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Domination and bondage number for double vertex graphs of some graphs

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Abstract. If a network modeled by a graph, then there are various graph theoretical parameters used to express the vulnerability and stability of communication networks. One of them is the concept of bondage number based on domination. The dominating set of a graph is a vertex set in that every vertex which is not in the dominating set is adjacent to at least one vertex of the dominating set. The domination number is the minimal cardinality among all dominating sets. The bondage number of any graph is the minimal cardinality among all sets of edges whose removal from the graph results in a graph with domination number greater than the domination number of the preliminary graph. In this paper, we investigate the domination and bondage numbers for double vertex graphs of some certain graphs.

1. Introduction

Graph theory has become one of the most powerful mathematical tools in the analysis and study of the architecture of networks whose vertices represent the components of the system and the edges represent connection between a pair of vertices that enable mutual communication. The vulnerability of a communication network measures the resistance of network to the disruption of operation after the failure of certain stations or communication links. For any communication network greater degrees of stability or less vulnerability is required. Vulnerability can be measured by certain parameters like domination, bondage number, connectivity, betweenness, binding number, toughness, scattering number, integrity etc. Colouring, choromatic index and bloking sets are also popular study topics in graph theory [7–12]. Graph theory is among the popular methods for solving many complex problems. Graph theory increases its development and usage area due to the easy modelling of daily problems and successful results of effective solution methods. The dominant nodes indicate the dominance of the people or objects modelled on the graph over each other. However, minimum domination set aims to connect all vertices in the graph with the least number of vertices selected on the graph. Determining the minimum dominating set in graphs is one of the most diffucult problems defined as NP-hard. When the usage areas of dominating sets in graphs examined, it is seen to provide significant gains in many areas such as social networks, transportation systmes, telecommunication, defence industry, health systems, etc. Domination varies when there are changes in edge or vertices in the domination set can be considered transmitters that cover a wide variety of communication links. The loss of certain links as a result of an attack on the graph may disrupt the

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connection. The question of re-establishing communication in the network after breaking the minimum number of connections has been arisen.
In a graph $G = (V(G), E(G))$, a subset $S \subseteq V(G)$ of vertices is a *dominating set* if every vertex in $V(G) - S$ is adjacent to at least one vertex of S. The *domination number* $\gamma(G)$ is the minimal cardinality of a dominating set. One of the vulnerability parameters based on domination number known as bondage number in a graph G examines the situation in which the domination number increases if some connections are broken. The opposite parameter, that is, examining the decrease in the domination number, is the reinforcement number. The *bondage number* $b(G)$ of a nonempty graph G is the cardinality of a smallest set of edges whose removal from G results in a graph with domination number greater than $\gamma(G)$. That is,

$$b(G) = min\{|S| : S \subseteq E(G), \gamma(G - S) > \gamma(G)\}.$$

We call such an edge set S that $\gamma(G - S) > \gamma(G)$ the *bondage set* and the minimum one the *minimum bondage set*. If $b(G)$ does not exist, for example empty graphs, then $b(G) = \infty$ is defined.
The bondage number was introduced by Bauer, Harary, Nieminen and Suffel [4], and has been further studied by Fink, Jacobson, Kinch and Roberts[6], Hartnell and Rall[3] and others. Later, the bondage number for middle graphs and complementary prism graphs was studied in [1, 2].
In this paper, the graph G is taken as a simple, finite and undirected graph with vertex set $V(G)$ and edge set $E(G)$. The distance $d(u, v)$ between two vertices u and v in G is the length of a shortest path joining them if any; otherwise $d(u, v) = \infty$. A shortest $u - v$ path is often called a *geodesic*. The *diameter* of G, denoted by $diam(G)$ is the largest distance between two vertices in $V(G)$. The number of the neighbor vertices of the vertex v is called degree of v and denoted by $deg_G(v)$. The minimum and maximum degrees of a vertex of G are denoted by $\delta(G)$ and $\Delta(G)$. A vertex v is said to be pendant vertex if $deg_G(v) = 1$. A vertex u is called support if u is adjacent to a pendant vertex [5]. Let u be a vertex of a graph $G = (V, E)$. Then $N(u) = \{v \in V(G), v \text{ and } u \text{ are adjacent}\}$ is the open neighborhood of u, and $N[u] = \{u\} \cup N(u)$ denotes the closed neighborhood of u. The *eccentricity* $e(v)$ of a vertex v in a connected graph G is max $d(u, v)$ for all u in G. The *radius* $r(G)$ is the minimum eccentricity of the vertices. Note that the maximum eccentricity is the diameter. A vertex v is a central vertex if $e(v) = r(G)$, and the *center* of G is the set of all central vertices [5]. So the central vertex c has degree $n - 1$ in a graph with n vertices. Furthermore, e_{uv} or (u, v) denotes the edges between the vertices u and v.
Aim of the paper is to establish the domination and bondage numbers for double vertex graphs of some certain graphs as the path P_n, the complete graph K_n, the star graph $S_{1,n}$, the complete bipartite graph $K_{2,m}$ and the wheel graph $W_{1,n}$. We calculated exact results for domination and bondage number of double vertex graphs and by comparing considered graphs with their double vertex graphs, we revealed the differences regarding the domination and bondage number parameters.

