Filomat 38:31 (2024), 10867–10878
https://doi.org/10.2298/FIL2431867M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Extremal problems for a polynomial and its polar derivative

Abdullah Mir[a], Adil Hussain[a]

[a]*Department of Mathematics, University of Kashmir, Srinagar-190006, Jammu and Kashmir, India.*

Abstract. If $P(z) := z^s(a_0 + a_1 z + ... + a_{n-s}z^{n-s}),\ 0 \leq s \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k,\ k \geq 1$, then for $|\delta| \geq k$, Govil and Kumar $\big($Appl. Anal. Discrete Math. 13 (2019) 711-720$\big)$ proved

$$\max_{|z|=1} |D_\delta P(z)| \geq (|\delta| - k)\left\{\frac{n+s}{1+k^n} + \frac{(k^{n-s}|a_{n-s}| - |a_0|)}{(1+k^n)(k^{n-s}|a_{n-s}| + |a_0|)}\right\}\max_{|z|=1} |P(z)|.$$

In a recent work, Mir $\big($Ramanujan J. 56 (2021) 1061-1071$\big)$ strengthened and generalised the inequality above and proved, under the same hypothesis, that

$$\begin{aligned}\max_{|z|=1} |D_\delta P(z)| \geq\ & \frac{n}{1+k^n}\left\{(|\delta| - k)\max_{|z|=1} |P(z)| + \left(|\delta| + \frac{1}{k^{n-1}}\right)m_k\right\}\\ &+ (|\delta| - k)\left\{\frac{s}{1+k^n} + \frac{(k^{n-s}|a_{n-s}| - |a_0| - m_k)}{(1+k^n)(k^{n-s}|a_{n-s}| + |a_0| - m_k)}\right\}\\ &\times \left\{\max_{|z|=1} |P(z)| - \frac{m_k}{k^n}\right\},\end{aligned}$$

where $m_k = \min_{|z|=k} |P(z)|$.
In this study, we generalize as well as improve upon the above inequalities and related results.

1. Introduction

Studying the extremal problems of the functions of a complex variable and generalizing the classical polynomial inequalities are topical in geometric function theory. Numerous inequalities of majorization between polynomials with complex coefficients form an essential part of the classical content of geometric function theory. These classical and fundamental inequalities are nowadays a widely studied topic and are equally important in modern papers that are devoted to developing techniques to generalize various well-known inequalities for polynomials and other analytic functions in approximation theory. The unit disk in the complex plane serves as the prototype of a bounded domain for studying the extremal properties of polynomials and their derivatives. If one is interested in how "big" a polynomial or its derivative can

2020 *Mathematics Subject Classification*. Primary 30A10; Secondary 30C10, 30C15.
Keywords. Maximum Modulus Principle, Turán's inequality, Laguerre theorem, Polar derivative of a polynomial.
Received: 21 February 2023; Accepted: 13 November 2023
Communicated by Miodrag Mateljević
The first author was supported by the National Board for Higher Mathematics (R.P), Department of Atomic Energy, Government of India (No. 02011/19/2022/R&D-II/10212) and by the Science & Engineering Research Board (SERB), Department of Science & Technology, Government of India (No. MTR/2022/000118).
Email addresses: mabdullah_mir@uok.edu.in (Abdullah Mir), malikadil6909@gmail.com (Adil Hussain)

be in the unit disk, then, because of the maximum modulus principle, it suffices to study the values on the boundary. In this paper, we shall establish some new lower bounds for the polar derivative of a polynomial on the unit disk and desire to look for better and improved bounds than those available in the literature.

For an arbitrary entire function f, let $\|f\| = \max_{|z|=1} |f(z)|$, the uniform-norm of f on the unit disk $|z| = 1$. Let $P(z) := \sum_{v=0}^{n} a_v z^v$ be an algebraic polynomial of degree n in the complex plane and $P'(z)$ is its derivative. One of the most known classical inequality with important applications in approximation theory is the Bernstein inequality (for reference, see [3]) for complex polynomials, namely if $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, then

$$\|P'(z)\| \leq n\|P(z)\|. \tag{1}$$

Equality holds in (1) if and only if $P(z)$ has all its zeros at the origin. One may easily notice that an improvement in (1) is implied by the restriction on the zeros of $P(z)$. It turns out that some control over the location of zeros in $P(z)$ is necessary in order to have any prospect of an improved lower bound. Turán in [23] took this into account and came up with a lower bound estimate for the size of the derivative of a polynomial on the unit disk in relation to the size of the polynomial when its zeros are restricted. In fact, Turán proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then

$$\|P'(z)\| \geq \frac{n}{2}\|P(z)\|. \tag{2}$$

Inequality (2) was refined by Aziz and Dawood [1] and proved under the same hypothesis that

$$\|P'(z)\| \geq \frac{n}{2}\Big\{\|P(z)\| + \min_{|z|=1} |P(z)|\Big\}. \tag{3}$$

Equality in (2) and (3) holds for any polynomial which has all its zeros on $|z| = 1$. Over the years, the inequalities (2) and (3) have been generalized and extended in several directions; see, for example ([6]-[10], [12], [14]-[20]). For a polynomial $P(z)$ of degree n having all its zeros in $|z| \leq k,\ k \geq 1$, Govil [6], proved that

$$\|P'(z)\| \geq \frac{n}{1+k^n}\|P(z)\|. \tag{4}$$

Equality in (4) holds for $P(z) = z^n + k^n$, one would expect that if we exclude the class of polynomials having all zeros on $|z| = k$, then it may be possible to improve the bound in (4). In this direction, it was shown by Govil [7] that if $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having all its zeros in $|z| \leq k,\ k \geq 1$, then

$$\|P'(z)\| \geq \frac{n}{1+k^n}\left\{\|P(z)\| + \min_{|z|=k} |P(z)|\right\}. \tag{5}$$

