepsilon B \\
&\leq 4\varepsilon B + \varepsilon AB_1.
\end{aligned}$$

Thus, $S : \overline{U} \to X$ is continuous and $(I - S)(\overline{U})$ resides in a compact subset of Y. Now, suppose that there is a $u \in \overline{U}$ so that $\|u\| = B$ and

$$u = \lambda(I - S)u$$

or

$$u = \lambda\varepsilon\,(u - 3B + S_2(u)), \tag{3.2}$$

for some $\lambda \in \left(0, \frac{1}{\varepsilon}\right)$. Then, using that $S_2(u)(t_0) \leq B$ and $u(t_0) \geq 0$, we get

$$u(t_0) = \lambda\varepsilon(u(t_0) - 3B + S_2(u)(t_0)) < \lambda\varepsilon u(t_0),$$

whereupon $\lambda\varepsilon > 1$, which is a contradiction. Consequently

$$\{u \in \overline{U} : u = \lambda_1(I - S)u, \ \|u\| = B\} = \emptyset$$

for any $\lambda_1 \in \left(0, \frac{1}{\varepsilon}\right)$. Then, from Theorem 2.5, it follows that the operator $T + S$ has a fixed point $u^* \in Y$. Therefore

$$\begin{aligned} u^*(t) &= Tu^*(t) + Su^*(t) \\ &= \varepsilon u^*(t) + u^*(t) - \varepsilon u^*(t) - \varepsilon S_2(u^*)(t) + 3\varepsilon B, \quad t \in I, \end{aligned}$$

whereupon

$$S_2(u^*)(t) = 3B, \quad t \in I.$$

From here, u^* is a solution to the problem (1.1). From here and from Lemma 3.3, it follows that u is a solution to the equation (1.1). This completes the proof. □

4. Existence of at Least Two Solutions

Let X be the space used in the previous section. Suppose

(A4) Let $m > 0$ be large enough and A, B, r, L, R_1 be positive constants that satisfy the following conditions

$$r < L < R_1, \quad AB_1 < \frac{L}{5}.$$

Our main result in this section is as follows.

Theorem 4.1. *Suppose that* (*A*1), (*A*2) *and* (*A*4) *hold. Then the equation* (1.1) *has at least two nonnegative solutions in X.*

Proof. Let

$$\widetilde{P} = \{u \in X : u \geq 0 \quad \text{on} \quad I\}.$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\widetilde{P}$. For $v \in X$, define the operators

$$\begin{aligned} T_1 v(t) &= (1 + m\varepsilon)v(t) - \varepsilon \frac{L}{10}, \\ S_3 v(t) &= -\varepsilon S_2(v)(t) - m\varepsilon v(t) - \varepsilon \frac{L}{10}, \end{aligned}$$

$t \in I$. Note that any fixed point $v \in X$ of the operator $T_1 + S_3$ is a solution to the equation (1.1). Define

$$\begin{aligned} \Omega &= \mathcal{P}, \\ U_1 &= \mathcal{P}_r = \{v \in \mathcal{P} : \|v\| < r\}, \\ U_2 &= \mathcal{P}_L = \{v \in \mathcal{P} : \|v\| < L\}, \\ U_3 &= \mathcal{P}_{R_1} = \{v \in \mathcal{P} : \|v\| < R_1\}. \end{aligned}$$

1. For $v_1, v_2 \in \Omega$, we have

$$\|T_1 v_1 - T_1 v_2\| = (1 + m\varepsilon)\|v_1 - v_2\|,$$

whereupon $T_1 : \Omega \to X$ is an expansive operator with a constant $h = 1 + m\varepsilon > 1$.

2. For $v \in \overline{\mathcal{P}}_{R_1}$, we get

$$\begin{aligned} \|S_3 v\| &\le \varepsilon\|S_2(v)\| + m\varepsilon\|v\| + \varepsilon\frac{L}{10} \\ &\le \varepsilon\Big(AB_1 + mR_1 + \frac{L}{10}\Big). \end{aligned}$$