Definition 1.1. *[14] Let G be a graph of order $n \geq 2$. The double vertex graph $U_2(G)$ of G is the graph whose vertex set consists of all 2-element subsets of V such that two distinct vertices x, y and u, v are adjacent if and only if $|\{x, y\} \cap \{u, v\}| = 1$ and if $x = u$, then y and v are adjacent in G.*

2. Basic Results

In this section, some well-known basic results are given with regard to domination and bondage number.

Theorem 2.1. *[13] The domination number of*
a) *a path graph P_n is $\lceil \frac{n}{3} \rceil$, for $n \geq 2$,*
b) *a star graph $S_{1,n}$ is 1,*
c) *a complete graph K_n is 1,*
d) *a wheel graph $W_{1,n}$ is 1,*
e) *a complete bipartite graph $K_{n,m}$ is 2,*

Theorem 2.2. *[6] For a complete graph K_n of order $n \geq 2$, then $b(K_n) = \lceil \frac{n}{2} \rceil$.*

Theorem 2.3. *[6] For a path graph P_n of order $n \geq 2$, then*

$$b(P_n) = \begin{cases} 2, & \text{if } n \equiv 1(mod3) \\ 1, & \text{otherwise} \end{cases}$$

Theorem 2.4. *[6] For a cycle graph C_n of order $n \geq 3$, then*

$$b(C_n) = \begin{cases} 3, & \text{if } n \equiv 1(mod3) \\ 2, & \text{otherwise} \end{cases}$$

Theorem 2.5. *[6] For a star graph $S_{1,n}$ of order $n+1$, where $n \geq 2$. Then, $b(S_{1,n}) = 1$.*

Theorem 2.6. *[6] If $G = K(n_1, n_2, n_3, ..., n_t)$ is a complete $t-partite$ graph, where $n_1 \leq n_2 \leq ...n_t$, then*

$$b(G) = \begin{cases} \lceil \frac{m}{2} \rceil, & \text{if } n_m = 1 \text{ and } n_{m+1} \geq 2, \text{for some } m, 1 \leq m < t, \\ 2t-1, & \text{if } n_1 = n_2 = ... = n_t = 2, \\ \sum_{i=1}^{t-1} n_i, & \text{otherwise} \end{cases}$$

Theorem 2.7. *[6] If G is a nonempty graph with a unique minimum dominating set, then $b(G) = 1$.*

Theorem 2.8. *[6] If T is a nontrivial tree, then $b(T) \leq 2$.*

Theorem 2.9. *[6] If G is a connected graph of order $n \geq 2$, then $b(G) \leq n-1$.*

Theorem 2.10. *[6] If G is a nonempty graph, then*

$$b(G) \leq min\{deg(u) + deg(v) - 1 : u \text{ and } v \text{ are adjacent vertices.}\}$$

Theorem 2.11. *[6] If $\Delta(G)$ and $\delta(G)$ denote respectively the maximum and minimum degree among all vertices of nonempty connected graph G, then $b(G) \leq \Delta(G) + \delta(G) - 1$.*

Theorem 2.12. *[6] If G is a nonempty graph with domination number $\gamma(G) \geq 2$, then $b(G) \leq (\gamma(G) - 1)\Delta(G) + 1$.*

Theorem 2.13. *[6] If G is a connected graph of order $n \geq 2$, then $b(G) \leq n - \gamma(G) + 1$.*

Theorem 2.14. *[6] If G is a nonempty graph , then $b(G) \leq \Delta(G) + 1$.*

3. Domination and Bondage Number of Some Double Vertex Graphs

In this section, we calculated the domination and bondage number for double vertex graphs of some certain graphs as path graph P_n, complete graph K_n, star graph $S_{1,n}$, complete bipartite graph $K_{2,m}$ and wheel graph $W_{1,n}$.
The double vertex graph $U_2(P_9)$ is illustrated in Figure 1 and also the figure below shows the selection of the $\gamma - set$ S for the graph $U_2(P_9)$.

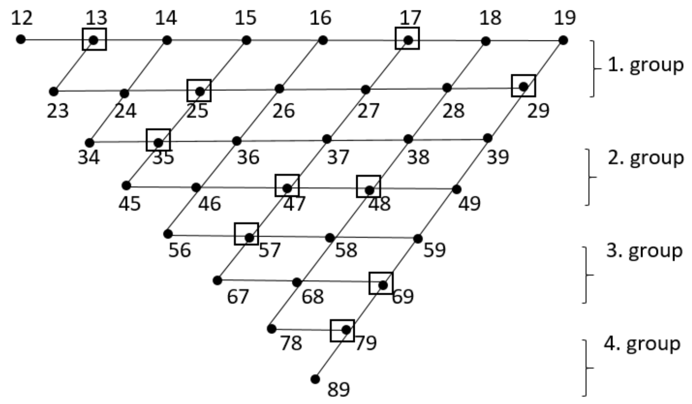

Figure 1: The graph $U_2(P_9)$

Theorem 3.1. *Let P_n be a path graph of order n and $U_2(P_n)$ be a double vertex graph of P_n. Then,*

$$\gamma(U_2(P_n)) = \begin{cases} \frac{n^2-1}{8}, & \text{if } n \text{ is odd} \\ \lceil \frac{n^2}{8} \rceil, & \text{if } n \text{ is even} \end{cases}$$

