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On rough deferred statistical convergence for sequences in neutrosophic normed space

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Abstract. In this study, we provide the notion of rough deferred statistical convergence for sequences within the framework of the neutrosophic normed space ($\mathfrak{NNS}$). We also study the notion of rough deferred statistical cluster points for sequences in $\mathfrak{NNS}$ and discuss how this cluster points set and the set of rough deferred statistical limit points related to the aforesaid convergence relate to each other.

1. Introduction

The notion of convergence of sequences has undergone several extensions as a result of the advent of distinct summability techniques. Statistical convergence, independently introduced by Steinhaus [28] and Fast [8], is a notion that extends the ordinary convergence of sequences of real and complex numbers. Another innovative approach to convergence, termed deferred statistical convergence of sequences, was investigated in [17]. This approach incorporates deferred density into the statistical convergence. Expanding on this, Et et al. [7] introduced the μ-deferred statistical convergence for real-valued functions, therefore significantly extending the notion. See [10, 15, 21, 24–26] for the fundamental characteristics and details of these novel ideas.

The concepts of roughness degree and rough convergence for sequences in a finite-dimensional normed linear space were first presented by Phu [22] who later extended this idea to an infinite-dimensional normed linear space [23]. Beyond investigating rough convergence, Phu delved into analytical properties like convexity and the proximity of the set of rough limits. Rough statistical convergence, which includes natural density, is a concept that Aytar [5] expanded upon. They also investigated the connection between the set of rough statistical limit points and the set of statistical cluster points. Expanding on the idea of rough convergence, several authors explored rough convergence and statistical rough convergence for sequences in different contexts. This exploration extended to the study of rough convergence and rough statistical convergence for double sequences in [18, 19].

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Zadeh [29] introduced the Theory of Fuzzy Sets ($\mathfrak{FS}$), which has significantly influenced various scientific fields. However, $\mathfrak{FS}$ encounters challenges in handling uncertain membership degrees. To address this issue, Atanassov [4] extended the theory to Intuitionistic Fuzzy Sets ($\mathfrak{IFS}$). Kramosil and Michalek [16] explored Fuzzy Metric Spaces ($\mathfrak{FMS}$) using concepts from fuzzy and probabilistic metric spaces. Kaleva and Seikkala [11] examined $\mathfrak{FMS}$ by considering the distance between two points as a non-negative fuzzy number. There are requirements for $\mathfrak{FMS}$ that George and Veeramani [9] listed. $\mathfrak{FMS}$ has garnered attention for its practical applications in fixed-point theory, medical imaging, and decision-making.

Smarandache [27] conducted an investigation into 'Neutrosophic set' ($\mathfrak{NS}$) as a generalization of $\mathfrak{FS}$ and $\mathfrak{IFS}$, aiming to address uncertainty in practical problem-solving. The $\mathfrak{NS}$ incorporates membership functions for falsehood ($\mathfrak{F}$), indeterminacy ($\mathfrak{I}$), and truth ($\mathfrak{T}$). Neutrosophy, reflecting impartial knowledge of thought, distinguishes $\mathfrak{NS}$ from fuzzy, neutral, logic, and intuitive fuzzy sets.

In $\mathfrak{NS}$, uncertainty is distinct from the values of $\mathfrak{T}$ and $\mathfrak{F}$, making $\mathfrak{NS}$ more encompassing than $\mathfrak{IFS}$ as there are no constraints among the degrees of $\mathfrak{T}$, $\mathfrak{F}$, and $\mathfrak{I}$. The term neutrosophy signifies impartial knowledge, and the notion of neutrality highlights the fundamental distinction from fuzzy, neutral, logic and intuitive fuzzy sets.

Menger [20] introduced triangular norms (t-norms) ($\mathfrak{TN}$) as a generalization of probability distributions, incorporating the triangle inequality in terms of metric spaces. Triangular conorms (t-conorms) ($\mathfrak{TC}$), identified as dual operations to $\mathfrak{TN}$, play a pivotal role in fuzzy operations, including intersections and unions. $\mathfrak{TN}$ and $\mathfrak{TC}$ serve as vital components for managing fuzzy operations within the framework of metric spaces.

The concept of a neutrosophic metric space, characterized by continuous t-norms and continuous t-conorms, was initially introduced by Kirişci and Şimşek [13]. Expanding on their work, Kirişci and Şimşek [14] further investigated neutrosophic normed spaces ($\mathfrak{NNS}$) and explored statistical convergence within the $\mathfrak{NNS}$ framework.

Antal et al. [2] introduced the notion of rough statistical convergence for sequences. Rahaman and Mursaleen [3] presented rough deferred statistical convergence for difference sequences in $\mathcal{L}$-fuzzy normed space. In another study [12], the authors proposed a modification to the definition of neutrosophic normed space, originally presented in [13]. Debnath et al. [6] presented the concept of deferred statistical convergence in the $\mathfrak{NNS}$. This study introduces the concept of rough deferred statistical convergence of sequences within this adapted space.

In certain instances, determining the exact values of terms in a convergent sequence (s_u) becomes challenging, particularly for large values of u. To address this challenge, an alternative sequence (v_u) is employed for approximation, introducing approximation errors. The concept of rough convergence emerged as a solution in such scenarios.

Our research aims to extend the concept of convergence to sequences within $\mathfrak{NNS}$ and explore various algebraic and topological properties. This unique convergence allows the limit to manifest as a set rather than a single point, prompting a thorough investigation into the topological (closedness) and geometric properties of the limit set. Additionally, we provided examples, for a given roughness degree $r > 0$, demonstrating that the set of all rough deferred statistical convergent sequences does not form a linear space. A rough deferred statistical cluster point in $\mathfrak{NNS}$ was also introduced, and a relationship between the cluster point set and the limit set under rough deferred statistical convergence was developed.

This study is driven by the need to enhance the analytical frameworks used to address sequences with uncertain behavior, which are commonly encountered in various real-life applications, including data science, engineering, and decision-making processes. By incorporating the principles of neutrosophy, the research broadens existing convergence theories to account for indeterminacy alongside truth and falsehood, offering a more comprehensive approach to sequence analysis. The introduction of rough deferred statistical cluster points and limit points in $\mathfrak{NNS}$ further enriches the theoretical foundation, providing new insights that have the potential to be applied across a wide range of disciplines where uncertainty is a critical factor. This work not only advances the theoretical understanding of sequence behavior in complex systems but also lays the groundwork for practical applications in areas such as artificial intelligence, economics, and environmental science, where precise decision-making under uncertain conditions is essential.

2. Auxiliary definitions and notations

A few necessary definitions are provided in this section.

Assuming $\mathcal{F}$ is a linear space over the field $\mathcal{V}$ and $\diamond$ and $*$ are $\mathfrak{TN}$ and $\mathfrak{TC}$, respectively. Let Θ, Ω and Ψ be single valued fuzzy sets on $\mathcal{F} \times (0,\infty)$. We designate the 6-tuple $(\mathcal{F}, \Theta, \Omega, \Psi, \diamond, *)$ as a $\mathfrak{NNS}$ if, for all $\omega, \gamma \in \mathcal{F}$ and $\tau, \kappa > 0$, the following conditions are satisfied:

(A1) $\Theta(\omega,\tau) + \Omega(\omega,\tau) + \Psi(\omega,\tau) \leq 3$,

(A2) $\Theta(\omega,\tau) = 1$, $\Omega(\omega,\tau) = 0$ and $\Psi(\omega,\tau) = 0$ iff $\omega = 0$,

(A3) $\Theta(\beta\omega,\tau) = \Theta\left(\omega, \frac{\tau}{|\beta|}\right)$, $\Omega(\beta\omega,\tau) = \Omega\left(\omega, \frac{\tau}{|\beta|}\right)$ and $\Psi(\beta\omega,\tau) = \Psi\left(\omega, \frac{\tau}{|\beta|}\right)$ for any $0 \neq \beta \in \mathcal{F}$,

(A4) $\Theta(\omega+\gamma, \tau+\kappa) \geq \Theta(\omega,\tau) \diamond \Theta(\gamma,\kappa)$, $\Omega(\omega+\gamma,\tau+\kappa) \leq \Omega(\omega,\tau) * \Omega(\gamma,\kappa)$ and $\Psi(\omega+\gamma,\tau+\kappa) \leq \Psi(\omega,\tau) * \Psi(\gamma,\kappa)$,

(A5) $\Theta(\omega,.)$, $\Omega(\omega,.)$ and $\Psi(\omega,.)$ are continuous on $(0,\infty)$,

(A6) $\lim_{\tau\to\infty}\Theta(\omega,\tau) = 1$, $\lim_{\tau\to\infty}\Omega(\omega,\tau) = 0$ and $\lim_{\mu\to\infty}\Psi(\omega,\tau) = 0$,

(A7) $\lim_{\tau\to 0}\Theta(\omega,\tau) = 0$, $\lim_{\mu\to 0}\Omega(\omega,\tau) = 1$ and $\lim_{\mu\to 0}\Psi(\omega,\tau) = 1$.

In this scenario, we denote the 3-tuple (Θ, Ω, Ψ) as a neutrosophic norm (shortly, $\mathfrak{NN}$) on $\mathcal{F}$.

Example 2.1. *Let $(\mathcal{F}, \|\|.)$ be a normed space. Consider $\gamma_1 \diamond \gamma_2 = \gamma_1 \cdot \gamma_2$ and $\gamma_1 * \gamma_2 = \min\{\gamma_1+\gamma_2, 1\}$, $\forall \gamma_1, \gamma_2 \in [0,1]$. Additionally, define Θ, Ω, and Ψ as follows:*

$$\Theta(\omega,\tau) = \frac{\tau}{\tau+\|\omega\|},\ \psi(u,\mu) = \frac{\|\omega\|}{\tau+\|\omega\|} \text{ and } \Psi(\omega,\mu) = \frac{2\|\omega\|}{\tau+2\|\omega\|}$$

for all $\omega \in \mathcal{F}$ and $\tau > 0$. Then $(\mathcal{F}, \Theta, \Omega, \Psi, \diamond, *)$ is a $\mathfrak{NNS}$.

Consider a $\mathfrak{NNS}$ $(\mathcal{F}, \Theta, \Omega, \Psi, \diamond, *)$ and let $\omega \in \mathcal{F}$. For a given $r > 0$ and $\tau \in (0,1)$, the set

$$\mathcal{B}_{\omega}^{(\Theta,\Omega,\Psi)}(r,\tau) = \{v \in \mathcal{F} : \Theta(\omega - v, r) > 1-\tau,\ \Omega(\omega - v, r) < \tau \text{ and } \Psi(\omega - v, r) < \tau\}$$

defines an open ball with centered at ω and radius r w.r.t $\tau \in (0,1)$. Define

$$\mathfrak{I}_{(\Theta,\Omega,\Psi)}(\mathcal{F}) = \left\{\mathcal{A} \subset \mathcal{F} :\ \text{for all } \omega \in \mathcal{A},\ \exists r > 0 \text{ and } \tau \in (0,1) : \mathcal{B}_{\omega}^{(\Theta,\Omega,\Psi)}(r,\tau) \subset \mathcal{A}\right\}.$$

Then $\mathfrak{I}_{(\Theta,\Omega,\Psi)}(\mathcal{F})$ defines a topology on $\mathcal{F}$, which is induced by NN (Θ,Ω,Ψ). Since

$$\left\{v \in \mathcal{F} : \Theta\left(\omega - v, \frac{1}{s}\right) > 1 - \frac{1}{s},\ \Omega\left(\omega - v, \frac{1}{s}\right) < \frac{1}{s} \text{ and } \Psi\left(\omega - v, \frac{1}{s}\right) < \frac{1}{s}\right\}$$

is a local base at $\omega \in \mathcal{F}$, the topology $\mathfrak{I}_{(\Theta,\Omega,\Psi)}(\mathcal{F})$ on $\mathcal{F}$ is first countable.