   Therefore $S_3(\overline{\mathcal{P}}_{R_1})$ is uniformly bounded. Since $S_3 : \overline{\mathcal{P}}_{R_1} \to X$ is continuous, we have that $S_3(\overline{\mathcal{P}}_{R_1})$ is equi-continuous. Consequently $S_3 : \overline{\mathcal{P}}_{R_1} \to X$ is a 0-set contraction.
3. Let $v_1 \in \overline{\mathcal{P}}_{R_1}$. Set

$$v_2 = v_1 + \frac{1}{m}S_2(v_1) + \frac{L}{5m}.$$

   Note that $S_2(v_1) + \frac{L}{5} \ge 0$ on I. We have $v_2 \ge 0$ on I. Therefore $v_2 \in \Omega$ and

$$-\varepsilon m v_2 = -\varepsilon m v_1 - \varepsilon S_2(v_1) - \varepsilon\frac{L}{10} - \varepsilon\frac{L}{10}$$

   or

$$\begin{aligned} (I - T_1)v_2 &= -\varepsilon m v_2 + \varepsilon\frac{L}{10} \\ &= S_3 v_1. \end{aligned}$$

   Consequently $S_3(\overline{\mathcal{P}}_{R_1}) \subset (I - T_1)(\Omega)$.
4. Assume that for any $v_0 \in \mathcal{P}^*$ there exist $\lambda \ge 0$ and $v \in \partial\mathcal{P}_r \cap (\Omega + \lambda v_0)$ or $v \in \partial\mathcal{P}_{R_1} \cap (\Omega + \lambda v_0)$ such that

$$S_3 v = (I - T_1)(v - \lambda v_0).$$

   Then

$$-\varepsilon S_2(v) - m\varepsilon v - \varepsilon\frac{L}{10} = -m\varepsilon(v - \lambda v_0) + \varepsilon\frac{L}{10}$$

   or

$$-S_2(v) = \lambda m v_0 + \frac{L}{5}.$$

   Hence,

$$\|S_2(v)\| = \left\|\lambda m v_0 + \frac{L}{5}\right\| > \frac{L}{5}.$$

   This is a contradiction.
5. Let $\varepsilon_1 = \frac{2}{5m}$. Suppose that there exist a $v_1 \in \partial\mathcal{P}_L$ and $\lambda_1 \ge 1 + \varepsilon_1$ such that

$$S_3 v_1 = (I - T_1)(\lambda_1 v_1). \tag{4.1}$$

   Moreover,

$$-\varepsilon S_2(v_1) - m\varepsilon v_1 - \varepsilon\frac{L}{10} = -\lambda_1 m\varepsilon v_1 + \varepsilon\frac{L}{10},$$

   or

$$S_2(v_1) + \frac{L}{5} = (\lambda_1 - 1)m v_1.$$

From here,

$$2\frac{L}{5} > \left\| S_2(v_1) + \frac{L}{5} \right\| = (\lambda_1 - 1)m\|v_1\| = (\lambda_1 - 1)mL$$

and

$$\frac{2}{5m} + 1 > \lambda_1,$$

which is a contradiction.

Therefore all conditions of Theorem 2.8 hold. Hence, the problem (1.1) has at least two solutions u_1 and u_2 so that

$$\|u_1\| = L < \|u_2\| < R_1$$

or

$$r < \|u_1\| < L < \|u_2\| < R_1.$$

□

5. Existence of at Least Three Solutions

(A5) $\epsilon mr > \frac{2L}{5}$, where $\epsilon \in \left(0, \frac{1}{2}\right)$ and the constants L, m and r are those which appear in $(A4)$.

Our main results for existence of at least three solutions of the problem (1.1) is as follows.

Theorem 5.1. *Under the hypotheses* $(A1)$, $(A2)$, $(A4)$ *and* $(A5)$, *the problem* (1.1) *has at least three nonnegative solutions* $u_1, u_2, u_3 \in X$.

Proof. 1. Assume that there are $\lambda_1 \geq 1 + \epsilon$, $u \in \partial U_1$ and $\lambda_1 u \in \Omega$ so that

$$S_3(u) = (I - T_1)(\lambda_1 u).$$

Then

$$-\epsilon D_2(u) - m\epsilon u - \epsilon \frac{L}{10} = -m\epsilon\lambda_1 u + \frac{L}{10}$$

or

$$S_2(u) = (\lambda_1 - 1)mu - \frac{L}{5}.$$

Hence,

$$\begin{aligned} \|S_2(u)\| &= \left\| m(\lambda_1 - 1)u - \frac{L}{5} \right\| \\ &\geq (\lambda_1 - 1)m\|u\| - \left\| \frac{L}{5} \right\| \\ &\geq \epsilon m\|u\| - \frac{L}{5} \\ &= \epsilon mr - \frac{L}{5} \\ &> \frac{L}{5}, \end{aligned}$$

which is a contradiction. Thus, the condition (**i**) of Theorem 2.9 holds.