Proof. The graph $U_2(P_n)$ has $\frac{n^2-n}{2}$ vertices. Let S be any minimal dominating set of $U_2(P_n)$. We term each vertex set of the graph, starting with $n-1$ vertices and decreasing by one, as a level. We assume that two consecutive levels form a group. If the vertex set $V(U_2(P_n))$ is partitioned into groups of two, then the rule of the vertex to be selected into the set S is similar for each group. The first level of the $U_2(P_n)$ starts with $n-1$ vertices and the number of the vertices decreases by one at each level until it reaches one. So, there are $n-1$ vertices at the first level, $n-2$ vertices at the second level, $n-3$ vertices at the third level,...and lastly 1 vertex at the $(n-1)$th. level. Hence, there are $\frac{n^2-n}{2}$ vertices in the double vertex graph of the path graph of order n. We choose the vertices to be selected from $n-1$ levels to S from the vertices that dominate the most vertices, that is, the vertices with the maximum degree. The graph $U_2(P_n)$ contains the graph $U_2(P_{n-1})$, the graph $U_2(P_{n-1})$ also contains the graph $U_2(P_{n-2})$. Hence, the double vertex graph $U_2(P_n)$ is a recursive structure and the induction method can be used for the proof. For example, in the figure above, the vertices that are dominate all the graph with minimum cardinality are shown in squares. If we continue to add vertices to S in this manner, we have, $\gamma(U_2(P_5))$=$\gamma(U_2(P_3)) + 2$, $\gamma(U_2(P_7))$=$\gamma(U_2(P_5)) + 3$, $\gamma(U_2(P_9))$=$\gamma(U_2(P_7))$+4, $\gamma(U_2(P_{11}))$=$\gamma(U_2(P_9))$+5, $\gamma(U_2(P_{13}))$=$\gamma(U_2(P_{11}))$+6, $\gamma(U_2(P_{15}))$=$\gamma(U_2(P_{13}))$+7,... for n is odd and $\gamma(U_2(P_6))$=$\gamma(U_2(P_4))$+3, $\gamma(U_2(P_8))$=$\gamma(U_2(P_6))$+3, $\gamma(U_2(P_{10}))$=$\gamma(U_2(P_8))$+5, $\gamma(U_2(P_{12}))$=$\gamma(U_2(P_{10}))$+ 5, $\gamma(U_2(P_{14}))$=$\gamma(U_2(P_{12}))$+7, $\gamma(U_2(P_{16}))$=$\gamma(U_2(P_{14}))$+7, $\gamma(U_2(P_{18}))$=$\gamma(U_2(P_{16}))$+9, $\gamma(U_2(P_{20}))$=$\gamma(U_2(P_{18}))$+9,... for n is even. We prove the theorem in two ways, depending on whether n is odd or even.

Case 1. $n \equiv 1(mod\ 2)$

In this case, there are $\frac{n-1}{2}$ groups in total and $\frac{n-1}{2}$ vertices from the first group; $\frac{n-3}{2}$ vertices from the second group; $\frac{n-5}{2}$ vertices from the third group; $\frac{n-7}{2}$ vertices from the fourth group; $\frac{n-9}{2}$ vertices from the fifth group;...$\frac{n-(n-2)}{2}$ = 1 vertex from the last group in S. Hence, the domination number of the graph $U_2(P_n)$, for

$A = \frac{n-1}{2}$

$$\begin{aligned}\gamma(U_2(P_n)) &= A + A - 1 + A - 2 + A - 3 + ... + A - (\frac{n-1}{2} - 1)\\ &= \frac{n-1}{2}A - (1 + 2 + 3 + ... + \frac{n-3}{2})\\ &= (\frac{n-1}{2})^2 - \sum_{i=1}^{\frac{n-3}{2}} i\\ &= (\frac{n-1}{2})^2 - \frac{\frac{n-3}{2}\frac{n-1}{2}}{2}\\ &= (\frac{n-1}{2})^2 - \frac{n^2-4n+3}{8}\\ &= \frac{n^2-1}{8}\\ &= \sum_{i=1}^{\frac{n-1}{2}} i\end{aligned}$$

So, we have $\gamma(U_2(P_3)) = 1$, $\gamma(U_2(P_5)) = 3$, $\gamma(U_2(P_7)) = 6$, $\gamma(U_2(P_9)) = 10$, $\gamma(U_2(P_{11})) = 15$, $\gamma(U_2(P_{13})) = 21$, $\gamma(U_2(P_{15})) = 28$, $\gamma(U_2(P_{17})) = 36$... We can prove our result using the induction method. Certainly $\gamma(U_2(P_3)) = 1$. Assume $\gamma(U_2(P_k)) = \frac{k^2-1}{8}$ for $1 < k < n-2$, then we need to show that $\gamma(U_2(P_n)) = \frac{n^2-1}{8}$.

$$\begin{aligned}\gamma(U_2(P_{k+2})) &= \gamma(U_2(P_k)) + \frac{k+1}{2}\\ &= \frac{k^2-1}{8} + \frac{4k+4}{8}\\ &= \frac{k^2+4k+3}{8}\\ &= \frac{(k+2)^2-1}{8}\end{aligned} \tag{1}$$