Turán-type inequalities have appeared in the literature in more generalized forms in which the underlying polynomial is replaced by more general classes of functions. One such generalization is the move from the domain of ordinary derivatives of polynomials to their polar derivatives. Let us explain the idea of the polar derivative involved before moving on to our primary findings. For a polynomial $P(z)$ of degree n, we define

$$D_\delta P(z) := nP(z) + (\delta - z)P'(z),$$

the polar derivative of $P(z)$ with respect to the point δ (see [11]). The polynomial $D_\delta P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\delta\to\infty}\left\{\frac{D_\delta P(z)}{\delta}\right\} := P'(z),$$

uniformly with respect to z for $|z| \leq R,\ R > 0$.
Various authors have produced a large number of different versions and generalizations of the aforementioned inequalities by introducing restrictions on the multiplicity of zero at $z = 0$, the modulus of largest root of $P(z)$, restrictions on coefficients etc. In many of these generalizations, different options for $P(z)$, δ and other parameters are compared with the polar derivative $D_\delta P(z)$. For more information on the polar derivative of polynomials, one can consult the comprehensive books of Gardner et al. [5], Marden [11], Milovanović et al. [13] or Rahman and Schmeisser [22].
In 1998, Aziz and Rather [2] established the polar derivative analogue of (4) by proving that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k,\ k \geq 1$, then for every $\delta \in \mathbb{C}$ with $|\delta| \geq k$,

$$\|D_\delta P(z)\| \geq n\left(\frac{|\delta| - k}{1 + k^n}\right)\|P(z)\|. \tag{6}$$

In the same paper, Aziz and Rather extended the inequality (3) to the polar derivative of a polynomial. In fact, they proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for any complex number δ with $|\delta| \geq 1$,

$$\|D_\delta P(z)\| \geq \frac{n}{2}\Big\{(|\delta| - 1)\|P(z)\| + (|\delta| + 1)\min_{|z|=1}|P(z)|\Big\}. \tag{7}$$

The corresponding polar derivative analogue of (5) and a refinement of (6) was given by Dewan et al. [4]. They proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k,\ k \geq 1$, then for any complex number δ with $|\delta| \geq k$,

$$\|D_\delta P(z)\| \geq \frac{n}{1 + k^n}\Big\{(|\delta| - k)\|P(z)\| + \left(|\delta| + \frac{1}{k^{n-1}}\right)\min_{|z|=k}|P(z)|\Big\}. \tag{8}$$

Recently, Kumar and Dhankhar [10] obtained a generalization of (4) and they proved that if $P(z) = z^s(a_0 + a_1 z + a_2 z^2 + ... + a_{n-s}z^{n-s}),\ 0 \leq s \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k,\ k \geq 1$, then

$$\|P'(z)\| \geq \left(\frac{n}{1 + k^{n-s}} + \frac{n(k^n|a_{n-s}| - k^s|a_0|)(k-1)}{2(1 + k^{n-s})(k^n|a_{n-s}| + k^{s+1}|a_0|)}\right)\|P(z)\|. \tag{9}$$

Equality in (9) holds for $P(z) = z^n + k^n$.
As a polar derivative analogue to (9), Kumar and Dhankhar in the same paper proved that if $P(z) = z^s(a_0 + a_1 z + a_2 z^2 + ... + a_{n-s}z^{n-s}),\ 0 \leq s \leq n$, be a polynomial of degree n having all its zeros in $|z| \leq k,\ k \geq 1$, then for any complex number δ with $|\delta| \geq k$,

$$\|D_\delta P(z)\| \geq \left(\frac{n(|\delta| - k)}{1 + k^{n-s}} + \frac{n(|\delta| - k)(k^n|a_{n-s}| - k^s|a_0|)(k-1)}{2(1 + k^{n-s})(k^n|a_{n-s}| + k^{s+1}|a_0|)}\right)\|P(z)\|. \tag{10}$$

In 2019, Govil and Kumar [8] established the following generalization and strengthening of (6). More precisely, they proved the following result.

Theorem 1.1. *If $P(z) = z^s(a_0 + a_1 z + ... + a_{n-s}z^{n-s}),\ 0 \leq s \leq n$, is a polynomial of degree n having all zeros in $|z| \leq k,\ k \geq 1$, then for every $\delta \in \mathbb{C}$ with $|\delta| \geq k$,*

$$\|D_\delta P(z)\| \geq (|\delta| - k)\left\{\frac{n + s}{1 + k^n} + \frac{(k^{n-s}|a_{n-s}| - |a_0|)}{(1 + k^n)(k^{n-s}|a_{n-s}| + |a_0|)}\right\}\|P(z)\|. \tag{11}$$

Dividing both sides of (11) by $|\delta|$ and let $|\delta| \to \infty$, we have the following refinement and generalization of (4).

Theorem 1.2. *If $P(z) = z^s(a_0 + a_1 z + ... + a_{n-s}z^{n-s}),\ 0 \leq s \leq n$, is a polynomial of degree n having all zeros in $|z| \leq k,\ k \geq 1$, then*

$$\|P'(z)\| \geq \left\{\frac{n + s}{1 + k^n} + \frac{(k^{n-s}|a_{n-s}| - |a_0|)}{(1 + k^n)(k^{n-s}|a_{n-s}| + |a_0|)}\right\}\|P(z)\|. \tag{12}$$

Very recently, Mir [17] generalized and sharpened both the above Theorems 1.1 and 1.2 in the form of the following results.