Definition 2.2. *Let $(\mathcal{F}, \Theta, \Omega, \Psi, \diamond, *)$ be a $\mathfrak{NNS}$. A sequence (s_u) in $\mathcal{F}$ then, if*

$$\Theta(s_u - s_0, \tau) \to 1,\ \Omega(s_u - s_0, \tau) \to 0 \text{ and } \Psi(s_u - s_0, \tau) \to 0 \text{ as } u \to \infty,$$

converges to s_0 w.r.t $\mathfrak{NN}$ (Θ,Ω,Ψ) for each $\tau > 0$. We write the limit by $(\Theta,\Omega,\Psi) - \lim s_u = s_0$.

We refer to the collections of all natural numbers and real numbers by $\mathbb{N}$ and $\mathbb{R}$, respectively, throughout this research. Assume that $A \subseteq \mathbb{N}$. The natural or asymptotic density of the set W, represented by $\delta(W)$, may be expressed as follows:

$$\delta(W) = \lim_{u\to\infty} \frac{1}{u} |\{t \leq u : t \in W\}|,$$

given the existence of the limit. Here the cardinality of the set $\{\cdots\}$ is shown by $|\{\cdots\}|$. If, for any $\varepsilon > 0$, we have

$$\delta(\{u \in \mathbb{N} : |s_u - w| \geq \varepsilon\}) = 0,$$

then a sequence (s_u) of numbers is said to be statistically convergent to w (see [8], [28]).

Definition 2.3. *A sequence (s_u) in $\mathcal{F}$ is statistically convergent to $s_0 \in \mathcal{F}$ w.r.t $\mathfrak{NN}$ (Θ, Ω, Ψ), if for all $\gamma \in (0,1)$ and $\tau > 0$,*

$$\lim_{t\to\infty} \frac{1}{t} \left|\{u \leq t : \Theta(s_u - s_0, \tau) \leq 1-\gamma \text{ or } \Omega(s_u - s_0, \tau) \geq \gamma \text{ or } \Psi(s_u - s_0, \tau) \geq \gamma\}\right| = 0.$$

We represent the limit as $(\Theta, \Omega, \Psi)_{st} - \lim s_u = s_0$.

Example 2.4. *Let $(\mathcal{F}, \|.\|)$ be a normed space. Suppose that $\gamma_1 \diamond \gamma_2 = \gamma_1 \cdot \gamma_2$ and $\gamma_1 * \gamma_2 = \min\{\gamma_1 + \gamma_2, 1\}$, $\forall \gamma_1, \gamma_2 \in [0,1]$. We select Θ, Ω, and Ψ as given in Example 2.1. Then, $(\mathcal{F}, \Theta, \Omega, \Psi, \diamond, *)$ forms a $\mathfrak{NN}\mathfrak{S}$. Define the sequence (s_u) as follows:*

$$s_u = \begin{cases} 1, & \text{if } u = m^2, m \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Now, consider the set

$$K_t(\gamma, \tau) = \{u \leq t : \Theta(s_u, \tau) \leq 1-\gamma \text{ or } \Omega(s_u, \tau) \geq \gamma \text{ or } \Psi(s_u, \tau) \geq \gamma\}$$

for every $\gamma \in (0,1)$ and for any $\tau > 0$. Then, we can express this set as

$$\begin{aligned} K_t(\gamma, \tau) &= \left\{u \leq t : \tfrac{\tau}{\tau + \|s_u\|} \leq 1-\gamma \text{ or } \tfrac{\|s_u\|}{\tau + \|s_u\|} \geq \gamma, \tfrac{\|s_u\|}{\tau} \geq \gamma\right\} \\ &= \left\{u \leq t : \|s_u\| \geq \tfrac{\tau\gamma}{1-\gamma} \text{ or } \|s_u\| \geq \tau\gamma\right\} \\ &= \{u \leq t : s_u = 1\} = \left\{u \leq t : u = m^2 \text{ and } m \in \mathbb{N}\right\}. \end{aligned}$$

Therefore,

$$\frac{1}{t}\left|K_t(\gamma, \tau)\right| = \frac{1}{t}\left|\left\{u \leq t : u = m^2 \text{ and } m \in \mathbb{N}\right\}\right| \leq \frac{\sqrt{t}}{t}.$$

This implies that as t becomes sufficiently large, the quantity $\Theta(s_u, \tau)$ becomes less than $1-\gamma$ and similarly the quantities $\Omega(s_u, \tau)$, $\Psi(s_u, \tau)$ become larger than γ. Hence, we have $\lim_{t\to\infty} \frac{1}{t}\left|K_t(\gamma, \tau)\right| = 0$ for every $\gamma \in (0,1)$ and for any $\tau > 0$.

Definition 2.5. *We define a sequence (s_u) in $\mathcal{F}$ as rough convergent to $s_0 \in \mathcal{F}$ w.r.t $\mathfrak{NN}$ (Θ, Ω, Ψ) for some $r \geq 0$ if, for any $\gamma \in (0,1)$ and $\tau > 0$, there exist $u_0 \in \mathbb{N}$ such that*

$$\Theta(s_u - s_0, r + \tau) > 1-\gamma, \ \Omega(s_u - s_0, r + \tau) < \gamma \text{ and } \Psi(s_u - s_0, r + \tau) < \gamma,$$

for all $u \geq u_0$.
The convergence of the sequence (s_u) is characterized by the limit expressed as $(\Theta, \Omega, \Psi)^r - \lim s_u = s_0$.

Agnew [1] defined postponed Cesàro mean as follows in 1932, expanding on the idea of Cesàro mean of real (or complex) sequences:
Let $(a_w), (b_w)$ be sequences of non-negative integers satisfying the conditions

$$\begin{aligned} a_w &< b_w \\ \lim_{w\to\infty} b_w &= \infty. \end{aligned} \tag{1}$$

The postponed Cesàro mean of a real (or complex) valued sequence (s_u) is defined by

$$(D_{a,b}(s_u))_w := \frac{1}{b_w - a_w} \sum_{u=a_w+1}^{b_w} s_u, \ w = 1, 2, \cdots.$$

If the limit is present,

$$D_{a,b}(U) := \lim_{w\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : a_w < u \le b_w,\ u \in U\}|,$$

defines the deferred density of U for $U \subseteq \mathbb{N}$. If, for any $\varepsilon > 0$, we have

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \le u \le b_w,\ |s_u - w| \ge \varepsilon\}| = 0,$$

then a sequence (s_u) of numbers is said to be deferred statistically convergent to w (see [17]).

The aforementioned definition aligns with the statistical convergence of (s_u) as shown in [8] for $a_w = 0$ and $b_w = w$.

3. Main Results

Within the context of $\mathfrak{NNS}$ $(\mathcal{F}, \Theta, \Omega, \Psi, \diamond, *)$, we introduce the concept of rough deferred statistical convergence for sequences in this section. (a_w) and (b_w) represent the sequences of non-negative integers that fulfill (1) throughout this investigation. Any more limitations on (a_w) and (b_w) (if any) will be provided in the instances and theorems that correspond to them.

Definition 3.1. *We say that a sequence (s_u) in $\mathcal{F}$ is rough deferred statistically convergent to $s_0 \in \mathcal{F}$ w.r.t. $\mathfrak{NN}$ (Θ, Ω, Ψ) for some $r \ge 0$, if*

$$\begin{aligned}\lim_{w\to\infty} \tfrac{1}{b_w - a_w} \Big|\{u \in \mathbb{N} : 1 + a_w \le u \le b_w,\ \Theta(s_u - s_0, r + \tau) > 1 - \gamma,\\ \Omega(s_u - s_0, r + \tau) < \gamma \text{ and } \Psi(s_u - s_0, r + \tau) < \gamma\}\Big| = 1\end{aligned} \tag{2}$$

supplies for each $\gamma \in (0, 1)$ and $\tau > 0$. We demonstrate the limit as $DS^r_{a,b}(\Theta, \Omega, \Psi) - \lim s_u = s_0$.

In the subsequent remark, we elaborate on how the $DS^r_{a,b}(\Theta, \Omega, \Psi)$-convergence encompasses certain regular convergence methods within $\mathfrak{NNS}$.

Remark 3.2. *Let $(\mathcal{F}, \Theta, \Omega, \Psi, \diamond, *)$ be an $\mathfrak{NNS}$ and $(s_u) \in \mathcal{F}$. Then*
(i) We term the $DS^r_{a,b}(\Theta, \Omega, \Psi)$-convergence of (s_u) as the deferred statistical convergence w.r.t. $\mathfrak{NN}$, provided that (2) holds true for $r = 0$.
(ii) For $a_w = 0$ and $b_w = w$ in (2), we term the $DS^r_{a,b}(\Theta, \Omega, \Psi)$-convergence of (s_u) as the rough statistical convergence w.r.t. $\mathfrak{NN}$.
(iii) Assume $a_w = u_{w-1}$ and $b_w = u_w$ in (2), where $\theta = (u_w)$ is an increasing sequence of integers such that $u_0 = 0$ and $u_w - u_{w-1} \to \infty$ as $w \to \infty$. Then, we term the $DS^r_{a,b}(\Theta, \Omega, \Psi)$-convergence of (s_u) as the rough lacunary statistical convergence w.r.t. $\mathfrak{NN}$.
(iv) Assume $a_w = w - \lambda_w$ and $b_w = w$ in (2), where (λ_w) is an increasing sequence of positive integers tending to ∞ such that $\lambda_1 = 1$ and $\lambda_w + 1 \ge \lambda_{w+1}$ for all $w \in \mathbb{N}$. Then, we term the $DS^r_{a,b}(\Theta, \Omega, \Psi)$-convergence of (s_u) as the rough λ-statistical convergence w.r.t. $\mathfrak{NN}$.

Notation 3.3. *Let $(\mathcal{F}, \Theta, \Omega, \Psi, \diamond, *)$ be an $\mathfrak{NNS}$ and $(s_u) \in \mathcal{F}$. In this context, both $(\Theta, \Omega, \Psi)^r - \lim s_u$ and $DS^r_{a,b}(\Theta, \Omega, \Psi) - \lim s_u$ may not be unique. Therefore, we use*

$$(\Theta, \Omega, \Psi) - \mathrm{LIM}^r(s_u) = \{s_0 \in \mathcal{F} : (\Theta, \Omega, \Psi)^r - \lim s_u = s_0\},$$

and

$$DS_{a,b}(\Theta, \Omega, \Psi) - \mathrm{LIM}^r(s_u) = \left\{s_0 \in \mathcal{F} : DS^r_{a,b}(\Theta, \Omega, \Psi) - \lim s_u = s_0\right\}$$

to demonstrate the set of all $(\Theta,\Omega,\Psi)^r - \lim s_u$ *and the set of all* $DS^r_{a,b}(\Theta,\Omega,\Psi) - \lim s_u$ *of the sequence* (s_u)*, respectively. We define the sequence* (s_u) *as rough convergent w.r.t.* $\mathfrak{NN}$ *if* $(\Theta,\Omega,\Psi) - \mathrm{LIM}^r(s_u) \neq \emptyset$ *and as rough deferred statistically convergent w.r.t.* $\mathfrak{NN}$ *if* $DS_{a,b}(\Theta,\Omega,\Psi) - \mathrm{LIM}^r(s_u) \neq \emptyset$ *for some* $r \geq 0$*. Certainly, it is evident from the definitions that* $0 \leq r_1 \leq r_2$*, then*

$$(\Theta,\Omega,\Psi) - \mathrm{LIM}^{r_1}(s_u) \subset (\Theta,\Omega,\Psi) - \mathrm{LIM}^{r_2}(s_u),$$

and

$$DS_{a,b}(\Theta,\Omega,\Psi) - \mathrm{LIM}^{r_1}(s_u) \subset DS_{a,b}(\Theta,\Omega,\Psi) - \mathrm{LIM}^{r_2}(s_u)$$

for a sequence (s_u) *in* $\mathcal{F}$.