Case 2. $n \equiv 0(mod\ 2)$
In this case, there are $\frac{n}{2} - 1$ groups in total and one pendant vertex apart from these groups. The rule for the vertices to be added to S is similar to Case 1. So, $\gamma(U_2(P_4)) = 2$, $\gamma(U_2(P_6)) = 5$, $\gamma(U_2(P_8)) = 8$, $\gamma(U_2(P_{10})) = 13$, $\gamma(U_2(P_{12})) = 18$, $\gamma(U_2(P_{14})) = 25$, $\gamma(U_2(P_{16})) = 32$, $\gamma(U_2(P_{18})) = 41$, $\gamma(U_2(P_{20})) = 50$, $\gamma(U_2(P_{22})) = 61$,...
So, for $k = \overline{0, \frac{n-6}{4}}$ we have

$$\begin{aligned}\gamma(U_2(P_{4k+6})) &= \gamma(U_2(P_{4k+4})) + \frac{4k+6}{2}\\ &= \gamma(U_2(P_{4k+4})) + 2k + 3\end{aligned}$$

and for $k = \overline{1, \frac{n-4}{4}}$

$$\begin{aligned}\gamma(U_2(P_{4k+4})) &= \gamma(U_2(P_{4k+2})) + \frac{4k+2}{2}\\ &= \gamma(U_2(P_{4k+2})) + 2k + 1\end{aligned}$$

To show this we use induction method. Certainly, $\gamma(U_2(P_4)) = 2$.
Case 2.1 For $k = \overline{0, \frac{n-6}{4}}$

We need to show that $\gamma(U_2(P_n)) = \lceil \frac{n^2}{8} \rceil$. Assume that $\gamma(U_2(P_{4k+4})) = \lceil \frac{(4k+4)^2}{8} \rceil$.

$$\begin{aligned}\gamma(U_2(P_{4k+6})) &= \lceil \frac{(4k+4)^2}{8} \rceil + 2k + 3 \\ &= \frac{16k^2 + 48k + 40}{8} \\ &= \frac{(4k+6)^2 + 4}{8} \\ &= \lceil \frac{(4k+6)^2}{8} \rceil \end{aligned} \tag{2}$$

Case 2.2 For $k = \overline{1, \frac{n-4}{4}}$
Assume that $\gamma(U_2(P_{4k+2})) = \lceil \frac{(4k+2)^2}{8} \rceil$, then we need to show that $\gamma(U_2(P_n)) = \lceil \frac{n^2}{8} \rceil$.

$$\begin{aligned}\gamma(U_2(P_{4k+4})) &= \lceil \frac{(4k+2)^2}{8} \rceil + 2k + 1 \\ &= \frac{16k^2 + 32k + 12}{8} \\ &= \frac{(4k+4)^2 - 4}{8}\end{aligned}$$

Since $\lceil \frac{(4k+4)^2-4}{8} \rceil = \lceil \frac{(4k+4)^2}{8} \rceil$ and from Case 1 and Case 2 we have

$$\gamma(U_2(P_n)) = \begin{cases} \frac{n^2-1}{8}, & \text{if n is odd} \\ \lceil \frac{n^2}{8} \rceil, & \text{if n is even} \end{cases}$$

□

The proof is completed.

Theorem 3.2. *The bondage number of the double vertex graph of P_n, $n \geq 3$, is $b(U_2(P_n)) = 1$.*

Proof. When we make a pendant vertex degree of 1 in the graph $U_2(P_n)$ disconnected removing the edge incident to this vertex from the graph, that remains is a graph consisting of $U_2(P_{n-1})$, the graph P_{n-2} and an unconnected vertex. The domination number of the connected graph is $\gamma(U_2(P_{n-1})) + \lceil \frac{n-2}{4} \rceil = \gamma(U_2(P_n))$. By adding the unconnected vertex to the dominating set S, the domination number increases by 1. Hence, we have $b(U_2(P_n)) = 1$. □

The complete graph K_5 and double vertex graph $U_2(K_5)$ are illustrated in Figure 2.

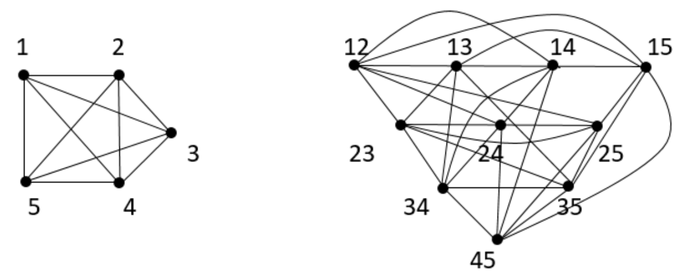

Figure 2: The graphs K_5 and $U_2(K_5)$

Theorem 3.3. *Let $U_2(K_n)$ be a double vertex graph of the complete graph K_n of order $\binom{n-1}{2} = \frac{(n-1)n}{2}$, then the domination number of $U_2(K_n)$ is $\gamma(U_2(K_n)) = \lfloor \frac{n}{2} \rfloor$.*