Theorem 1.3. *If $P(z) = z^s(a_0 + a_1 z + ... + a_{n-s}z^{n-s})$, $0 \le s \le n$, is a polynomial of degree n having all zeros in $|z| \le k$, $k \ge 1$, then for any complex number δ with $|\delta| \ge k$,*

$$\begin{aligned}\|D_\delta P(z)\| \ge\ & \frac{n}{1+k^n}\left\{(|\delta|-k)\|P(z)\| + \left(|\delta| + \frac{1}{k^{n-1}}\right)m_k\right\}\\ & + (|\delta|-k)\left\{\frac{s}{1+k^n} + \frac{(k^{n-s}|a_{n-s}| - |a_0| - m_k)}{(1+k^n)(k^{n-s}|a_{n-s}| + |a_0| - m_k)}\right\}\\ & \times\left\{\|P(z)\| - \frac{m_k}{k^n}\right\}, \end{aligned} \tag{13}$$

where $m_k = \min_{|z|=k}|P(z)|$.

Dividing both sides of inequality (13) by $|\delta|$ and let $|\delta| \to \infty$, we get the following refinement of Theorem 1.2.

Theorem 1.4. *If $P(z) = z^s(a_0 + a_1 z + ... + a_{n-s}z^{n-s})$, $0 \le s \le n$, is a polynomial of degree n having all zeros in $|z| \le k$, $k \ge 1$, then*

$$\begin{aligned}\|P'(z)\| \ge\ & \frac{n}{1+k^n}\Big(\|P(z)\| + m_k\Big)\\ & + \left\{\frac{s}{1+k^n} + \frac{(k^{n-s}|a_{n-s}| - |a_0| - m_k)}{(1+k^n)(k^{n-s}|a_{n-s}| + |a_0| - m_k)}\right\}\left(\|P(z)\| - \frac{m_k}{k^n}\right), \end{aligned} \tag{14}$$

where $m_k = \min_{|z|=k}|P(z)|$.

The research on the estimation of various norms of derivatives and the underlying polynomial has been active in recent years; there are many research papers published in a variety of journals each year, and different approaches have been taken for different purposes. The present article is concerned with extending the aforementioned inequalities by establishing norm estimates for the derivative and polar derivative of a polynomial under certain constraints for its zeros.

2. The Main Result and its Applications

In this note, we further generalize and sharpen (13) and (14). Besides, the obtained inequality gives generalizations and refinements of (8)-(12) as well.

Theorem 2.1. *If $P(z) = z^s(a_0 + a_1 z + ... + a_{n-s}z^{n-s})$, $0 \le s \le n$, is a polynomial of degree n having all zeros in $|z| \le k$, $k \ge 1$, then for any complex number δ with $|\delta| \ge k$ and $0 \le l \le 1$,*

$$\begin{aligned}&\|D_\delta P(z)\|\\ &\ge \frac{n(|\delta|-k)}{1+k^{n-s}}\left[\left(1 + \frac{(k-1)}{2}W_k(s,l)\right)\|P(z)\| + \frac{1}{2k^n}\Big(k^{n-s} - 1 - (k-1)W_k(s,l)\Big)lm_k\right]\\ &+ n\left(\frac{|\delta|+k)}{2k^n}\right)lm_k + (|\delta|-k)V_k(s,l)\left(1 + \frac{(k-1)}{2}W_k(s,l)\right)\left(\|P(z)\| - \frac{1}{k^n}lm_k\right), \end{aligned} \tag{15}$$

where

$$V_k(s,l) = \left\{\frac{s}{1+k^{n-s}} + \frac{(k^{n-s}|a_{n-s}| - |a_0| - lm_k)}{(1+k^{n-s})(k^{n-s}|a_{n-s}| + |a_0| - lm_k)}\right\},$$

$$W_k(s,l) = \frac{k^n|a_{n-s}| - k^s|a_0| - lm_k}{k^n|a_{n-s}| + k^{s+1}|a_0| - lm_k}$$

and $m_k = \min_{|z|=k}|P(z)|$.

Dividing both sides of inequality (15) by $|\delta|$ and let $|\delta| \to \infty$, we get the following result.

Corollary 2.2. *If $P(z) = z^s(a_0 + a_1 z + ... + a_{n-s}z^{n-s})$, $0 \le s \le n$, is a polynomial of degree n having all zeros in $|z| \le k$, $k \ge 1$, then for $0 \le l \le 1$,*

$$\begin{aligned}
&\|P'(z)\| \\
&\ge \frac{n}{1+k^{n-s}}\left[\left(1 + \frac{(k-1)}{2}W_k(s,l)\right)\|P(z)\| + \frac{1}{k^n}\left(k^{n-s} - \frac{(k-1)}{2}W_k(s,l)\right)lm_k\right] \\
&+ V_k(s,l)\left(1 + \frac{(k-1)}{2}W_k(s,l)\right)\left(\|P(z)\| - \frac{1}{k^n}lm_k\right),
\end{aligned} \tag{16}$$

where $V_k(s,l)$, $W_k(s,l)$ and m_k are as defined in Theorem 2.1.

If we take $l = 0$ in (15) and (16), we get the following results:

Corollary 2.3. *If $P(z) = z^s(a_0 + a_1 z + ... + a_{n-s}z^{n-s})$, $0 \le s \le n$, is a polynomial of degree n having all zeros in $|z| \le k$, $k \ge 1$, then for any complex number δ with $|\delta| \ge k$,*

$$\|D_\delta P(z)\| \ge \left\{\frac{n(|\delta| - k)}{1+k^{n-s}} + (|\delta| - k)V_k(s)\right\}\left(1 + \frac{(k-1)}{2}W_k(s)\right)\|P(z)\|, \tag{17}$$

where

$$V_k(s) = \left\{\frac{s}{1+k^{n-s}} + \frac{(k^{n-s}|a_{n-s}| - |a_0|)}{(1+k^{n-s})(k^{n-s}|a_{n-s}| + |a_0|)}\right\}$$

and

$$W_k(s) = \frac{k^n|a_{n-s}| - k^s|a_0|}{k^n|a_{n-s}| + k^{s+1}|a_0|}.$$

Corollary 2.4. *If $P(z) = z^s(a_0 + a_1 z + ... + a_{n-s}z^{n-s})$, $0 \le s \le n$, is a polynomial of degree n having all zeros in $|z| \le k$, $k \ge 1$, then*

$$\|P'(z)\| \ge \left\{\frac{n}{1+k^{n-s}} + V_k(s)\right\}\left(1 + \frac{(k-1)}{2}W_k(s)\right)\|P(z)\|, \tag{18}$$

where $V_k(s)$ and $W_k(s)$ are as defined in Corollary 2.3.