2. Now, assume that there are $\lambda_1 \geq 1+\epsilon$, $u \in \partial U_3$ and $\lambda_1 u \in \Omega$ so that

$$S_3(u) = (I - T_1)(\lambda_1 u).$$

As above,

$$\begin{aligned} \|S_2(u)\| &\geq (\lambda_1 - 1)m\|u\| - \left\|\frac{L}{5}\right\| \\ &\geq \epsilon m\|u\| - \frac{L}{5} \\ &= \epsilon m R_1 - \frac{L}{5} \\ &> \epsilon m r - \frac{L}{5} \\ &> \frac{L}{5}, \end{aligned}$$

which is a contradiction. Hence, the condition (**iii**) of Theorem 2.9 holds.

3. Assume that for any $u_0 \in \mathcal{P}^*$ there exist $\lambda_1 \geq 0$ and $u \in \partial\mathcal{P}_L \cap (\Omega + \lambda_1 u_0)$ such that

$$S_3(u) = (I - T_1)(u - \lambda_1 u_0).$$

Then

$$-\epsilon S_2(u) - m\epsilon u - \epsilon\frac{L}{10} = -m\epsilon(u - \lambda_1 u_0) + \epsilon\frac{L}{10}$$

or

$$-S_2(u) = \lambda_1 m u_0 + \frac{L}{5}.$$

Hence,

$$\|S_2 u\| = \left\|\lambda_1 m u_0 + \frac{L}{5}\right\| > \frac{L}{5}.$$

This is a contradiction. Form here, the condition (**ii**) of Theorem 2.9 holds.

Now, by Theorem 2.9, it follows that the problem (1.1) has at least three classical solutions u_1, u_2 and u_3 such that

$$u_1 \in \partial U_1 \cap \Omega \text{ and } u_2 \in (U_2 \setminus \overline{U}_1) \cap \Omega \text{ and } u_3 \in (\overline{U}_3) \setminus \overline{U}_2) \cap \Omega,$$

or

$$u_1 \in U_1 \cap \Omega \text{ and } u_2 \in (U_2 \setminus \overline{U}_1) \cap \Omega \text{ and } u_3 \in (\overline{U}_3) \setminus \overline{U}_2) \cap \Omega.$$

□

6. An Example

Below, we will illustrate our main results. Let

$$g(t) = \frac{A(1+t_0)}{(1+t)(1+\sigma(t))f(t)}, \quad t \in I,$$

where the positive constant A will be determined below. Hence,

$$\begin{aligned}\int_{t_0}^{t} g(s)f(s)\Delta s &= \int_{t_0}^{t} \frac{A(1+t_0)f(s)}{(1+s)(1+\sigma(s))f(s)}\Delta s\\ &= A(1+t_0)\int_{t_0}^{t}\frac{1}{(1+s)(1+\sigma(s))}\Delta s\\ &= A(1+t_0)\left(-\frac{1}{1+s}\Big|_{s=t_0}^{s=t}\right)\\ &= A(1+t_0)\left(\frac{1}{1+t_0}-\frac{1}{1+t}\right)\\ &\le A(1+t_0)\frac{1}{1+t_0}\\ &= A, \quad t\in I.\end{aligned}$$

We have that g satisfies $(A2)$. Let

$$R_1 = 10, \quad L = 5, \quad r = 4, \quad m = 10^{50}, \quad B = 1, \quad A = \frac{1}{10B_1}, \quad \varepsilon = \frac{1}{4},$$

$p = 2$. Then $B_1 = 1$, $A = \frac{1}{10}$ and

$$AB_1 = \frac{1}{10} < B, \quad \varepsilon B_1(1+A) = \frac{1}{4}\cdot 1\cdot\left(1+\frac{1}{10}\right) < 1,$$

i.e., $(A3)$ holds. Next,

$$r < L < R_1, \quad \varepsilon > 0, \quad R_1 > \left(\frac{2}{5m}+1\right)L, \quad AB_1 < \frac{L}{5}.$$

i.e., $(A4)$ holds. Moreover,

$$\epsilon m r = \frac{1}{4}\cdot 10^{50}\cdot 4 > 2 = \frac{2L}{5},$$

i.e., $(A5)$ holds. Let

$$\begin{aligned} a(t) &= 1,\\ C(t,s) &= 1,\\ H(t,y) &= y^2, \quad t,s\in I, \quad y\in\mathbb{R}.\end{aligned}$$

Therefore for the equation

$$u(t) = 1 + \int_{t_0}^{t}(u(s)+(u(s))^2)\Delta s, \quad t\in I,$$

are fulfilled all conditions of Theorem 3.4, Theorem 4.1 and Theorem 5.1.

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