Proof. The degree of each vertex of $U_2(K_n)$ is $2n-4$. So, the graph $U_2(K_n)$ is $(2n-4)-regular$. We can label the vertices as $V(K_n) = \{1, 2, ..., n\}$ and $V(U_2(K_n)) = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} ij$. So, $S = \{ij, xy, km, ...| \quad k \neq m \neq i \neq j \neq x \neq y \quad and \quad A = \{i, j, k, m, x, y, ...\} = V(K_n) = \{1, 2, 3, ...n\}$ if n is even and $S = \{ij, xy, km, ...| \quad k \neq m \neq i \neq j \neq x \neq y \quad and \quad A = \{i, j, k, m, x, y, ...\} \quad and \quad |V(K_n)| - |A| = 1\}$ if n is odd, where S is the minimal dominating set of $U_2(K_n)$. So, for example the sets $\{12, 34, 56\}, \{16, 23, 45\}, \{16, 24, 35\}, \{14, 23, 56\}, ...$ are the minimal dominating sets for $U_2(K_6)$ and $\{17, 23, 45\}, \{12, 35, 67\}, ...$ are the minimal dominating sets for $U_2(K_7)$. Since, the cardinality of these sets is $\lfloor \frac{n}{2} \rfloor$, we have $\gamma(U_2(K_n)) = \lfloor \frac{n}{2} \rfloor$. □

Theorem 3.4. *The bondage number of the double vertex graph of the complete graph K_n, $n > 4$, is $b(U_2(K_n)) = 2n-4$.*

Proof. In order to increase the cardinality of the $\gamma - set$ of the graph $U_2(K_n)$, any vertex must dominate itself. This yields deleting the edges to isolate any vertex. Therefore, the domination number of the remaining graph results with $\lfloor \frac{n}{2} \rfloor + 1$, which is 1 more than $\gamma(U_2(K_n))$. Since $deg(v) = 2n-4$ for $v \in U_2(K_n)$, $b(U_2(K_n)) = 2n-4$. □

The star graph $S_{1,5}$ and double vertex graph $U_2(S_{1,5})$ are illustrated in Figure 3.

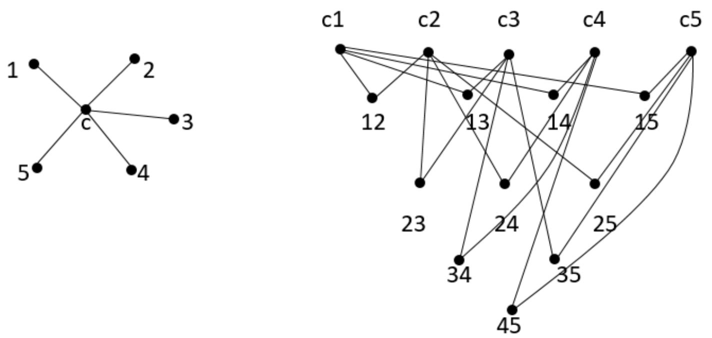

Figure 3: The graphs $S_{1,5}$ and $U_2(S_{1,5})$

Theorem 3.5. *Let $S_{1,n}$ be a star graph of order $n+1$ and $U_2(S_{1,n})$ be a double vertex graph of $S_{1,n}$ of order $\frac{n^2+3n}{2}$. Then, $\gamma(U_2(S_{1,n})) = n-1$.*

Proof. We denote the central vertex degree of n with c and the pendant vertices degree of 1 with $1, 2, ..., n$ of the graph $S_{1,n}$; n major vertices degree of $n-1$ with ci, $i = \overline{1, n}$ and $\sum_{i=1}^{n-1} i = \frac{n^2-n}{2}$ minor vertices degree of 2 with $\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} ij$ of the graph $U_2(S_{1,n})$. Any vertex ci dominates the all $n-1$ minor vertices ij. The minor vertex ij also dominates the vertices ci and cj. Hence, in the minimal dominating set of $U_2(S_{1,n})$ there are $n-1$ vertices in total, $n-2$ of which are from major vertices and 1 from minor vertices like $S = \{c_i, xy \mid i \neq x \neq y \ and \ |\{c_i\}| = n-2\}$, where S is a minimal dominating set. For example, the sets $\{c1, c2, c3, c4, 56\}$, $\{c1, c4, c5, c6, 23\}$, $\{c2, c3, c4, c5, 16\}$ are the minimal dominating sets for $U_2(S_{1,6})$. So, we have $\gamma(U_2(S_{1,n})) = n-1$. This completes the proof. □

Theorem 3.6. *The bondage number of the double vertex graph of the star graph $S_{1,n}$ is $b(U_2(S_{1,n})) = 2$.*