Remark 2.5. *It may be remarked that, in general for any polynomial of degree n of the form $P(z) = z^s(a_0 + a_1 z + ... + a_{n-s}z^{n-s})$, $0 \le s \le n$, having all its zeros in $|z| \le k$, $k \ge 1$, the inequalities* (15) *and* (16) *would give improvements over the bounds obtained from the inequalities* (13) *and* (14) *respectively, excepting the case when $P(z)$ has all its zeros on $|z| = k$. For the class of polynomials having a zero on $|z| = k$ and $k \ne 1$, the inequalities* (17) *and* (18) *will give bounds that are sharper than obtainable from the inequalities* (10) *and* (9) *respectively. One can also observe that for the class of polynomials having all their zeros in $|z| \le k$, the inequalities* (17) *and* (18) *respectively improves the inequalities* (11) *and* (12) *considerably when $k^{n-s}|a_{n-s}| - |a_0| \ne 0$ and $k > 1$.*

Remark 2.6. *Recall that the polynomial $P(z)$ has all its zeros in $|z| \le k$, $k \ge 1$, with s-fold zeros at the origin. If $P(z)$ has all its zeros at the origin, that is, if we suppose $s = n$, then clearly $W_k(s,l) = \frac{-1}{k} < 0$, and in this case there is no significant improvements of* (15) *and* (16) *over* (13) *and* (14) *respectively. We now suppose that $0 \le s < n$, and for this we show that $W_k(s,l) \ge 0$. To prove this, we show that*

$$k^s|a_0| + lm_k \le k^n|a_{n-s}|, \quad 0 \le s < n \text{ and } 0 \le l \le 1. \tag{19}$$

We can write

$$P(z) = z^s h(z),$$

where $h(z) = a_0 + a_1 z + a_2 z^2 + ... + a_{n-s} z^{n-s}$, *has all its zeros in* $|z| \le k,\ k \ge 1$, *with* $h(0) \ne 0$. *If* $h(z)$ *has a zero on* $|z| = k$, *then* $P(z)$ *has a zero on* $|z| = k$, *and* $m_k = \min_{|z|=k} |P(z)| = 0$, *and in this case Theorem 2.1 reduces to Corollary 2.3 and Corollary 2.2 reduces to Corollary 2.4. Henceforth, we suppose that* $h(z)$ *has all its zeros in* $|z| < k,\ k \ge 1$, *so that* $m_k > 0$. *Now* $m_k \le |P(z)|$ *for* $|z| = k$, *therefore, if* λ *is any complex number with* $|\lambda| < 1$, *then*

$$|\lambda m_k (z/k)^s| = |\lambda| m_k |z/k|^s < |P(z)| \quad \text{for} \quad |z| = k.$$

It follows by Rouché's theorem that all the zeros of

$$P(z) + \lambda m_k (z/k)^s = z^s \Big((a_0 + \lambda m_k / k^s) + a_1 z + a_2 z^2 + ... + a_{n-s} z^{n-s} \Big)$$

lie in $|z| < k$, *with a zero of order* s, $0 \le s < n$ *at the origin. If* $z_1, z_2, ..., z_{n-s}$, *are the zeros of*

$$(a_0 + \lambda m_k / k^s) + a_1 z + a_2 z^2 + ... + a_{n-s} z^{n-s},$$

then $|z_v| < k,\ v = 1, 2, ..., n - s$, *and hence*

$$\left| \frac{a_0 + \lambda m_k / k^s}{a_{n-s}} \right| = |z_1 z_2 ... z_{n-s}| < k^{n-s}. \tag{20}$$

If in (20), *we choose the argument of* λ *suitably, so that*

$$|a_0 + \lambda m_k / k^s| = |a_0| + |\lambda| m_k / k^s,$$

we get

$$|a_0| + |\lambda| m_k / k^s \le k^{n-s} |a_{n-s}|. \tag{21}$$

For λ *with* $|\lambda| = 1$, *the above inequality follows by continuity. The inequality* (19) *follows by letting* $|\lambda| = l$ *in* (21). *By using* (19), *it easily follows that*

$$\frac{k^n |a_{n-s}| - k^s |a_0| - l m_k}{k^n |a_{n-s}| + k^{s+1} |a_0| - l m_k} = W_k(s, l) \ge 0.$$

By using this fact, one can easily see that (15) *and* (16) *improves the bounds of* (13) *and* (14) *respectively when* $k \ne 1$. *Also, by Lemma 3.6, we have* $|a_{n-s}| \ge m_k / k^n$, *which further implies that*

$$\|P(z)\| = \max_{|z|=1} |P(z)| \ge |a_{n-s}| \ge m_k / k^n \ge l m_k / k^n, \quad 0 \le l \le 1.$$

Using this and the fact that for $|\delta| \ge k$ *and* $k \ge 1$, *one can easily check that*

$$\begin{aligned} \psi(x) = \frac{n\,(|\delta| - k)}{1 + k^{n-s}} \left[\left(1 + \frac{k-1}{2} x\right) \|P(z)\| + \frac{1}{2k^n} \left(k^{n-s} - 1 - (k-1)x \right) l m_k \right] \\ + (|\delta| - k) V_k(s, l) \left(1 + \frac{(k-1)}{2} x\right) \left(\|P(z)\| - \frac{1}{k^n} l m_k \right) \end{aligned}$$

is an increasing function of x. *Thus from Theorem 2.1, we get the following refinement as well as generalization of* (10).