Example 3.4. *Take* $\mathfrak{NN}\mathfrak{S}(\mathbb{R},\Theta,\Omega,\Psi,\diamond,*)$*, where* $(\mathbb{R},\|.\|)$ *is the usual normed space. Consider* $\gamma_1\diamond\gamma_2 = \gamma_1\diamond\gamma_2$ *and* $\gamma_1 * \gamma_2 = \min\{\gamma_1+\gamma_2,1\}$, $\forall\gamma_1,\gamma_2 \in [0,1]$*. Furthermore,* $\mathfrak{N}_{(\Theta,\Omega,\Psi)}$ *represents the neutrosophic fuzzy set on* $\mathbb{R}\times(0,\infty)$ *characterized by*

$$\mathfrak{N}_{(\Theta,\Omega,\Psi)} = \left(\frac{\tau}{\tau+\|\omega\|}, \frac{\|\omega\|}{\tau+\|\omega\|}, \frac{2\|\omega\|}{\tau+2\|\omega\|}\right)$$

for all $w \in \mathcal{F}$ *and* $\tau > 0$*. The sequence* (s_u) *in* $\mathbb{R}$ *is defined as follows:*

$$s_u = \begin{cases} 1, & \text{if } u = 2t-1 \\ -1, & \text{if not} \end{cases}, \ t \in \mathbb{N}.$$

Let $s_0 \in (\Theta,\Omega,\Psi) - \mathrm{LIM}^r(s_u)$ *for some* $r > 0$*. Then* $\Theta(s_u - s_0, r+\tau) > 1-\gamma$, $\Omega(s_u - s_0, r+\tau) < \gamma$ *and* $\Psi(s_u - s_0, r+\tau) < \gamma$*. So, we obtain*

$$(\tau+r)\frac{\gamma}{(1-\gamma)} > |s_u - s_0|, \ \forall\gamma \in (0,1) \text{ and } \tau > 0.$$

Consider $\kappa = \frac{\tau\gamma}{1-\gamma}$ *as infinitesimal small and* $r' = \frac{r\gamma}{1-\gamma}$*. Then*

$$r' + \kappa > |s_u - s_0| \Rightarrow s_0 \in [s_u - r', s_u + r'].$$

For $u = 2t-1$*, we get* $s_0 \in [1-r', 1+r']$*. When* $u \neq 2t-1$*, then* $s_0 \in [-1-r', -1+r']$*. Now*

$$[1-r', 1+r'] \cap [-1-r', -1+r'] = \begin{cases} \emptyset, & \text{if } r' < 1 \\ [1-r', r'-1], & \text{if } r' \geq 1. \end{cases}$$

Hence

$$(\Theta,\Omega,\Psi) - \mathrm{LIM}^r(s_u) = \begin{cases} [1-r, r-1], & \text{if } r \geq 1 \\ \emptyset, & \text{if not.} \end{cases}$$

Establish the sequence (v_u) *in* $\mathbb{R}$

$$v_u = \begin{cases} u, & \text{if } u = 2^t \\ -1, & \text{if not} \end{cases}, \ t \in \mathbb{N}.$$

Take $a_w = w$ *and* $b_w = w^2+1$, $\forall w \in \mathbb{N}$*. Then*

$$\begin{aligned} &\tfrac{1}{b_w - a_w}\Big|\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, \ \Theta(s_u - s_0, r+\tau) > 1-\gamma, \\ &\qquad \Omega(s_u - s_0, r+\tau) < \gamma \text{ and } \Psi(s_u - s_0, r+\tau) < \gamma\}\Big| \\ &= \tfrac{1}{b_w - a_w}\left|\left\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, \ (\tau+r)\tfrac{\gamma}{(1-\gamma)} > |v_u - s_0|\right\}\right| \\ &= \tfrac{1}{b_w - a_w}\left|\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, \ r' + \kappa > |v_u - s_0|\}\right|. \end{aligned}$$

Since $r > 0$, *we have* $r' > 0$. *So, for each* $r' > 0$, *we obtain*

$$r' + \kappa > |v_u - s_0| \text{ implies } r' + \kappa > |1 + s_0| \tag{3}$$

whenever $1 + a_w \leq u \leq b_w$ *and* $u \neq 2^t$. *Since*

$$\frac{1}{b_w - a_w}\left|\left\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w,\ u \neq 2^t\right\}\right| \to 1 \text{ as } s \to \infty$$

from (3), we write

$$\lim_{s\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w,\ s_0 \in [-1 - r', r' - 1]\}| = 1, \forall r' > 0.$$

As a result

$$DS_{a,b}(\Theta, \Omega, \Psi) - \mathrm{LIM}^r (v_u) = \begin{cases} [-1 - r, r - 1], & \text{if } r > 0 \\ \emptyset, & \text{if not.} \end{cases}$$

In relation to $\mathfrak{NN}$ (Θ, Ω, Ψ), *none of the sequences* (s_u) *nor* (v_u) *converge in the ordinary sense. Furthermore, the limit* $(\Theta, \Omega, \Psi)^r - \lim v_u$ *does not valid for* $r > 0$.

Notation 3.5. *Unlike the ordinary convergence observed in an* $\mathfrak{NN}\mathfrak{S}$, *the rough convergence of a sequence* (s_u) *w.r.t.* $\mathfrak{NN}$ *does not generally imply the rough convergence of a subsequence of* (s_u) *w.r.t. the same. Considering* $(s_u) = (u)$ *in the* $\mathfrak{NN}\mathfrak{S}$ *defined in (3.4), it is evident that* $(\Theta, \Omega, \Psi) - \mathrm{LIM}^r (s_u) = [1 - r, 1 + r]$ *for* $r > 0$. *However, when examining the subsequence* $(s_{u^2}) = \left(u^2\right)$ *of* (s_u), *the* $(\Theta, \Omega, \Psi) - \mathrm{LIM}^r (s_{u^2})$ *does not exist for any* $r > 0$. *This reasoning similarly applies to the* $DS^r_{a,b}(\Theta, \Omega, \Psi)$*-convergence of a sequence* (s_u) *in* $\mathfrak{NN}\mathfrak{S}$.

Example 3.6. *Take* $\mathfrak{NN}\mathfrak{S}$ $(\mathbb{R}, \Theta, \Omega, \Psi, \diamond, *)$, *where* $(\mathbb{R}, \|.\|)$ *is the usual normed space. Consider* $\gamma_1 \diamond \gamma_2 = \gamma_1 \cdot \gamma_2$ *and* $\gamma_1 * \gamma_2 = \min\{\gamma_1 + \gamma_2, 1\}$, $\forall \gamma_1, \gamma_2 \in [0, 1]$ *and* Θ, Ω, Ψ *is defined in Example 3.4. Define the sequence* (s_u) *in* $\mathbb{R}$ *as follows:*

$$(s_u) = \begin{cases} u, & \text{if } u = t^2, \\ 0, & \text{if not} \end{cases}, t \in \mathbb{N}.$$

Then $\left(s_{u_j}\right) = (1, 4, 9, 16, ...)$. *Take* $a_w = 0$ *and* $b_w = w$, $\forall w \in \mathbb{N}$. *Then, we have*

$$DS_{a,b}(\Theta, \Omega, \Psi) - \mathrm{LIM}^r (s_u) = [-r, r], \forall r \geq 0.$$

and

$$DS_{a,b}(\Theta, \Omega, \Psi) - \mathrm{LIM}^r \left(s_{u_j}\right) = \emptyset.$$

We can now give our auxiliary theorem, which plays an important role in the proofs of the following results.

Lemma 3.7. *Assume that* $(\mathcal{F}, \Theta, \Omega, \Psi, \diamond, *)$ *is a* $\mathfrak{NN}\mathfrak{S}$ *and that* (s_u) *is a sequence in* $\mathcal{F}$. *For any* $\gamma \in (0, 1)$ *and* $\tau > 0$, *the following statements are interchangeable:*
(i) $DS^r_{a,b}(\Theta, \Omega, \Psi) - \lim s_u = s_0$.
(ii)

$$\lim_{w\to\infty} \tfrac{1}{b_w - a_w} \left|\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w,\ \Theta(s_u - s_0, r + \tau) \leq 1 - \gamma,\right.$$
$$\left.\Omega(s_u - s_0, r + \tau) \geq \gamma \text{ or } \Psi(s_u - s_0, r + \tau) \geq \gamma\}\right| = 0.$$

Proof. The results are self-evident, and therefore, the proof is omitted. □

Theorem 3.8. *Suppose that $(\mathcal{F}, \Theta, \Omega, \Psi, \diamond, *)$ be a $\mathfrak{N}\mathfrak{N}\mathfrak{S}$. Then*

$$(\Theta, \Omega, \Psi) - \text{LIM}^r(s_u) \subset DS_{a,b}(\Theta, \Omega, \Psi) - \text{LIM}^r(s_u).$$

supplies for each sequence (s_u) in $\mathcal{F}$ and $r > 0$.

Proof. Let $s_0 \in (\Theta, \Omega, \Psi) - \text{LIM}^r(s_u)$. For all $\gamma \in (0,1)$ and $\tau > 0$, $\exists u_0 \in \mathbb{N}$ such that

$$\Theta(s_u - s_0, r + \tau) > 1 - \gamma, \ \Omega(s_u - s_0, r + \tau) < \gamma \text{ and } \Psi(s_u - s_0, r + \tau) < \gamma, \forall u \geq u_0.$$

Thus

$$\begin{aligned}&\{u \in \mathbb{N} : \Theta(s_u - s_0, r + \tau) \leq 1 - \gamma, \Omega(s_u - s_0, r + \tau) \geq \gamma \text{ or } \Psi(s_u - s_0, r + \tau) \geq \gamma\}\\ &\subset \{1, 2, ..., u_0 - 1\}.\end{aligned}$$

Since

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, \ u \in \{1, 2, ..., u_0 - 1\}\}| = 0,$$

we obtain

$$\begin{aligned}\lim_{w\to\infty} \tfrac{1}{b_w - a_w} \Big|\{&u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, \ \Theta(s_u - s_0, r + \tau) \leq 1 - \gamma,\\ &\Omega(s_u - s_0, r + \tau) \geq \gamma \text{ or } \Psi(s_u - s_0, r + \tau) \geq \gamma\}\Big| = 0.\end{aligned}$$

Therefore, $s_0 \in DS_{a,b}(\Theta, \Omega, \Psi) - \text{LIM}^r(s_u)$. As a result, we obtain

$$(\Theta, \Omega, \Psi) - \text{LIM}^r(\omega_{uv}) \subset DS_{a,b}(\Theta, \Omega, \Psi) - \text{LIM}^r(s_u).$$

□

The inclusion connection indicated above is in fact rigorous, as Example 3.4 shows.