Proof. If one of the edges connecting the vertex ij, $i = \overline{1, n-1}$ and $j = \overline{2, n}$, to the vertices ci or cj, for example the edge (ij, cj) is removed from the graph $U_2(S_{1,n})$, then the vertex set $S = \{ci, ck, xy \mid k \in \{1, 2, ..., n\}, |\{ck\}| = n-3 \ and \ i \neq k \neq x \neq y\}$ dominates the remaining structure. If we denote the set of deleted edges by S'

and $S' = (ij, cj)$, then $|S| = n-1$ and $\gamma(U_2(S_{1,n}) - S') = \gamma(U_2(S_{1,n})$. The removal of an edge set S' of $U_2(S_{1,n})$ of cardinality less than two results in a graph with the same domination number as that of $U_2(S_{1,n})$. If we remove the both edges (ij, ci) and (ij, cj), then the remaining graph, except the vertex ij, is dominated by $S^* = \{ck, xy \mid \{k\} \cup x \cup y = \{1, 2, ..., n\},\ k \neq x \neq y \ and \ |\{ck\}| = n-2\}$. Since the edges (ij, ci) and (ij, cj) incident to the vertex ij are deleted, the vertex ij must be included in the minimal dominating set S. $\gamma(U_2(S_{1,n}) - S'') = n$, $S'' = \{(ij, ci), (ij, cj)\}$, which is greater by 1 than that of $\gamma(U_2(S_{1,n})) = n-1$. Therefore, the bondage number of $U_2(S_{1,n})$ is $b(U_2(S_{1,n})) = 2$. □

The complete bipartite graph $K_{2,4}$ and double vertex graph $U_2(K_{2,4})$ are illustrated in Figure 4.

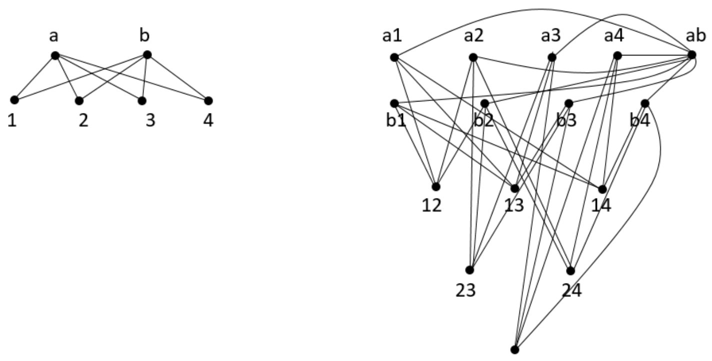

Figure 4: The graphs $K_{2,4}$ and $U_2(K_{2,4})$

Theorem 3.7. *Let $K_{n,m}$ be a complete bipartite graph of order $n + m$ and $U_2(K_{n,m})$ be a double vertex graph of $K_{n,m}$ of order $n^2 + nm$. Then, the domination number of $U_2(K_{2,m})$ is $\gamma(U_2(K_{2,m})) = m$.*

Proof. We can denote the vertices of $K_{n,m}$ and $U_2(K_{n,m})$ with $A = \{a, b, c, ..., n\}$, $|A| = n$; $B = 1, 2, 3, ..., m$, $|B| = m$ and $a1, a2, ..., am$; $b1, b2, ..., bm$; $c1, c2, ..., cm$;...,$n1, n2, ..., nm$; $ab, ac, ..., an$; $bc, ..., bn$; $12, 13, ..., 1m$; $23, 24, ..., 2m$; $34, 35, ..., 3m$; ...$(m-1)m$, respectively. We can split into three subsets of the vertex set $V(U_2(K_{n,m}))$ as $V(U_2(K_{n,m})) = V_1(U_2(K_{n,m})) \cup V_2(U_2(K_{n,m})) \cup V_3(U_2(K_{n,m}))$:

$V_1(U_2(K_{n,m})) = \{a1, a2, ..., am; b1, b2, ..., bm; c1, c2, ..., cm; ..., n1, n2, ..., nm \mid the\ set\ that\ consists\ of\ the\ vertices\ degree\ of\ n + m + 2\}$

$V_2(U_2(K_{n,m})) = \{ab, ac, ..., an; bc, ..., bn \mid the\ set\ that\ consists\ of\ the\ vertices\ degree\ of\ 2m\}$

$V_3(U_2(K_{n,m})) = \{12, 13, ..., 1m; 23, 24, ..., 2m; 34, 35, ..., 3m; ...(m-1)m \mid the\ set\ that\ consists\ of\ the\ vertices\ degree\ of\ 2n\}$

We can easily see that specially $deg(v) = m\ if\ v \in V_1(U_2(K_{2,m}))$, $deg(v) = 2m\ if\ v \in V_2(U_2(K_{2,m}))$ and $deg(v) = 4\ if\ v \in V_3(U_2(K_{2,m}))$ for $n = 2$.

By definition, the $\gamma - set$ includes the vertex ab with maximum degree. Hence, all vertices in $V_1(U_2(K_{2,m}))$ are dominated by ab. The vertices $12, 13, 14, ..., 1m$; $23, 24, ..., 2m$; $34, 35, ..., 3m$;...; $(m-2)(m-1), (m-2)m$ are dominated by the vertices $a1$ or $b1$; $a2$ or $b2$; $a3$ or $b3$;...; $a(m-2)$ or $b(m-2)$, respectively. Lastly, the vertex $(m-1)m$ is dominated by $a(m-1)$ or am or $b(m-1)$ or bm. The dominating set S thus obtained is the minimal dominatig set. $S_1 \subset V_1(U_2(K_{2,m}))$, $|S_1| = m-1$, $S_2 = \{ab\} = V_2(U_2(K_{2,m}))$ and $S = S_1 \cup S_2$. Hence, we have $|S| = \gamma(U_2(K_{2,m})) = m$. □

Theorem 3.8. *The bondage number of the double vertex graph of $K_{2,m}$, $m \geq 3$, is $b(U_2(K_{2,m})) = 3$.*