Corollary 2.7. *If* $P(z) = z^s (a_0 + a_1 z + ... + a_{n-s} z^{n-s})$, $0 \le s \le n$, *is a polynomial of degree* n *having all zeros in* $|z| \le k,\ k \ge 1$, *then for any complex number* δ *with* $|\delta| \ge k$ *and* $0 \le l \le 1$,

$$\begin{aligned} \|D_\delta P(z)\| \ge \frac{n\,(|\delta| - k)}{1 + k^{n-s}} \left[\|P(z)\| + \frac{(k^{n-s} - 1)}{2k^n} l m_k \right] \\ + n \left(\frac{|\delta| + k}{2k^n} \right) l m_k + (|\delta| - k) V_k(s, l) \left(\|P(z)\| - \frac{1}{k^n} l m_k \right), \end{aligned} \tag{22}$$

where $V_k(s, l)$, $W_k(s, l)$ *and* m_k *are as defined in Theorem 2.1.*

If we divide both sides of inequality (22) by $|\delta|$ and let $|\delta| \to \infty$, we get a refinement of inequality (9).

3. Auxiliary Results

For the proof of the theorem, we shall make use of the following lemmas. The first lemma is a simple deduction from the Maximum Modulus Principle (see [21]).

Lemma 3.1. *If $P(z)$ is a polynomial of degree at most n, then for $R \geq 1$,*

$$\max_{|z|=R} |P(z)| \leq R^n \|P(z)\|.$$

The following lemma is due to Mir et al. [20].

Lemma 3.2. *If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for $R \geq 1$ and $0 \leq l \leq 1$, we have*

$$\max_{|z|=R} |P(z)| \leq \left(\frac{(1+R^n)(|a_0| + R|a_n| - lm_1)}{(1+R)(|a_0| + |a_n| - lm_1)} \right) \|P(z)\| - \left(\frac{(1+R^n)(|a_0| + R|a_n| - lm_1)}{(1+R)(|a_0| + |a_n| - lm_1)} - 1 \right) lm_1, \tag{23}$$

where $m_1 = \min_{|z|=1} |P(z)|$.
Equality in (23) *holds for $P(z) = \frac{a+bz^n}{2}$, $|a| = |b| = 1$.*

Lemma 3.3. *If $P(z) = z^s(a_0 + a_1 z + a_2 z^2 + ... + a_{n-s} z^{n-s})$, $0 \leq s \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then*

$$\max_{|z|=k} |P(z)| \geq \frac{2k^n}{1+k^{n-s}} \left\{ \left(1 + \frac{k-1}{2} W_k(s,l) \right) \|P(z)\| + \frac{1}{2k^n} \Big[k^{n-s} - 1 - (k-1) W_k(s,l) \Big] lm_k \right\}, \tag{24}$$

where $W_k(s,l)$ and m_k are as defined in Theorem 2.1. Equality in (24) *holds for $P(z) = z^n + k^n$.*

Proof of Lemma 3.3. Let $T(z) = P(kz)$. Since $P(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$, the polynomial $T(z)$ has all its zeros in $|z| \leq 1$. Let $H(z) = z^n T\left(\frac{1}{z}\right)$ be the reciprocal polynomial of $T(z)$, then $H(z)$ is a polynomial of degree $n-s$ having no zeros in $|z| < 1$. Hence applying (23) of Lemma 3.2 to the polynomial $H(z)$, we get for $k \geq 1$ and $0 \leq l \leq 1$,

$$\max_{|z|=k} |H(z)| \leq \frac{(1+k^{n-s})(k^n|a_{n-s}| + k^{s+1}|a_0| - lm^*)}{(1+k)(k^n|a_{n-s}| + k^s|a_0| - lm^*)} \|H(z)\| - \left(\frac{(1+k^{n-s})(k^n|a_{n-s}| + k^{s+1}|a_0| - lm^*)}{(1+k)(k^n|a_{n-s}| + k^s|a_0| - lm^*)} - 1 \right) lm^*, \tag{25}$$

where $m^* = \min_{|z|=1} |H(z)|$.
Since $|H(z)| = |T(z)|$ on $|z| = 1$, therefore,

$$m^* = \min_{|z|=1} |H(z)| = \min_{|z|=1} \left| z^n P\left(\frac{k}{z}\right) \right| = \min_{|z|=k} |P(z)| = m_k,$$

$$\max_{|z|=1} |H(z)| = \max_{|z|=1} |T(z)| = \max_{|z|=k} |P(z)|$$

and

$$\max_{|z|=k} |H(z)| = \max_{|z|=k} \left| z^n P\left(\frac{k}{z}\right) \right| = k^n \max_{|z|=1} |P(z)| = k^n \|P(z)\|.$$

The above when substituted in (25) gives

$$\max_{|z|=k}|P(z)| \geq \left(\frac{(1+k)(k^n|a_{n-s}|+k^s|a_0|-lm_k)}{(1+k^{n-s})(k^n|a_{n-s}|+k^s|a_0|-lm_k)}\right)k^n\|P(z)\| + \left(1-\frac{(1+k)(k^n|a_{n-s}|+k^s|a_0|-lm_k)}{(1+k^{n-s})(k^n|a_{n-s}|+k^{s+1}|a_0|-lm_k)}\right)lm_k. \tag{26}$$

Using the fact that

$$\frac{(1+k)(k^n|a_{n-s}|+k^s|a_0|-lm_k)}{(1+k^{n-s})(k^n|a_{n-s}|+k^{s+1}|a_0|-lm_k)} = \frac{2}{1+k^{n-s}} + \frac{(k^n|a_{n-s}|-k^s|a_0|-lm_k)(k-1)}{(1+k^{n-s})(k^n|a_{n-s}|+k^{s+1}|a_0|-lm_k)},$$

in (26), we get

$$\max_{|z|=k}|P(z)| \geq \frac{2k^n}{1+k^{n-s}}\left\{\left(1+\frac{k-1}{2}W_k(s,l)\right)\|P(z)\| + \frac{1}{2k^n}\Big[k^{n-s}-1-(k-1)W_k(s,l)\Big]lm_k\right\},$$

which is (24) and this completes the proof of Lemma 3.3.