Theorem 3.9. *Let $(\mathcal{F}, \Theta, \Omega, \Psi, \diamond, *)$ be a $\mathfrak{N}\mathfrak{N}\mathfrak{S}$ and (s_u) be a sequence in $\mathcal{F}$. It follows that for all $r > 0$ and $\gamma \in (0,1)$, there is no pair of elements $s_1, s_2 \in DS_{a,b}(\Theta, \Omega, \Psi) - \text{LIM}^r(s_u)$ such that $\Theta(s_1 - s_2, qr) \leq 1 - \gamma$ or $\Omega(s_1 - s_2, qr) \geq \gamma$ or $\Psi(s_1 - s_2, qr) \geq \gamma$ for $q > 2$.*

Proof. Given $\gamma \in (0,1)$, there exists $\gamma_1 \in (0,1)$ such that $(1 - \gamma_1) \diamond (1 - \gamma_1) > 1 - \gamma$ and $\gamma_1 * \gamma_1 < \gamma$. We derive this outcome through a contradiction. Subsequently, there are elements $s_1, s_2 \in DS_{a,b}(\Theta, \Omega, \Psi) - \text{LIM}^r$ such that

$$\Theta(s_1 - s_2, qr) \leq 1 - \gamma \text{ or } \Omega(s_1 - s_2, qr) \geq \gamma \text{ or } \Psi(s_1 - s_2, qr) \geq \gamma, \tag{4}$$

for $q > 2$. For each $\tau > 0$ and establish the following sets

$$\begin{aligned}K \ &= \Big\{u \in \mathbb{N} : \Theta\Big(s_u - s_1, r + \tfrac{\tau}{2}\Big) \leq 1 - \gamma_1,\\ &\Omega\Big(s_u - s_1, r + \tfrac{\tau}{2}\Big) \geq \gamma_1 \text{ or } \Psi\Big(s_u - s_1, r + \tfrac{\tau}{2}\Big) \geq \gamma_1\Big\},\end{aligned}$$

and

$$\begin{aligned}L = \ &\Big\{u \in \mathbb{N} : \Theta\Big(s_u - s_2, r + \tfrac{\tau}{2}\Big) \leq 1 - \gamma_1,\\ &\Omega\Big(s_u - s_2, r + \tfrac{\tau}{2}\Big) \geq \gamma_1 \text{ or } \Psi\Big(s_u - s_2, r + \tfrac{\tau}{2}\Big) \geq \gamma_1\Big\},\end{aligned}$$

Hence, based on Lemma 3.7, we get

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, u \in K\}| = 0,$$

and

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, \, u \in L\}| = 0.$$

Now

$$\begin{aligned} &\lim_{w\to\infty} \tfrac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, \ u \in K \cup L\}| \\ &\leq \lim_{w\to\infty} \tfrac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, \ u \in K\}| \\ &+ \lim_{w\to\infty} \tfrac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, \ u \in L\}| \\ &= 0. \end{aligned}$$

Hence

$$M = \{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, \ u \notin K \cup L\} \neq \emptyset.$$

Since $q > 2$, take $qr = 2r + \tau$ for some $\tau > 0$. Let $u \in M = K^c \cap L^c$. Take $\Theta(s_1 - s_2, qr) \leq 1 - \gamma$ for $q > 2$. Then, we write

$$\begin{aligned} 1 - \gamma \geq \Theta(s_1 - s_2; 2r + \tau) \quad &\geq \Theta\left(s_u - s_1, r + \tfrac{\tau}{2}\right) \diamond \Theta\left(s_u - s_2, r + \tfrac{\tau}{2}\right) \\ &> (1 - \gamma_1) \diamond (1 - \gamma_1) > 1 - \gamma, \end{aligned}$$

which is absurd. If $\Omega(s_1 - s_2, qr) \geq \gamma$ for $q > 2$, then we have

$$\begin{aligned} \gamma \leq \Omega(s_1 - s_2; 2r + \tau) \quad &\leq \Omega\left(s_u - s_1, r + \tfrac{\tau}{2}\right) * \Omega\left(s_u - s_2, \tfrac{\tau}{2}\right) \\ &< \gamma_1 * \gamma_1 < \gamma, \end{aligned}$$

which is absurd. If $\Psi(s_1 - s_2, qr) \geq \gamma$ for $q > 2$, then

$$\begin{aligned} \gamma \leq \Psi(\omega_{uv} - w, r + \tau) \quad &\leq \Psi\left(s_u - s_1, r + \tfrac{\tau}{2}\right) * \Psi\left(s_u - s_2, \tfrac{\tau}{2}\right) \\ &< \gamma_1 * \gamma_1 < \gamma. \end{aligned}$$

which is absurd. Hence,

$$\Theta(s_1 - s_2; 2r + \tau) > 1 - \gamma \text{ and } \Omega(s_1 - s_2; 2r + \tau) < \gamma, \ \Psi(s_1 - s_2; 2r + \tau) < \gamma. \tag{5}$$

Then, from (5) we get

$$\Theta(s_1 - s_2; qr) > 1 - \gamma \text{ and } \Psi(s_1 - s_2; qr) < \gamma, \Omega(s_1 - s_2; qr) < \gamma \text{ for } q > 2$$

which is a contradiction to (4). Therefore, there does not exists elements $s_1, s_2 \in DS_{a,b}(\Theta, \Omega, \Psi) - \text{LIM}^r$ such that $\Theta(s_1 - s_2; qr) \leq 1 - \gamma$ or $\Psi(s_1 - s_2; qr) \geq \gamma$, $\Omega(s_1 - s_2; qr) \geq \gamma$ for $q > 2$. □

Proposition 3.10. *Let $(\mathcal{F}, \Theta, \Omega, \Psi, \diamond, *)$ be a $\mathfrak{N}\mathfrak{N}\mathfrak{S}$ and (s_u), (v_u) be sequences in $\mathcal{F}$. If $DS^{r_1}_{a,b}(\Theta, \Omega, \Psi) - \lim s_u = s_0$ and $DS^{r_2}_{a,b}(\Theta, \Omega, \Psi) - \lim v_u = v_0$ for some $r_1, r_2 \geq 0$, then*

$$DS^{(r_1+r_2)}_{a,b}(\Theta, \Omega, \Psi) - \lim (s_u + v_u) = s_0 + v_0.$$

Proof. Given $\gamma \in (0, 1)$, there exists $\gamma_1 \in (0, 1)$ such that $(1 - \gamma_1) \diamond (1 - \gamma_1) > 1 - \gamma$ and $\gamma_1 * \gamma_1 < \gamma$. Suppose $DS^{r_1}_{a,b}(\Theta, \Omega, \Psi) - \lim s_u = s_0$ and $DS^{r_2}_{a,b}(\Theta, \Omega, \Psi) - \lim v_u = v_0$ for a certain $r_1, r_2 \geq 0$. For any $\tau > 0$, take into

$$\begin{aligned} P = \Big\{&u \in \mathbb{N} : \Theta\left(s_u - s_0, r_1 + \tfrac{\tau}{2}\right) > 1 - \gamma_1, \\ &\Omega\left(s_u - s_0, r_1 + \tfrac{\tau}{2}\right) < \gamma_1 \text{ and } \Psi\left(s_u - s_0, r_1 + \tfrac{\tau}{2}\right) < \gamma_1\Big\}, \end{aligned}$$

and

$$Q = \Big\{u \in \mathbb{N} : \Theta\Big(v_u - v_0, r_2 + \tfrac{\tau}{2}\Big) > 1 - \gamma_1,$$
$$\Omega\Big(v_u - v_0, r_2 + \tfrac{\tau}{2}\Big) < \gamma_1 \text{ and } \Psi\Big(v_u - v_0, r_2 + \tfrac{\tau}{2}\Big) < \gamma_1\Big\}.$$

Then, we deduce

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} \left|\{u \in \mathbb{N} : 1 + a_w \le u \le b_w,\ u \in P\}\right| = 1,$$

and

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} \left|\{u \in \mathbb{N} : 1 + a_w \le u \le b_w,\ u \in Q\}\right| = 1.$$

Determine

$$R = \{u \in \mathbb{N} : u \in P \cap Q\}.$$

Then, we get

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} \left|\{u \in \mathbb{N} : 1 + a_w \le u \le b_w,\ u \in R\}\right| = 1.$$

This yields that $R \neq \emptyset$. Let $u \in R$. Then

$$\begin{aligned}
&\Theta\left((s_u + v_u) - (s_0 + v_0), r_1 + r_2 + \tau\right)\\
&\ge \Theta\Big(s_u - s_0, r_1 + \tfrac{\tau}{2}\Big) \diamond \Theta\Big(v_u - v_0, r_2 + \tfrac{\tau}{2}\Big)\\
&> (1 - \gamma_1) \diamond (1 - \gamma_1)\\
&> 1 - \gamma,
\end{aligned}$$

$$\begin{aligned}
&\Omega\left((s_u + v_u) - (s_0 + v_0), r_1 + r_2 + \tau\right)\\
&\le \Omega\Big(s_u - s_0, r_1 + \tfrac{\tau}{2}\Big) * \Omega\Big(v_u - v_0, r_2 + \tfrac{\tau}{2}\Big)\\
&< \gamma_1 * \gamma_1\\
&< \gamma,
\end{aligned}$$

and

$$\begin{aligned}
&\Psi\left((s_u + v_u) - (s_0 + v_0), r_1 + r_2 + \tau\right)\\
&\le \Psi\Big(s_u - s_0, r_1 + \tfrac{\tau}{2}\Big) * \Psi\Big(v_u - v_0, r_2 + \tfrac{\tau}{2}\Big)\\
&< \gamma_1 * \gamma_1\\
&< \gamma.
\end{aligned}$$

Hence

$$\begin{aligned}
&P \cap Q \subseteq \{u \in \mathbb{N} : \Theta\left((s_u + v_u) - (s_0 + v_0), r_1 + r_2 + \tau\right) > 1 - \gamma,\\
&\Omega\left((s_u + v_u) - (s_0 + v_0), r_1 + r_2 + \tau\right) < \gamma\\
&\text{and } \Psi\left((s_u + v_u) - (s_0 + v_0), r_1 + r_2 + \tau\right) < \gamma\}.
\end{aligned}$$

This means that

$$\begin{aligned}
&\lim_{w\to\infty} \tfrac{1}{b_w - a_w} \left|\{u \in \mathbb{N} : 1 + a_w \le u \le b_w, u \in P \cap Q\}\right|\\
&\le \lim_{w\to\infty} \tfrac{1}{b_w - a_w} \Big|\{u \in \mathbb{N} : 1 + a_w \le u \le b_w,\ \Theta\left((s_u + v_u) - (s_0 + v_0), r_1 + r_2 + \tau\right) > 1 - \gamma,\\
&\Omega\left((s_u + v_u) - (s_0 + v_0), r_1 + r_2 + \tau\right) < \gamma \text{ and } \Psi\left((s_u + v_u) - (s_0 + v_0), r_1 + r_2 + \tau\right) < \gamma\}\Big|.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\lim_{w\to\infty} \tfrac{1}{b_w - a_w} \Big|\{u \in \mathbb{N} : 1 + a_w \le u \le b_w,\ \Theta\left((s_u + v_u) - (s_0 + v_0), r_1 + r_2 + \tau\right) > 1 - \gamma,\\
&\quad \Omega\left((s_u + v_u) - (s_0 + v_0), r_1 + r_2 + \tau\right) < \gamma\\
&\quad \text{and } \Psi\left((s_u + v_u) - (s_0 + v_0), r_1 + r_2 + \tau\right) < \gamma\}\Big| = 1,
\end{aligned}$$

As a result, $DS_{a,b}^{(r_1+r_2)}(\Theta, \Omega, \Psi) - \lim\left(s_u + v_u\right) = s_0 + v_0$. □

Remark 3.11. *Proposition 3.10 is not valid for* $0 < r < r_1 + r_2$ *when at least one of* r_1 *and* r_2 *is non-zero, namely, for* $r_1 \neq 0$ *or* $r_2 \neq 0$ *when* $DS^{r_1}_{a,b}(\Theta,\Omega,\Psi) - \lim s_u = s_0$ *and* $DS^{r_2}_{a,b}(\Theta,\Omega,\Psi) - \lim v_u = v_0$*, then* $DS^{(r_1+r_2)}_{a,b}(\Theta,\Omega,\Psi) - \lim (s_u + v_u)$ *need not to be equal to* $s_0 + v_0$ *for* $0 < r < r_1 + r_2$.