Proof. If we denote the set of the deleted edges by S', then $\gamma(U_2(K_{2,m}) - S') > \gamma(U_2(K_{2,m})$. Assume that this set is $S' = \{(ab, ai), (ab, b(i+1))\}$, $1 \leq i < m$. In this case the vertices ai and $b(i+1)$ can be dominated by the vertex $i(i+1)$. So, sets of the form $\{a1, b2, ..., a(m-2), i(i+1), ab\}$, $\{b1, a2, b3, ..., b(m-2), i(i+1), ab\}$ are minimal dominating sets and $|S_1| = |S_2| = m$. So, if less than three edges of $U_2(K_{2,m})$ are deleted, then $\gamma(U_2(K_{2,m}))$ is unchanged. In order to increase the cardinality of the $\gamma - set$, we need to delete the edge $(ai, i(i+1))$ from the graph. So, the vertex ai can not be dominated by $i(i+1)$. In this case another vertex ij, $j \neq i+1$ must be added to $\gamma - set$. Furthermore, the vertex $b(i+1)$ can also be dominated by any vertex $j(i+1)$, $1 \leq j \leq m$. Therefore, sets of the form $S_3 = \{ij, j(i+1), ab, a1, b2, ..., a(m-2)\}$, $S_4 = \{ij, i(i+1), ab, b1, b2, ..., b(m-2)\}$... are the minimal dominating sets and their cardinality is $m+1$. Hence, removing the set $S' = \{(ab, ai), (ab, b(i+1)), (ai, i(i+1))\}$ from the graph $U_2(K_{2,m}))$ increases the domination number by 1 and $b(U_2(K_{2,m})) = 3$ is obtained. □

The wheel graph $W_{1,5}$ and double vertex graph $U_2(W_{1,5})$ are illustrated in Figure 5.

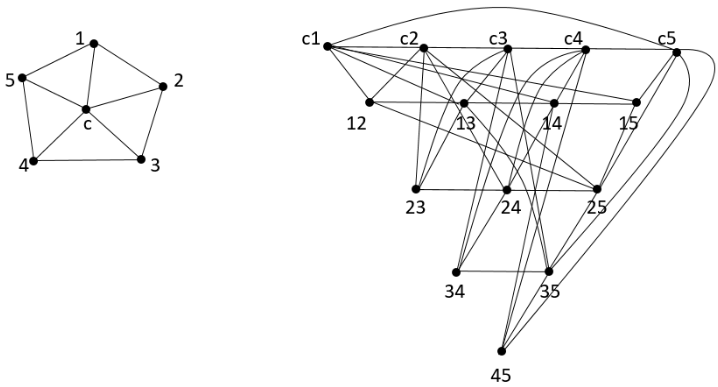

Figure 5: The graphs $W_{1,5}$ and $U_2(W_{1,5})$

Theorem 3.9. *Let $W_{1,n}$ be a wheel graph of order $n+1$ and $U_2(W_{1,n})$ be a double vertex graph of $W_{1,n}$. Then,*

$$\gamma(U_2(W_{1,n}) = \begin{cases} 2, & \text{if } n{=}4 \\ n-2, & \text{if } n > 4 \end{cases}$$

Proof. There are n vertices ci, $i = \overline{1,n}$, degree of $n+1$; $n-1$ vertices $i(i+1)$ and the vertex $1n$ degree of 4 and $\frac{n^2-3n}{2}$ remaining vertices degree of 6 in $U_2(W_{1,n})$ for $n \geq 4$. We can denote the vertex set $V(U_2(W_{1,n})) = V_1(U_2(W_{1,n})) \cup V_2(U_2(W_{1,n})) \cup V_3(U_2(W_{1,n}))$, where $v \in V_1(U_2(W_{1,n}))$ if $deg(v) = n+1$, $v \in V_2(U_2(W_{1,n}))$ if $deg(v) = 6$ and $v \in V_3(U_2(W_{1,n}))$ if $deg(v) = 4$.

Let S be a minimal dominating set of $U_2(W_{1,n})$. $|V_1(U_2(W_{1,4}))| = 4$, $|V_2(U_2(W_{1,4}))| = 2$, $|V_3(U_2(W_{1,4}))| = 4$. There isn't any vertex from $V_1(U_2(W_{1,4}))$ in S and $V_2(U_2(W_{1,4})) = \{13, 24\} = S$. So, $\gamma(U_2(W_{1,4})) = 2$.