The following lemma is due to Govil and Kumar [8].

Lemma 3.4. *If $P(z) = z^s(a_0 + a_1z + ... + a_{n-s}z^{n-s}),\ 0 \leq s \leq n$, is a polynomial of degree n having all zeros in $|z| \leq 1$, then for any complex number δ with $|\delta| \geq 1$,*

$$\|D_\delta P(z)\| \geq (|\delta|-1)\left\{\frac{n+s}{2} + \frac{(|a_{n-s}|-|a_0|)}{2(|a_{n-s}|+|a_0|)}\right\}\|P(z)\|.$$

Lemma 3.5. *If $P(z) = z^s(a_0 + a_1z + ... + a_{n-s}z^{n-s}),\ 0 \leq s \leq n$, is a polynomial of degree n having all zeros in $|z| \leq 1$, then for any complex number δ with $|\delta| \geq 1$ and $0 \leq l \leq 1$,*

$$\|D_\delta P(z)\| \geq \frac{n}{2}\Big((|\delta|-1)\|P(z)\| + (|\delta|+1)lm_1\Big) + \left(\frac{|\delta|-1}{2}\right)\left\{s + \frac{|a_{n-s}|-lm_1-|a_0|}{|a_{n-s}|-lm_1+|a_0|}\right\}(\|P(z)\|-lm_1),$$

where $m_1 = \min_{|z|=1}|P(z)|$.

Proof of Lemma 3.5. Recall that $P(z) = z^s(a_0 + a_1z + ... + a_{n-s}z^{n-s}),\ 0 \leq s \leq n$, has all its zeros in $|z| \leq 1$. If the polynomial $h(z) = a_0 + a_1z + ... + a_{n-s}z^{n-s}$ has a zero on $|z| = 1$, then $m_1 = \min_{|z|=1}|P(z)| = 0$ and the result follows by Lemma 3.4 in this case. Henceforth, we assume that all the zeros of $P(z) = z^sh(z)$ lie in $|z| < 1$, so that $m_1 > 0$. Therefore, we have $m_1 \leq |P(z)|$ for $|z| = 1$. This implies for any complex number γ with $|\gamma| < 1$, that

$$m_1|\gamma z^n| < |P(z)| \text{ for } |z| = 1.$$

Since all the zeros of $P(z)$ lie in $|z| < 1$, it follows by Rouché's theorem that all the zeros of

$$P(z) - \gamma m_1 z^n = z^s\Big(a_0 + a_1z + ... + (a_{n-s} - \gamma m_1)z^{n-s}\Big)$$

also lie in $|z| < 1$. Hence, by Lemma 3.4, we get for $|\delta| \geq 1$ and $|z| = 1$,

$$\left|D_\delta(P(z) - \gamma m_1 z^n)\right| \geq (|\delta| - 1)\left\{\frac{n+s}{2} + \frac{(|a_{n-s} - \gamma m_1| - |a_0|)}{2(|a_{n-s} - \gamma m_1| + |a_0|)}\right\} \times |P(z) - \gamma m_1 z^n|. \tag{27}$$

For every $\gamma \in \mathbb{C}$, we have

$$|a_{n-s} - \gamma m_1| \geq |a_{n-s}| - |\gamma| m_1,$$

and since the function $\frac{x-|a_0|}{x+|a_0|}$ is a non-decreasing function of x, it follows from (27) that for every γ with $|\gamma| < 1$ and $|z| = 1$,

$$\left|D_\delta P(z) - \gamma \delta n m_1 z^{n-1}\right| \geq (|\delta| - 1)\left\{\frac{n+s}{2} + \frac{(|a_{n-s}| - |\gamma| m_1 - |a_0|)}{2(|a_{n-s}| - |\gamma| m_1 + |a_0|)}\right\} \times |P(z) - \gamma m_1 z^n|. \tag{28}$$

It is a simple deduction of Laguerre theorem (see [11], p.52) on the polar derivative of a polynomial that for any δ with $|\delta| \geq 1$, the polynomial

$$D_\delta(P(z) - \gamma m_1 z^n) = D_\delta P(z) - \gamma \delta n m_1 z^{n-1}$$

has all its zeros in $|z| < 1$. This implies that

$$|D_\delta P(z)| \geq m_1 n |\delta| |z|^{n-1} \text{ for } |z| \geq 1. \tag{29}$$

Now choosing the argument of γ suitably on the left hand side of (28) such that

$$\left|D_\delta P(z) - \gamma \delta n m_1 z^{n-1}\right| = |D_\delta P(z)| - |\gamma||\delta| n m_1 \text{ for } |z| = 1,$$

which is possible by (29), we get

$$|D_\delta P(z)| - m_1 n |\gamma||\delta| \geq (|\delta| - 1)\left\{\frac{n+s}{2} + \frac{(|a_{n-s}| - |\gamma| m_1 - |a_0|)}{2(|a_{n-s}| - |\gamma| m_1 + |a_0|)}\right\} \times (||P(z)|| - |\gamma| m_1|) \text{ for } |z| = 1. \tag{30}$$

For γ with $|\gamma| = 1$, the above inequality follows by continuity. Putting $|\gamma| = l$ in (30), we get for $|z| = 1$,

$$||D_\delta P(z)|| \geq \frac{n}{2}\Big((|\delta| - 1)||P(z)|| + (|\delta| + 1) l m_1\Big) + \left(\frac{|\delta| - 1}{2}\right)\left\{s + \frac{|a_{n-s}| - l m_1 - |a_0|}{|a_{n-s}| - l m_1 + |a_0|}\right\}(||P(z)|| - l m_1),$$

and the proof of Lemma 3.5 is thus completed.