Example 3.12. *Take into consideration the* $\mathfrak{NRS}$ $(\mathbb{R},\Theta,\Omega,\Psi,\diamond,*)$ *as defined in Example 3.4. Establish*

$$s_u = \begin{cases} u, & \text{if } u = 5^t, \\ (-1)^t, & \text{if not} \end{cases}, \quad t \in \mathbb{N}$$

and

$$v_u = \begin{cases} 0, & \text{if } u = 5^t, \\ (-2)^t, & \text{if not} \end{cases}, \quad t \in \mathbb{N}.$$

Let $a_w = w$ *and* $b_w = 3w$, $\forall w \in \mathbb{N}$*. Following the approach seen in Example 3.4, we obtain*

$$DS_{a,b}(\Theta,\Omega,\Psi) - \mathrm{LIM}^{r_1}(s_u) = \begin{cases} [1 - r_1, r_1 - 1], & \text{if } r_1 \geq 1 \\ \emptyset, & \text{if not,} \end{cases}$$

and

$$DS_{a,b}(\Theta,\Omega,\Psi) - \mathrm{LIM}^{r_2}(v_u) = \begin{cases} [2 - r_2, r_2 - 2], & \text{if } r_2 \geq 2 \\ \emptyset, & \text{if not.} \end{cases}$$

Now

$$(s_u + v_u) = \begin{cases} u, & \text{if } u = 5^t, \\ (-3)^t, & \text{if not} \end{cases}, \quad t \in \mathbb{N}.$$

Then

$$DS_{a,b}(\Theta,\Omega,\Psi) - \mathrm{LIM}^{r}(s_u + v_u) = \begin{cases} [3 - r, r - 3], & \text{if } r \geq 3 \\ \emptyset, & \text{if not.} \end{cases}$$

$DS^{r_1}_{a,b}(\Theta,\Omega,\Psi) - \lim s_u$ *and* $DS^{r_2}_{a,b}(\Theta,\Omega,\Psi) - \lim v_u$ *are equal to* 0 *if* $r_1 = 1$ *and* $r_2 = 2$*. We obtain* $DS_{a,b}(\Theta,\Omega,\Psi) - \mathrm{LIM}^{r}(s_u + v_u) = \emptyset$ *for* $0 < r < r_1 + r_2 = 3$.

Proposition 3.13. *Suppose that* $(\mathcal{F},\Theta,\Omega,\Psi,\diamond,*)$ *be a* $\mathfrak{NRS}$ *and* (s_u) *be a sequence in* $\mathcal{F}$*. If* $DS^{r}_{a,b}(\Theta,\Omega,\Psi) - \lim s_u = s_0$ *for some* $r \geq 0$*, then* $DS^{|c|r}_{a,b}(\Theta,\Omega,\Psi) - \lim cs_u = cs_0$ *for any* $c \in \mathbb{R}$.

Proof. When $0 = c \in \mathbb{R}$, the outcome is clear. Let $0 \neq c \in \mathbb{R}$. For given $\gamma \in (0,1)$, one has $\gamma_2 \in (0,1)$ such that $1 - \gamma_2 > 1 - \gamma$. Since $DS^{r}_{p,q}(\Theta,\Omega,\Psi) - \lim s_u = s_0$, we can consider the set

$$U = \Big\{u \in \mathbb{N} : \Theta\left(s_u - s_0, r + \tfrac{\tau}{2|c|}\right) > 1 - \gamma_2, \\ \Omega\left(s_u - s_0, r + \tfrac{\tau}{2|c|}\right) < \gamma_2 \text{ and } \Psi\left(s_u - s_0, r + \tfrac{\tau}{2|c|}\right) < \gamma_2\Big\}$$

with

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w,\ u \in U\}| = 1.$$

Consider $u \in U$. Then

$$\begin{aligned} \Theta(cs_u - cs_0, |c|r + \tau) &= \Theta\left(s_u - s_0, r + \tfrac{\tau}{|c|}\right) \\ &\geq \Theta\left(s_u - s_0, r + \tfrac{\tau}{2|c|}\right) \\ &> 1 - \gamma_2 > 1 - \gamma, \end{aligned}$$

$$\begin{aligned}\Omega\left(cs_u - cs_0, |c|r + \tau\right) &= \Omega\left(s_u - s_0, r + \tfrac{\tau}{|c|}\right)\\ &\leq \Omega\left(s_u - s_0, r + \tfrac{\tau}{2|c|}\right)\\ &< \gamma_2 < \gamma,\end{aligned}$$

and

$$\begin{aligned}\Psi\left(cs_u - cs_0, |c|r + \tau\right) &= \Psi\left(s_u - s_0, r + \tfrac{\tau}{|c|}\right)\\ &\leq \Psi\left(s_u - s_0, r + \tfrac{\tau}{2|c|}\right)\\ &< \gamma_2 < \gamma.\end{aligned}$$

Consequently,

$$\begin{aligned}U \;&\subset \{u \in \mathbb{N} : \Theta\left(cs_u - cs_0, |c|r + \tau\right) > 1 - \gamma,\\ &\Omega\left(cs_u - cs_0, |c|r + \tau\right) < \gamma \text{ and } \Psi\left(cs_u - cs_0, |c|r + \tau\right) < \gamma\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\lim_{w\to\infty} \tfrac{1}{b_w - a_w} \left|\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w : \Theta\left(cs_u - cs_0, |c|r + \tau\right) > 1 - \gamma\right.\\ &\left.\Omega\left(cs_u - cs_0, |c|r + \tau\right) < \gamma \text{ and } \Psi\left(cs_u - cs_0, |c|r + \tau\right) < \gamma\}\right| = 1.\end{aligned}$$

Consequently, $DS^{|c|r}_{a,b}(\Theta, \Omega, \Psi) - \lim cs_u = cs_0$. □

Remark 3.14. *When $0 < t < |c|r$, Proposition 3.13 is invalid for $r > 0$, namely, for some $r > 0$ if $DS^{r}_{a,b}(\Theta, \Omega, \Psi) - \lim s_u = s_0$, then $DS^{t}_{a,b}(\Theta, \Omega, \Psi) - \lim cs_u$ need not to be equal cs_0 for $0 < t < |c|r$ and $0 \neq c \in \mathbb{R}$.*

Example 3.15. *Take a look at Example 3.12 and assume $c = 2$. It is obvious that*

$$2s_u = \begin{cases} 2u, & \text{if } u = 5^m \\ (-1)^m\, 2, & \text{if not} \end{cases}, \quad m \in \mathbb{N},$$

and

$$DS_{a,b}(\Theta, \Omega, \Psi) - \mathrm{LIM}^t\left(2s_u\right) = \begin{cases} [2 - t, t - 2], & \text{if } t \geq 2 \\ \emptyset, & \text{if not.} \end{cases}$$

Let $r_1 = 2$. Then

$$DS_{a,b}(\Theta, \Omega, \Psi) - \mathrm{LIM}^2\left(s_u\right) = [-1, 1]$$

and

$$DS_{a,b}(\Theta, \Omega, \Psi) - \mathrm{LIM}^{2c}\left(cs_u\right) = [-2, 2] = c\,[-1, 1].$$

Conversely, $DS_{a,b}(\Theta, \Omega, \Psi) - \mathrm{LIM}^t\left(cs_u\right) = [2 - t, t - 2] \neq c[-1, 1]$ is obtained if $2 \leq t < 4$.

Theorem 3.16. *Let $(\mathcal{F}, \Theta, \Omega, \Psi, \diamond, *)$ be a $\mathfrak{NNS}$ and (s_u) be a sequence in $\mathcal{F}$. If there is a sequence (v_u) in $\mathcal{F}$ with $DS_{a,b}(\Theta, \Omega, \Psi) - \lim v_u = s_0$ such that for each $\gamma \in (0, 1)$ we have $\Theta\left(s_u - v_u, r\right) > 1 - \gamma$, $\Omega\left(s_u - v_u, r\right) < \gamma$ and $\Psi\left(s_u - v_u, r\right) < \gamma$ for all $u \in \mathbb{N}$, then $DS^{r}_{a,b}(\Theta, \Omega, \Psi) - \lim s_u = s_0$ for some $r \geq 0$.*

Proof. For given $\gamma \in (0, 1)$, choose $\gamma_1 \in (0, 1)$ such that $(1 - \gamma_1) \diamond (1 - \gamma_1) > 1 - \gamma$ and $\gamma_1 * \gamma_1 < \gamma$. Assume $DS_{a,b}(\Theta, \Omega, \Psi) - \lim v_u = s_0$ and

$$\Theta\left(s_u - v_u, r\right) > 1 - \gamma, \Omega\left(s_u - v_u, r\right) < \gamma \text{ and } \Psi\left(s_u - v_u, r\right) < \gamma$$

for each $\gamma \in (0,1)$ and for every $u \in \mathbb{N}$. For all $\tau > 0$ and the sets

$$U = \{u \in \mathbb{N} : \Theta(v_u - s_0, \tau) \leq 1 - \gamma_1, \\ \Omega(v_u - s_0, \tau) \geq \gamma_1 \text{ or } \Psi(v_u - s_0, \tau) \geq \gamma_1\},$$

and

$$V = \{u \in \mathbb{N} : \Theta(s_u - v_u, r) \leq 1 - \gamma_1, \\ \Omega(s_u - v_u, r) \geq \gamma_1 \text{ or } \Psi(s_u - v_u, r) \geq \gamma_1\}$$

we get

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, u \in U\}| \\ = \lim_{w\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, u \in V\}| = 0.$$

$$\Longrightarrow \lim_{w\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, u \in U^c\}| \\ = \lim_{w\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, u \in V^c\}| = 1.$$

Evidently, $U^c \cap V^c \neq \emptyset$ and

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, u \in U^c \cap V^c\}| = 1.$$

Consider $u \in U^c \cap V^c$. Then

$$\begin{aligned} \Theta(s_u - s_0, r + \tau) &\geq \Theta(s_u - v_u, r) \diamond \Theta(v_u - s_0, \tau) \\ &> (1 - \gamma_1) \diamond (1 - \gamma_1) > 1 - \gamma, \end{aligned}$$

$$\begin{aligned} \Omega(s_u - s_0, r + \tau) &\leq \Omega(s_u - v_u, r) * \Omega(v_u - s_0, \tau) \\ &< \gamma_1 * \gamma_1 < \gamma, \end{aligned}$$

and

$$\Psi(s_u - s_0, r + \tau) \leq \Psi(s_u - v_u, r) * \Psi(v_u - s_0, \tau) < \gamma_1 * \gamma_1 < \gamma.$$

So,

$$\begin{aligned} U^c \cap V^c &\subset \{u \in \mathbb{N} : \Theta(s_u - s_0, r + \tau) > 1 - \gamma, \\ &\Omega(s_u - s_0, r + \tau) < \gamma \text{ and } \Psi(s_u - s_0, r + \tau) < \gamma\}, \end{aligned}$$

follows, implying

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} \Big|\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, \Theta(s_u - s_0, r + \tau) > 1 - \gamma, \\ \Omega(s_u - s_0, r + \tau) < \gamma \text{ and } \Psi(s_u - s_0, r + \tau) < \gamma\}\Big| = 1.$$

Thus $DS^r_{p,q}(\Theta, \Omega, \Psi) - \lim s_u = s_0$. □

Definition 3.17. *We say that a sequence (s_u) in $\mathcal{F}$ is deferred statistically bounded w.r.t. $\mathfrak{NN}$ (Θ, Ω, Ψ), if for all $\gamma \in (0,1)$, $\exists \alpha > 0$ such that*

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} \Big|\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, \Theta(s_u, \alpha) \leq 1 - \gamma \text{ or} \\ \Omega(s_u, \alpha) \geq \gamma, \Psi(s_u, \alpha) \geq \gamma\}\Big| = 0.$$