There are $n-4$ or $n-5$ vertices from $V_1(U_2(W_{1,n}))$ and two or three vertices from $V_2(U_2(W_{1,n}))$ in S, respectively for $n > 4$. So, we have $\gamma(U_2(W_{1,n})) = n-2$, for $n > 4$. For example $\{c1, c2, c4, 36, 57\}$, $\{c1, c2, c3, 46, 57\}$, $\{c1, c5, c7, 24, 36\}$, $\{c1, c2, 35, 47, 57\}$, $\{c2, c6, 14, 35, 47\}$, $\{c4, c5, c6, 13, 27\}$,... are minimal dominating sets for $U_2(W_{1,7})$. The graph $U_2(W_{1,n})$ contains the graph $U_2(W_{1,n-1})$ except for the edge $(c1, c(n-1))$. The equalities $\gamma(U_2(W_{1,4})) = 2$, $\gamma(U_2(W_{1,5})) = 3$, $\gamma(U_2(W_{1,6})) = 4$,...hold. Since, $U_2(W_{1,n})$ is a recursive structure, we can express the domination number as $\gamma(U_2(W_{1,n})) = \gamma(U_2(W_{1,n-1})) + 1$. We assume that $\gamma(U_2(W_{1,n})) = n-2$ and using our assumption

$$\begin{aligned} \gamma(U_2(W_{1,n})) &= \gamma(U_2(W_{1,n-1})) + 1 \\ &= n-3+1 \\ &= n-2 \end{aligned}$$

This completes the proof. □

Theorem 3.10. *The bondage number of the double vertex graph of $W_{1,n}$ is*

$$b(U_2(W_{1,n})) = \begin{cases} 1, & \text{if } n=4 \\ 4, & \text{if } n > 4 \end{cases}$$

Proof. **Case 1.** $n = 4$
In this case, the minimal dominating set $S = \{13, 24\}$ is unique. So, $b(U_2(W_{1,4})) = 1$ from the Theorem 2.7.
Case 2. $n > 4$
In this case, any vertex $v \in V_3(U_2(W_{1,n}))$ degree of 4 can be dominated by at most two vertices $ci \in V_1(U_2(W_{1,n}))$ or by at most two vertices $ij \in V_2(U_2(W_{1,n}))$, $i \in \{1, 2, ..., n-2\}$ and $j \in \{3, 4, ..., n\}$ in S. Even if the two edges connecting the vertex $v \in V_3(U_2(W_{1,n}))$ to two distinct vertices in S are removed from the graph $U_2(W_{1,n})$, the vertex v can be dominated by another minimal dominating set of cardinality $n-2$ without these two vertices since each pair of minimal dominating sets has at most $\lceil \frac{n-2}{2} \rceil$ common vertices. Since the vertex v degree of 4 can be dominated by at least one of adjacent four vertices, the domination number only increases by 1 removing the edges that are incident to the vertex v from the graph. Hence, we have $b(U_2(W_{1,n})) = 4$.
□

4. Conclusion

In this paper, we have concentrated on domination and bondage number, a measure of network vulnerability. We have computed the domination and bondage numbers of double vertex graphs of some certain graphs as the path P_n, the complete graph K_n, the star graph $S_{1,n}$, the complete bipartite graph $K_{2,m}$ and the wheel graph $W_{1,n}$. Double vertex graphs are taken to model the network system and the domination and bondage number values of them reveal that how network can be made more stable than earlier. Although the number of the vertices and the domination number of double vertex graphs of considered graphs are increase in high number, bondage number values are not. This means that the double vertex graph of a graph can be more strong according to the graph itself. These results can help the network designers to choose a suitable topology for the network. Consequently if we design a communication network, then we can prefer the double vertex graphs.

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References

[1] A.Aytaç and T.Turacı, On the bondage, strong and weak bondage numbers in complementary prism graphs, *Computer Science Journal of Moldova*, **29:1(85)**, 59–75 (2021).
[2] A. Aytaç, T.Turacı and Z.N. Odabaş, *On the bondage number of middle graphs*, Mathematical Notes. **93:6** (2013), 795–801.
[3] D.F.Hartnell and D.F.Rall, *Bounds on the bondage number of a graph*, Discrete Math. **128:1-3** (1994), 173–177.
[4] D. Bauer, F.Harary, J. Nieminen and C.L. Suffel, *Domination alteration sets in graph*, Discrete Math., **47:2-3** (1983), 153–161.
[5] F. Harary, *Graph Theory*, Addition-Wesley Publishing Co., Reading, MA/Menlo Park, CA/London, 1969.
[6] J.F.Fink, M.S.Jacobson, L.F. Kinch and J.Roberts, *The bondage number of a graph*, Discrete Math. **86:1-3** (1990), 47–57.
[7] P. Bonacini and L. Marino,*Equitable block colourings*, Ars. Comb. **120** (2015), 255–258.
[8] P. Bonacini, M.G. Cinquegrani and L. Marino,*Homogeneous edge-colourings of graphs*, Utilitas Mathematica. **100** (2016).
[9] P. Bonacini and L. Marino,*Equitable block colourings for 8-cycle systems*, Australasian Journal of Combinatorics. **69(2)** (2017), 184–196.
[10] P. Bonacini and L. Marino,*On voloshin colourings in 3-hypergraph designs*, Discrete Mathematics. **343(10)** (2020), 112030.
[11] P. Bonacini and L. Marino,*Blocking sets for cycles and path designs*, Applicable Analysis and Discrete Mathematics. **14** (2020), 183–197.
[12] P. Bonacini, M. Gionfriddo and L. Marino,*Block colourings of 6-cycle systems*, Opuscula Mathematica. **37(5)** (2017), 647–664.
[13] V. Singh Nisha *On domination numbers of some classes of graceful graphs*, International Journal of Creative Research Thoughts. **11(6)** (2023), 871–878.
[14] Y. Alavi, Don R. Lick and J. Liu *Survey of double vertex graphs*, Graph and Combinatorics. **18** (2002), 709–715.