Lemma 3.6. *If $P(z) = z^s(a_0 + a_1 z + a_2 z^2 + ... + a_{n-s} z^{n-s})$, $0 \leq s \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$, $k > 0$, then*

$$|a_{n-s}| \geq \frac{m_k}{k^n},$$

where $m_k = \min_{|z|=k} |P(z)|$.

Proof of Lemma 3.6. Since $P(z) = z^s(a_0 + a_1 z + a_2 z^2 + ... + a_{n-s} z^{n-s})$ has all its zeros in $0 < |z| \leq k$, therefore, the reciprocal polynomial
$Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)} = \overline{a_0} z^{n-s} + \overline{a_1} z^{n-s-1} + ... + \overline{a_{n-s}}$ does not vanish in $|z| < 1/k$. We can assume without loss of generality that $Q(z)$ has no zeros on $|z| = 1/k$, for otherwise the result holds trivially. Since $Q(z)$ is analytic in $|z| \leq 1/k$, and has no zeros in $|z| \leq 1/k$, by the Minimum Modulus Principle,

$$\min_{|z|=\frac{1}{k}} |Q(z)| \leq |Q(z)| \text{ for } |z| \leq 1/k,$$

which implies

$$\frac{1}{k^n} \min_{|z|=k} |P(z)| \leq |Q(z)| \text{ for } |z| \leq 1/k,$$

which in particular implies

$$\frac{m_k}{k^n} \leq |Q(0)| = |a_{n-s}|,$$

which completes the proof of Lemma 3.6.

4. Proof of Main Result

Proof of Theorem 2.1. By hypothesis $P(z)$ has all its zeros in $|z| \leq k,\ k \geq 1$, therefore, all the zeros of the polynomial $J(z) = P(kz)$ lie in $|z| \leq 1$. Since $J(z)$ is a polynomial of degree n and having all zeros in $|z| \leq 1$, therefore applying Lemma 3.5 to the polynomial $J(z)$ and noting that $\frac{|\delta|}{k} \geq 1$, we get for $|z| = 1$,

$$\begin{aligned}\left|D_{\frac{\delta}{k}} J(z)\right| \geq \frac{n}{2}&\left\{\left(\frac{|\delta|}{k} - 1\right)\|J(z)\| + \left(\frac{|\delta|}{k} + 1\right) l m^*\right\} \\ &+ \left(\frac{|\delta|}{k} - 1\right)\left\{\frac{s}{2} + \frac{(k^{n-s}|a_{n-s}| - |a_0| - lm^*)}{2(k^{n-s}|a_{n-s}| + |a_0| - lm^*)}\right\} \\ &\times (\|J(z)\| - lm^*)\,, \end{aligned} \tag{31}$$

where $m^* = \min_{|z|=1} |J(z)| = \min_{|z|=1} |P(kz)| = \min_{|z|=k} |P(z)| = m_k$.
The above inequality (31) is equivalent to

$$\begin{aligned}&\left\|nP(kz) + \left(\frac{\delta}{k} - z\right) kP'(kz)\right\| \\ &\quad\geq \frac{n}{2}\left\{\left(\frac{|\delta| - k}{k}\right)\|P(kz)\| + \left(\frac{|\delta| + k}{k}\right) l m_k\right\} \\ &\quad+ \left(\frac{|\delta| - k}{k}\right)\left\{\frac{s}{2} + \frac{(k^{n-s}|a_{n-s}| - |a_0| - lm_k)}{2(k^{n-s}|a_{n-s}| + |a_0| - lm_k)}\right\}(\|P(kz)\| - lm_k)\,. \end{aligned}$$

The last inequality yields

$$\begin{aligned}\max_{|z|=k} |D_\delta P(z)| \geq \frac{n}{2}&\left\{\left(\frac{|\delta| - k}{k}\right)\max_{|z|=k} |P(z)| + \left(\frac{|\delta| + k}{k}\right) l m_k\right\} \\ &+ \left(\frac{|\delta| - k}{k}\right)\left\{\frac{s}{2} + \frac{(k^{n-s}|a_{n-s}| - |a_0| - lm_k)}{2(k^{n-s}|a_{n-s}| + |a_0| - lm_k)}\right\} \\ &\times \left(\max_{|z|=k} |P(z)| - lm_k\right). \end{aligned} \tag{32}$$