Theorem 3.18. *Let (ω_{uv}) be a sequence in $\mathcal{F}$ and $(\mathcal{F}, \Theta, \Omega, \Psi, \diamond, *)$ be a $\mathfrak{NNS}$. In such case, for some $r \geq 0$, (s_u) is deferred statistically bounded iff $DS_{p,q}(\Theta, \Omega, \Psi) - \mathrm{LIM}^r s_u \neq \emptyset$ for some $r \geq 0$.*

Proof. Assume that (s_u) is deferred statistically bounded. For each $\gamma \in (0,1)$,there exists $\alpha > 0$ such that the set

$$\mathcal{K} = \{u \in \mathbb{N} : \Theta(s_u, \alpha) \leq 1 - \gamma \text{ or } \Omega(s_u, \alpha) \geq \gamma, \Psi(s_u, \alpha) \geq \gamma\}$$

has

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, u \in \mathcal{K}\}| = 0.$$

Thus, we obtain

$$\mathcal{K}^c = \{u \in \mathbb{N} : \Theta(s_u, \alpha) > 1 - \gamma \text{ and } \Omega(s_u, \alpha) < \gamma, \Psi(s_u, \alpha) < \gamma\}$$

and

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, u \in \mathcal{K}^c\}| = 1.$$

Consider $u \in \mathcal{K}^c$. For each $\tau > 0$, we have

$$\begin{aligned} \Theta(s_u, \alpha + \tau) &\geq \Theta(s_u, \alpha) \diamond \Theta(0, \tau) \\ &> (1 - \gamma) \diamond 1 \\ &= 1 - \gamma, \end{aligned}$$

$$\begin{aligned} \Omega(s_u, \alpha + \tau) &\leq \Omega(s_u, \alpha) * \Omega(0, \tau) \\ &< \gamma * 0, \\ &= \gamma, \end{aligned}$$

and

$$\begin{aligned} \Psi(s_u, \alpha + \tau) &\leq \Psi(s_u, \alpha) * \Psi(0, \tau) \\ &< \gamma * 0, \\ &= \gamma. \end{aligned}$$

So, we obtain

$$\begin{aligned} \mathcal{K}^c \subset \{u \in \mathbb{N} : &\Theta(s_u, \alpha + \tau) > 1 - \gamma, \\ &\Omega(s_u, \alpha + \tau) < \gamma \text{ and } \Psi(s_u, \alpha + \tau) < \gamma\} \end{aligned}$$

and

$$\begin{aligned} \lim_{w\to\infty} \tfrac{1}{b_w - a_w} \Big| \{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, \Theta(s_u, \alpha + \tau) > 1 - \gamma, \\ \Omega(s_u, \alpha + \tau) < \gamma \text{ and } \Psi(s_u, \alpha + \tau) < \gamma\} \Big| = 1. \end{aligned}$$

Consequently, we have $0 \in DS_{a,b}(\Theta, \Omega, \Psi) - \text{LIM}^{\alpha} s_u$. Consequently, $DS_{a,b}(\Theta, \Omega, \Psi) - \text{LIM}^{\alpha} s_u \neq \emptyset$. On the contrary, assume $DS_{a,b}(\Theta, \Omega, \Psi) - \text{LIM}^{r} s_u \neq \emptyset$ for some $r \geq 0$. So, there exist $s_0 \in \mathcal{F}$ such that $s_0 \in DS_{a,b}(\Theta, \Omega, \Psi) - \text{LIM}^{r} s_u$. Therefore, for each $\gamma \in (0,1)$ and $\tau > 0$, we get

$$\begin{aligned} \mathcal{L} = \{u \in \mathbb{N} : \Theta(s_u - s_0, r + \tau) > 1 - \gamma \\ \Omega(s_u - s_0, r + \tau) < \gamma \text{ and } \Psi(s_u - s_0, r + \tau) < \gamma\} \end{aligned}$$

with

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, u \in \mathcal{L}\}| = 1.$$

Select an $T > 0$ large enough such that $W = T - (r + \tau) > 0$, $\Theta(s_0, W) = 1$ and $\Omega(s_0, W) = \Psi(s_0, W) = 0$. Let $u \in \mathcal{L}$. Then

$$\begin{aligned}\Theta(s_u, T) &\geq \Theta(s_u - s_0, r + \tau) \diamond \Theta(s_0, W) \\ &> (1-\gamma) \diamond 1 \\ &= 1 - \gamma,\end{aligned}$$

$$\begin{aligned}\Omega(s_u, T) &\leq \Omega(s_u - s_0, r + \tau) * \Omega(s_0, W) \\ &< \gamma * 0 \\ &= \gamma.\end{aligned}$$

Likewise, we obtain $\Psi(s_u, T) < \gamma$. Thus,

$$\mathcal{L} \subset \{u \in \mathbb{N} : \Theta(s_u, T) > 1 - \gamma, \Omega(s_u, T) < \gamma, \Psi(s_u, T) < \gamma\}.$$

and so, we get

$$\begin{aligned}\lim_{w\to\infty} \tfrac{1}{b_w - a_w} \big|\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, \Theta(s_u, T) > 1 - \gamma, \\ \Omega(s_u, T) < \gamma \text{ and } \Psi(s_u, T) < \gamma\} \big| = 1.\end{aligned}$$

As a result, the sequence (s_u) is deferred statistically bounded. □

Theorem 3.19. *Let $(\mathcal{F}, \Theta, \Omega, \Psi, \diamond, *)$ be a $\mathfrak{NNS}$ and (s_u) be a sequence in $\mathcal{F}$. Then, $DS_{a,b}(\Theta, \Omega, \Psi) - \mathrm{LIM}^r(s_u)$ is a convex set for each $r \geq 0$.*

Proof. Assume that $\gamma \in (0, 1)$ and $s_1, s_2 \in DS_{a,b}(\Theta, \Omega, \Psi) - \mathrm{LIM}^r(s_u)$. Then, there exists $\gamma_1 \in (0, 1)$ such that $(1 - \gamma_1) \diamond (1 - \gamma_1) > 1 - \gamma$ and $\gamma_1 * \gamma_1 < \gamma$. We show that

$$\beta s_1 + (1 - \beta)s_2 \in DS_{p,q}(\Theta, \Omega, \Psi) - \mathrm{LIM}^r(s_u)$$

for any $\beta \in [0, 1]$. The proof is straightforward when $\beta = 0$ and $\beta = 1$. Consider $\beta \in (0, 1)$. For any $\tau > 0$, we define

$$\begin{aligned}T = \Big\{u \in \mathbb{N} : \Theta\Big(s_u - s_1, r + \tfrac{\tau}{2\beta}\Big) > 1 - \gamma_1 \\ \Omega\Big(s_u - s_1, r + \tfrac{\tau}{2\beta}\Big) < \gamma_1 \text{ and } \Psi\Big(s_u - s_1, r + \tfrac{\tau}{2\beta}\Big) < \gamma_1\Big\},\end{aligned}$$

and

$$\begin{aligned}V = \Big\{u \in \mathbb{N} : \Theta\Big(s_u - s_2, r + \tfrac{\tau}{2(1-\beta)}\Big) > 1 - \gamma_1, \\ \Omega\Big(s_u - s_2, r + \tfrac{\tau}{2(1-\beta)}\Big) < \gamma_1 \text{ and } \Psi\Big(s_u - s_2, r + \tfrac{\tau}{2(1-\beta)}\Big) < \gamma_1\Big\}.\end{aligned}$$

Since $s_1, s_2 \in DS_{a,b}(\Theta, \Omega, \Psi) - \mathrm{LIM}^r(s_u)$, we get

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, u \in T\}| = 1,$$

and

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, u \in V\}| = 1.$$

So, $T \cap V \neq \emptyset$ and

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, u \in T \cap V\}| = 1.$$

Consider $u \in T \cap V$. Next,

$$\begin{aligned}
&\Theta\left(s_u - [\beta s_1 + (1-\beta)s_2], r + \tau\right) \\
&= \Theta\left((1-\beta)(s_u - s_2) + \beta(s_u - s_1), (1-\beta)r + \beta r + \tau\right) \\
&\geq \Theta\left((1-\beta)(s_u - s_2), (1-\beta)r + \tfrac{\tau}{2}\right) \diamond \Theta\left(\beta(s_u - s_1), \beta r + \tfrac{\tau}{2}\right) \\
&= \Theta\left(s_u - s_2, r + \tfrac{\tau}{2(1-\beta)}\right) \diamond \Theta\left(s_u - s_1, r + \tfrac{\tau}{2\beta}\right) \\
&> (1-\gamma_1) \diamond (1-\gamma_1) > 1 - \gamma,
\end{aligned}$$

and

$$\begin{aligned}
&\Omega\left(s_u - [\beta s_1 + (1-\beta)s_2], r + \tau\right) \\
&= \Omega\left((1-\beta)(s_u - s_2) + \beta(s_u - s_1), (1-\beta)r + \beta r + \tau\right) \\
&\leq \Omega\left((1-\beta)(s_u - s_2), (1-\beta)r + \tfrac{\tau}{2}\right) * \Omega\left(\beta(s_u - s_1), \beta r + \tfrac{\tau}{2}\right) \\
&= \Omega\left(s_u - s_2, r + \tfrac{\tau}{2(1-\beta)}\right) * \Omega\left(s_u - s_1, r + \tfrac{\tau}{2\beta}\right) \\
&< \gamma_1 * \gamma_1 < \gamma.
\end{aligned}$$

In a similar vein

$$\Psi\left(s_u - [\beta s_1 + (1-\beta)s_2], r + \tau\right) < \gamma.$$

This indicates that the set

$$\begin{aligned}
\{u \in \mathbb{N} : \Theta\left(s_u - [\beta s_1 + (1-\beta)s_2], r + \tau\right) > 1 - \gamma,\\
\Omega\left(s_u - [\beta w_1 + (1-\beta)s_2], r + \tau\right) < \gamma\\
\text{and } \Psi\left(s_u - [\beta w_1 + (1-\beta)s_2], r + \tau\right) < \gamma\}
\end{aligned}$$

contains $T \cap V$ as a subset. Consequently, we have

$$\begin{aligned}
&\lim_{w\to\infty} \tfrac{1}{b_w - a_w} \Big|\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, \Theta\left(s_u - [\beta s_1 + (1-\beta)s_2], r + \tau\right) \\
&> 1 - \gamma,\ \Omega\left(s_u - [\beta w_1 + (1-\beta)s_2], r + \tau\right) < \gamma \\
&\text{ and } \Psi\left(s_u - [\beta w_1 + (1-\beta)s_2], r + \tau\right) < \gamma\}\Big| = 1.
\end{aligned}$$

Thus, $\beta s_1 + (1-\beta)s_2 \in DS_{a,b}(\Theta, \Omega, \Psi) - \mathrm{LIM}^r(s_u)$. □

Theorem 3.20. *Let $(\mathcal{F}, \Theta, \Omega, \Psi, \diamond, *)$ be a $\mathfrak{N}\mathfrak{N}\mathfrak{S}$ and (s_u) be a sequence in $\mathcal{F}$. Then for each $r \geq 0$, $DS_{a,b}(\Theta, \Omega, \Psi) -$ LIM^r is closed.*

Proof. We do not need to establish a proof since $DS_{a,b}(\Theta, \Omega, \Psi) - \mathrm{LIM}^r(s_u)$ is an empty set. Suppose $DS_{a,b}(\Theta, \Omega, \Psi) - \mathrm{LIM}^r(s_u) \neq \emptyset$ for some $r > 0$. Let $s_0 \in \overline{DS_{a,b}(\Theta, \Omega, \Psi) - \mathrm{LIM}^r(s_u)}$. Then, we have a convergent sequence (w_u) in $DS_{a,b}(\Theta, \Omega, \Psi) - \mathrm{LIM}^r(s_u)$ w.r.t. $\mathfrak{N}\mathfrak{N}$ such that $w_u \overset{(\Theta,\Omega,\Psi)}{\rightarrow} w_0$. Select $\gamma_1 \in (0,1)$ for $\gamma \in (0,1)$ such that $(1-\gamma_1) \diamond (1-\gamma_1) > (1-\gamma)$ and $\gamma_1 * \gamma_1 < \gamma$. Since $w_u \overset{(\Theta,\Omega,\Psi)}{\rightarrow} w_0$, then, for all $\tau > 0$, $\gamma_1 \in (0,1)$ and $\exists u_0 \in \mathbb{N}$ such that

$$\Theta\left(w_u - s_0, \frac{\tau}{2}\right) > 1 - \tau_1, \Omega\left(w_u - s_0, \frac{\tau}{2}\right) < \tau_1 \text{ and } \Psi\left(w_u - s_0, \frac{\tau}{2}\right) < \tau_1$$

for all $u \geq u_0$. Adjust $t > u_0$ so that $w_t \in DS_{a,b}(\Theta, \Omega, \Psi) - \mathrm{LIM}^r(s_u)$. Thus, using

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, u \in \mathcal{K}\}| = 1, \tag{6}$$

we obtain

$$\begin{aligned}
\mathcal{K} = \Big\{u \in \mathbb{N} : \Theta\left(s_u - w_t, r + \tfrac{\tau}{2}\right) > 1 - \tau_1,\\
\Omega\left(s_u - w_t, r + \tfrac{\tau}{2}\right) < \tau_1 \text{ and } \Psi\left(s_u - w_t, r + \tfrac{\tau}{2}\right) < \tau_1\Big\}.
\end{aligned}$$

If $u \in \mathcal{K}$, then we get

$$\begin{aligned}\Theta\left(s_u - s_0, r+\tau\right) &\geq \Theta\left(s_u - w_t, r+\tfrac{\tau}{2}\right) \diamond \Theta\left(w_t - s_0, \tfrac{\tau}{2}\right)\\ &> (1-\gamma_1) \diamond (1-\gamma_1) > 1-\gamma,\\ \Omega\left(s_u - s_0, r+\tau\right) &\leq \Omega\left(s_u - w_t, r+\tfrac{\tau}{2}\right) * \Omega\left(w_t - s_0, \tfrac{\tau}{2}\right)\\ &< \gamma_1 * \gamma_1 < \gamma,\end{aligned}$$

and

$$\begin{aligned}\Psi\left(s_u - s_0, r+\tau\right) &\leq \Psi\left(s_u - w_t, r+\tfrac{\tau}{2}\right) * \Psi\left(w_t - s_0, \tfrac{\tau}{2}\right)\\ &< \gamma_1 * \gamma_1 < \gamma.\end{aligned}$$

As a result

$$\mathcal{K} \subseteq \{u \in \mathbb{N} : \Theta\left(s_u - s_0, r+\tau\right) > 1-\gamma,\\ \Omega\left(s_u - s_0, r+\tau\right) < \gamma \text{ and } \Psi\left(s_u - s_0, r+\tau\right) < \gamma\}.$$

According to the (6), we obtain

$$\lim_{w\to\infty} \tfrac{1}{b_w - a_w} \Big|\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, \Theta\left(s_u - s_0, r+\tau\right) > 1-\gamma,\\ \Omega\left(s_u - s_0, r+\tau\right) < \gamma \text{ and } \Psi\left(s_u - s_0, r+\tau\right) < \gamma\}\Big| = 1,$$

or $s_0 \in DS_{a,b}(\Theta, \Omega, \Psi) - \text{LIM}^r(s_u)$. Consequently, the outcome guarantees. □

Theorem 3.21. *In the event that $DS_{a,b}(\Theta, \Omega, \Psi) - \lim(s_u) = s_0$, $\tau \in (0,1)$ occurs such that, for some $r > 0$, $\overline{\mathcal{B}_{s_0}^{(\Theta,\Omega,\Psi)}}(r,\tau) \subset DS^r_{a,b}(\Theta, \Omega, \Psi) - \lim(s_u)$.*

Proof. If $\gamma \in (0,1)$ is known, find $\exists\gamma_1 \in (0,1)$ such that $(1-\gamma_1) \diamond (1-\gamma_1) > 1-\gamma$ and $\gamma_1 * \gamma_1 < \gamma$. Assume that $DS_{a,b}(\Theta, \Omega, \Psi) - \lim(s_u) = s_0$. For every $\tau > 0$ and consider the set

$$\mathcal{L} = \{u \in \mathbb{N} : \Theta\left(s_u - s_0, \tau\right) > 1-\gamma_1\\ \Omega\left(s_u - s_0, \tau\right) < \gamma_1 \text{ and } \Psi\left(s_u - s_0, \tau\right) < \gamma_1\}.$$

Then, we get

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, u \in \mathcal{L}\}| = 1.$$

Select p such that $p \in \overline{\mathcal{B}_{s_0}^{(\Theta,\Omega,\Psi)}(r,\tau_1)}$, $r > 0$.

$$\Theta(s_0 - p, r) \geq 1-\gamma_1,\ \Omega(s_0 - p, r) \leq \gamma_1 \text{ and } \Psi(s_0 - p, r) \leq \gamma_1$$

in such case. Likewise, for $u \in \mathcal{L}$, we get

$$\Theta\left(s_u - p, r+\tau\right) > 1-\gamma,\ \Omega\left(s_u - p, r+\tau\right) < \gamma \text{ and } \Psi\left(s_u - p, r+\tau\right) < \gamma.$$

Consequently,

$$\mathcal{L} \subset \{u \in \mathbb{N} : \Theta\left(s_u - p, r+\tau\right) > 1-\gamma\\ \Omega\left(s_u - p, r+\tau\right) < \gamma \text{ and } \Psi\left(s_u - p, r+\tau\right) < \gamma\}.$$

So, we obtain

$$\lim_{w\to\infty} \tfrac{1}{b_w - a_w} \Big|\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, \Theta\left(s_u - p, r+\tau\right) > 1-\gamma,\\ \Omega\left(s_u - p, r+\tau\right) < \gamma \text{ and } \Psi\left(s_u - p, r+\tau\right) < \gamma\}\Big| = 1.$$

Thus, $p \in DS^r_{a,b}(\Theta, \Omega, \Psi)\,\text{LIM}^r(s_u)$. This gives that

$$\overline{\mathcal{B}_{s_0}^{(\Theta,\Omega,\Psi)}(r,\tau)} \subset DS^r_{a,q}(\Theta, \Omega, \Psi)\,\text{LIM}^r(s_u).$$

□

Now, let's introduce and explore the concept of a rough deferred statistical cluster point in a $\mathfrak{NNS}$ as stated below:

Definition 3.22. *Let $(\mathcal{F}, \Theta, \Omega, \Psi, \diamond, *)$ be a $\mathfrak{NNS}$ and consider (s_u) as a sequence in $\mathcal{F}$. For each $r \geq 0$, we define $p \in \mathcal{F}$ as a rough deferred statistical cluster point of (s_u) w.r.t. $\mathfrak{NN}$ (Θ, Ω, Ψ) if*

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} \left|\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, \Theta(s_u - p, r + \tau) > 1 - \gamma, \Omega(s_u - p, r + \tau) < \gamma \text{ and } \Psi(s_u - p, r + \tau) < \gamma\}\right| \neq 0$$

holds for every $\tau > 0$ and $\gamma \in (0, 1)$. We use $\Gamma^r_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)$ to represent the collection of all $DS^r_{a,b}(\Theta, \Omega, \Psi)$-cluster points of the sequence (s_u).

When r equals 0, we refer to the rough deferred statistical cluster point of a sequence (ω_{uv}) in $\mathcal{F}$ as the deferred statistical cluster point of (s_u) w.r.t $\mathfrak{NN}$ (Θ, Ω, Ψ), denoted as $DS_{a,b}(\Theta, \Omega, \Psi)$-cluster point. In this scenario, we represent the collection of all $DS_{a,b}(\Theta, \Omega, \Psi)$-cluster points of (s_u) by $\Gamma_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)$.

This is how we now display the set $\Gamma^r_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)$'s topological property:

Theorem 3.23. *Let $(\mathcal{F}, \Theta, \Omega, \Psi, \diamond, *)$ be a $\mathfrak{NNS}$ and consider (s_u) as a sequence in $\mathcal{F}$. It follows that for any $r \geq 0$, $\Gamma^r_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)$ is a closed set.*

Proof. Assume that $\gamma \in (0, 1)$. Then, there exists $\gamma_1 \in (0, 1)$ such that

$$(1 - \gamma_1) \diamond (1 - \gamma_1) > 1 - \gamma \text{ and } \gamma_1 * \gamma_1 < \gamma.$$

Suppose $p \in \overline{\Gamma^r_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)}$. Then, there is a sequence (p_u) of members in $\Gamma^r_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)$ such that $(\Theta, \Omega, \Psi)-\lim p_u = p$. Thus, for each $\tau > 0$, $\exists u \in \mathbb{N}$ such that

$$\Theta\left(p_u - p, \frac{\tau}{2}\right) > 1 - \gamma_1, \ \Omega\left(p_u - p, \frac{\tau}{2}\right) < \gamma_1 \text{ and } \Psi\left(p_u - p, \frac{\tau}{2}\right) < \gamma_1$$

for all $u \geq u_0$. Assign $t \geq u_0$. Next,

$$\Theta\left(p_t - p, \frac{\tau}{2}\right) > 1 - \gamma_1, \ \Omega\left(p_t - p, \frac{\tau}{2}\right) < \gamma_1 \text{ and } \Psi\left(p_t - p, \frac{\tau}{2}\right) < \gamma_1.$$

Also, we have

$$W = \left\{u \in \mathbb{N} : \Theta\left(s_u - p_t, r + \tfrac{\tau}{2}\right) > 1 - \gamma_1, \Omega\left(s_u - p_t, r + \tfrac{\tau}{2}\right) < \gamma_1 \text{ and } \Psi\left(s_u - p_t, r + \tfrac{\tau}{2}\right) < \gamma_1\right\}$$

with

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \leq u \leq b_w, u \in W\}| \neq 0.$$

If $u \in W$, then we get

$$\begin{aligned} \Theta(s_u - p, r + \tau) &\geq \Theta\left(s_u - p_t, r + \tfrac{\tau}{2}\right) \diamond \Theta\left(p_t - p, \tfrac{\tau}{2}\right) \\ &> (1 - \gamma_1) \diamond (1 - \gamma_1) > 1 - \gamma, \end{aligned}$$

$$\begin{aligned} \Omega(s_u - p, r + \tau) &\leq \Omega\left(s_u - p_t, r + \tfrac{\tau}{2}\right) * \Omega\left(p_t - p, \tfrac{\tau}{2}\right) \\ &< \gamma_1 * \gamma_1 < \gamma, \end{aligned}$$

and

$$\begin{aligned} \Psi(s_u - p, r + \tau) &\leq \Psi\left(s_u - p_t, r + \tfrac{\tau}{2}\right) * \Psi\left(p_t - p, \tfrac{\tau}{2}\right) \\ &< \gamma_1 * \gamma_1 < \gamma. \end{aligned}$$

Thus, we get

$$\begin{aligned} W & \subset \{u \in \mathbb{N} : \Theta(s_u - p, r+\tau) > 1-\gamma \\ & \Omega(s_u - p, r+\tau) < \gamma \text{ and } \Psi(s_u - p, r+\tau) < \gamma\}. \\ \implies & \lim_{w\to\infty} \tfrac{1}{b_w - a_w} \left|\{u \in \mathbb{N} : 1 + a_w \le u \le b_w, \Theta(s_u - p, r+\tau) > 1-\gamma \right. \\ & \left. \Omega(s_u - p, r+\tau) < \gamma \text{ and } \Psi(s_u - p, r+\tau) < \gamma\}\right| \neq 0. \end{aligned}$$

$p \in \Gamma^r_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)$ as a result, and $\Gamma^r_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)$ is closed. □

Theorem 3.24. *Let $(\mathcal{F}, \Theta, \Omega, \Psi, \diamond, *)$ be a $\mathfrak{NNS}$ and let (s_u) be a sequence in $\mathcal{F}$. Assume $\kappa \in \Gamma_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)$. If, for each $\gamma \in (0,1)$,*

$$\Theta(\varsigma - \kappa, r) > 1-\gamma,\ \Omega(\varsigma - \kappa, r) < \gamma \text{ and } \Psi(\varsigma - \kappa, r) < \gamma.$$

hold for some $r \ge 0$, then $\varsigma \in \Gamma^r_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)$.

Proof. For given $\gamma \in (0,1)$, $\exists \gamma_1 \in (0,1)$ such that $(1-\gamma_1)\diamond(1-\gamma_1) > 1-\gamma$ and $\gamma_1 * \gamma_1 < \gamma$. Suppose that $\kappa \in \Gamma_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)$. Then for every $\tau > 0$, the set

$$\begin{aligned} T & = \{u \in \mathbb{N} : \Theta(s_u - \kappa, \tau) > 1-\gamma_1, \\ & \Omega(s_u - \kappa, \tau) < \gamma_1 \text{ and } \Psi(s_u - \kappa, \tau) < \gamma_1\} \end{aligned}$$

has

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \le u \le b_w, u \in T\}| \neq 0. \tag{7}$$

Consider $\varsigma \in \mathcal{F}$ such that

$$\Theta(\varsigma - \kappa, r) > 1-\gamma_1,\ \Omega(\varsigma - \kappa, r) < \gamma_1 \text{ and } \Psi(\varsigma - \kappa, r) < \gamma_1$$

for some $r \ge 0$. For any pair $u \in T$, following a similar approach as mentioned above, we derive

$$\Theta(s_u - \varsigma, r+\tau) > 1-\gamma,\ \Omega(s_u - \varsigma, r+\tau) < \gamma \text{ and } \Psi(s_u - \varsigma, r+\tau) < \gamma.$$

Therefore,

$$\begin{aligned} T & \subset \{u \in \mathbb{N} : \Theta(s_u - \varsigma, r+\tau) > 1-\gamma, \\ & \Omega(s_u - \varsigma, r+\tau) < \gamma \text{ and } \Psi(s_u - \varsigma, r+\tau) < \gamma\}. \\ \implies & \lim_{w\to\infty} \tfrac{1}{b_w - a_w} \left|\{u \in \mathbb{N} : 1 + a_w \le u \le b_w, \Theta(s_u - \varsigma, r+\tau) > 1-\gamma, \right. \\ & \left. \Omega(s_u - \varsigma, r+\tau) < \gamma \text{ and } \Psi(s_u - \varsigma, r+\tau) < \gamma\}\right| \neq 0. \end{aligned}$$

$$\begin{aligned} T & \subset \{u \in \mathbb{N} : \Theta(s_u - \varsigma, r+\tau) > 1-\gamma, \\ & \Omega(s_u - \varsigma, r+\tau) < \gamma \text{ and } \Psi(s_u - \varsigma, r+\tau) < \gamma\}. \\ \implies & \lim_{w\to\infty} \tfrac{1}{b_w - a_w} |\{u \in \mathbb{N} : 1 + a_w \le u \le b_w, u \in T\}| \\ & \le \lim_{w\to\infty} \tfrac{1}{b_w - a_w} \left|\{u \in \mathbb{N} : 1 + a_w \le u \le b_w, \Theta(s_u - \varsigma, r+\tau) > 1-\gamma, \right. \\ & \left. \Omega(s_u - \varsigma, r+\tau) < \gamma \text{ and } \Psi(s_u - \varsigma, r+\tau) < \gamma\}\right|. \end{aligned}$$

Since the (7) holds, we obtain

$$\begin{aligned} & \lim_{w\to\infty} \tfrac{1}{b_w - a_w} \left|\{u \in \mathbb{N} : 1 + a_w \le u \le b_w, \Theta(s_u - \varsigma, r+\tau) > 1-\gamma, \right. \\ & \quad \left. \Omega(s_u - \varsigma, r+\tau) < \gamma \text{ and } \Psi(s_u - \varsigma, r+\tau) < \gamma\}\right| \neq 0. \end{aligned}$$

So, $\varsigma \in \Gamma^r_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)$. □

The aforementioned theorem makes it abundantly evident that there is a corresponding rough deferred statistical cluster point for each deferred statistical cluster point in a sequence in a $\mathfrak{NNS}$. The following theorem is presented in view of this fact.

Theorem 3.25.

$$\Gamma^r_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u) = \bigcup_{s_0\in\Gamma_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)} \overline{\mathcal{B}^{(\Theta,\Omega,\Psi)}_{s_0}(r,\gamma)}$$

exists for some $r > 0$ and $\gamma \in (0,1)$.

Proof. Suppose $\gamma \in (0,1)$ is given. So, $\exists\gamma_1 \in (0,1)$ such that $(1-\gamma_1)\diamond(1-\gamma_1) > 1-\gamma$ and $\gamma_1 * \gamma_1 < \gamma$. For some $r > 0$, let

$$\varsigma \in \bigcup_{s_0\in\Gamma_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)} \overline{\mathcal{B}^{(\Theta,\Omega,\Psi)}_{s_0}(r,\gamma_1)}.$$

Then, $\exists s_0 \in \Gamma_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)$ such that $\varsigma \in \overline{\mathcal{B}^{(\Theta,\Omega,\Psi)}_{s_0}(r,\gamma_1)}$, that is,

$$\Theta(s_0-\varsigma,r) \geq 1-\gamma_1,\ \Omega(s_0-\varsigma,r) \leq \gamma_1 \text{ and } \Psi(s_0-\varsigma,r) \leq \gamma_1.$$

By $s_0 \in \Gamma_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)$, for each $\tau > 0$ and the set

$$\begin{aligned} H = \{u \in \mathbb{N} : \Theta(s_u - s_0,\tau) > 1-\gamma_1,\\ \Omega(s_u-s_0,\tau) \leq \gamma_1 \text{ and } \Psi(s_u-s_0,\tau) \leq \gamma_1\}, \end{aligned}$$

we get

$$\lim_{w\to\infty} \frac{1}{b_w-a_w} \left|\{u \in \mathbb{N} : 1+a_w \leq u \leq b_w, u \in H\}\right| \neq 0.$$

Consider $u \in H$. In a similar vein, we get

$$\Theta(s_u-\varsigma,r+\tau) > 1-\gamma,\ \Omega(s_u-\varsigma,r+\tau) < \gamma \text{ and } \Psi(s_u-\varsigma,r+\tau) < \gamma,$$

as mentioned earlier. Thus,

$$\begin{aligned} H \quad &\subset \{u \in \mathbb{N} : \Theta(s_u-\varsigma,r+\tau) > 1-\gamma,\\ &\quad \Omega(s_u-\varsigma,r+\tau) < \gamma \text{ and } \Psi(s_u-\varsigma,r+\tau) < \gamma\}\\ \Rightarrow \quad &\lim_{w\to\infty} \tfrac{1}{b_w-a_w} \big|\{u \in \mathbb{N} : 1+a_w \leq u \leq b_w, \Theta(s_u-\varsigma,r+\tau) > 1-\gamma,\\ &\quad \Omega(s_u-\varsigma,r+\tau) < \gamma \text{ and } \Psi(s_u-\varsigma,r+\tau) < \gamma\}\big| \neq 0, \end{aligned}$$

that is, $\varsigma \in \Gamma^r_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)$. So, we have

$$\bigcup_{s_0\in\Gamma_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)} \overline{\mathcal{B}^{(\Theta,\Omega,\Psi)}_{s_0}(r,\gamma_1)} \subset \Gamma^r_{(\varphi,\psi,\eta)_{S_{\theta_2}}}(\omega_{uv}). \tag{8}$$

Conversely, if $\varsigma \in \Gamma^r_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)$, then let on contrary

$$\varsigma \notin \bigcup_{s_0\in\Gamma_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)} \overline{\mathcal{B}^{(\Theta,\Omega,\Psi)}_{s_0}(r,\gamma_1)}.$$

So, for all $s_0 \in \Gamma_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)$, we obtain $\varsigma \notin \overline{\mathcal{B}^{(\Theta,\Omega,\Psi)}_{s_0}(r,\gamma_1)}$, i.e.,

$$\Theta(s_0-\varsigma,r) < 1-\gamma_1, \Omega(s_0-\varsigma,r) > \gamma_1 \text{ and } \Psi(s_0-\varsigma,r) > \gamma_1.$$

Therefore, by Theorem 3.24, we have $\varsigma \notin \Gamma^r_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)$, which goes against what we assumed. Thus

$$\Gamma^r_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u) \subset \bigcup_{s_0 \in \Gamma_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)} \overline{\mathcal{B}^{(\Theta,\Omega,\Psi)}_{s_0}(r,\gamma_1)} \tag{9}$$

Combining (8) and (9), yields the following outcome. □

Theorem 3.26. *Let $(\mathcal{F},\Theta,\Omega,\Psi,\diamond,*)$ be a $\mathfrak{N}\mathfrak{N}\mathfrak{S}$. Given a sequence (s_u) in $\mathcal{F}$, let $DS_{a,b}(\Theta,\Omega,\Psi) - \lim s_u = s_0$. Then $\Gamma^r_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u) \subset DS_{a,b}(\Theta,\Omega,\Psi) - \mathrm{LIM}^r(s_u)$ for some $r > 0$.*

Proof. Assume $DS_{a,b}(\Theta,\Omega,\Psi) - \lim s_u = s_0$. Thus $s_0 \in \Gamma_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u)$. By Theorem 3.25, for some $r > 0$ and $\gamma \in (0,1)$,

$$\Gamma^r_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u) = \overline{\mathcal{B}^{(\Theta,\Omega,\Psi)}_{s_0}(r,\gamma)}. \tag{10}$$

Also, by Theorem 3.21,

$$\overline{\mathcal{B}^{(\Theta,\Omega,\Psi)}_{s_0}(r,\gamma_1)} \subset DS_{a,b}(\Theta,\Omega,\Psi) - \mathrm{LIM}^r(s_u). \tag{11}$$

Hence by (10) and (11), we have

$$\Gamma^r_{DS_{a,b}(\Theta,\Omega,\Psi)}(s_u) \subset DS_{a,b}(\Theta,\Omega,\Psi) - \mathrm{LIM}^r(s_u).$$

□

4. Conclusion

When dealing with a convergent sequence (s_u) where the terms become challenging to estimate for sufficiently large u, an auxiliary sequence (v_u) is employed to approximate these values, introducing approximation errors. Rough convergence emerged as a solution to address this challenge. Many mathematicians are actively exploring the relationship between statistical convergence and various convergence concepts within neutrosophic normed space. However, the more general idea in this theory has yet to be thoroughly investigated with consideration given to the Pringsheim limit. This study, through the extension of neutrosophic theory, significantly enhances the existing body of literature. Two valuable additions are introduced by this study in the realm of neutrosophic theory for sequences in $\mathfrak{N}\mathfrak{N}\mathfrak{S}$: (i) a type of rough deferred statistical convergence; (ii) rough deferred statistical limit and cluster points. These concepts and findings can serve as theoretical tools for examining optimal approaches within the framework of turnpike theory in a fuzzy environment.

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