Since $D_\delta P(z)$ is a polynomial of degree at most $n-1$, we have by Lemma 3.1 for $R = k \geq 1$,

$$\max_{|z|=k} |D_\delta P(z)| \leq k^{n-1} \|D_\delta P(z)\|,$$

which gives by using the inequality (32) and Lemma 3.3, that

$$\begin{aligned}
&k^{n-1}\|D_\delta P(z)\| \geq \max_{|z|=k} |D_\delta P(z)| \\
&\geq \frac{n}{2}\left(\frac{|\delta|-k}{k}\right)\frac{2k^n}{1+k^{n-s}}\Big\{\Big(1+\frac{k-1}{2}W_k(s,l)\Big)\|P(z)\| \\
&+\frac{1}{2k^n}\Big[k^{n-s}-1-(k-1)W_k(s,l)\Big]lm_k\Big\} + n\left(\frac{|\delta|+k}{2k}\right)lm_k \\
&+\left(\frac{|\delta|-k}{k}\right)\left\{\frac{s}{2}+\frac{(k^{n-s}|a_{n-s}|-|a_0|-lm_k)}{2(k^{n-s}|a_{n-s}|+|a_0|-lm_k)}\right\} \\
&\times\left[\frac{2k^n}{1+k^{n-s}}\Big\{\Big(1+\frac{k-1}{2}W_k(s,l)\Big)\|P(z)\|\right. \\
&\left.+\frac{1}{2k^n}\Big[k^{n-s}-1-(k-1)W_k(s,l)\Big]lm_k\Big\} - lm_k\right],
\end{aligned} \tag{33}$$

where

$$W_k(s,l) = \frac{k^n|a_{n-s}| - k^s|a_0| - lm_k}{k^n|a_{n-s}| + k^{s+1}|a_0| - lm_k}.$$

Inequality (33) after simplification is equivalent to

$$\begin{aligned}
&\|D_\delta P(z)\| \\
&\geq \frac{n\,(|\delta|-k)}{1+k^{n-s}}\left[\Big(1+\frac{(k-1)}{2}W_k(s,l)\Big)\|P(z)\| + \frac{1}{2k^n}\Big(k^{n-s}-1-(k-1)W_k(s,l)\Big)lm_k\right] \\
&+ n\left(\frac{|\delta|+k)}{2k^n}\right)lm_k + (|\delta|-k)V_k(s,l)\Big(1+\frac{(k-1)}{2}W_k(s,l)\Big)\Big(\|P(z)\| - \frac{1}{k^n}lm_k\Big),
\end{aligned}$$

which is exactly (15), where

$$V_k(s,l) = \left\{\frac{s}{1+k^{n-s}} + \frac{(k^{n-s}|a_{n-s}|-|a_0|-lm_k)}{1+k^{n-s}(k^{n-s}|a_{n-s}|+|a_0|-lm_k)}\right\}.$$

This completes the proof of Theorem 2.1.

5. Acknowledgements

The authors would like to express their gratitude to the anonymous referee for carefully reading the work and offering some insightful recommendations.

References

[1] A. Aziz and Q. M. Dawood, *Inequalities for a polynomial and its derivative*, J. Approx. Theory, **54** (1988), 306-313.
[2] A. Aziz and N. A. Rather, *A refinement of a theorem of Paul Turán concerning polynomials*, Math. Inequal. Appl., **1** (1998), 231-238.
[3] S. Bernstein, *Sur l'ordre de la meilleure approximation des functions continues par des polynômes de degré donné*, Mem. Acad. R. Belg., **4** (1912), 1-103.

[4] K. K. Dewan, N. Singh, A. Mir and A. Bhat, *Some inequalities for the polar derivative of a polynomial,* Southeast Asian Bull. Math., **34** (2010), 69-77.
[5] R. B. Gardner, N. K. Govil, G. V. Milovanović, *Extremal Problems and Inequalities of Markov-Bernstein Type for Algebraic Polynomials,* Elsevier/Academic Press, London (2022).
[6] N. K. Govil, *On the derivative of a polynomial,* Proc. Amer. Math. Soc., **41** (1973), 543-546.
[7] N. K. Govil, *Some inequalities for derivative of polynomials*, J. Approx. Theory, **66** (1991), 29-35.
[8] N. K. Govil and P. Kumar, *On sharpening of an inequality of Turán,* Appl. Anal. Discrete Math., **13** (2019), 711-720.
[9] A. Hussain, A. Mir, A. Ahmad, *On Bernstein-type inequalities for polynomials involving the polar derivative,* J. Classical Anal. **16** (2020), 9-15.
[10] P. Kumar and R. Dhankhar, *Some refinements of inequalities for polynomials,* Bull. Math. Soc. Sci. Math. Roumanie, **63** (111) (2020), 359-367.
[11] M. Marden, *Geometry of Polynomials,* Math. Surveys, **3** (1966).
[12] G. V. Milovanović, A. Mir and A. Hussain, *Extremal problems of Bernstein-type and an operator preserving inequalities between polynomials*, Siberian Math. J., **63** (2022), 138-148.
[13] G. V. Milovanović, D. S. Mitrinović and T. M. Rassias, *Topics in Polynomials, Extremal Problems, Inequalities, Zeros,* World Scientific, Singapore (1994).
[14] A. Mir, *On an operator preserving inequalities between polynomials*, Ukrainian Math. J., **69** (2018), 1234-1247.
[15] A. Mir, *Bernstein type integral inequalities for a certain class of polynomials*, Mediterranean J. Math., **16** (2019), (Art. 143) pp. 1-11.
[16] A. Mir, *Generalizations of Bernstein and Turán-type inequalities for the polar derivative of a complex polynomial*, Mediterranean J. Math., **17** (2020), (Art. 14) pp. 1-12.
[17] A. Mir, *A note on an inequality of Paul Turán concerning polynomials*, Ramanujan J., **56** (2021), 1061-1071.
[18] A. Mir, D. Breaz, *Bernstein and Turán-type inequalities for a polynomial with constraints on its zeros*, RACSAM, **115** (2021), (Art. 124) pp. 1-12.
[19] A. Mir, A. Hussain, *Generalizations of some Bernstein-type inequalities for the polar derivative of a polynomial,* Kragjevac J. Math., **49** (2025), 31-41.
[20] A. Mir, A. Wani and I. Hussain, *A note on Ankeny-Rivlin theorem*, J. Anal., **27** (2019), 1103-1107.
[21] G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis,* Springer-Verlag, Berlin (1925).
[22] Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials,* Oxford University Press (2002).
[23] P. Turán, *Über die Ableitung von Polynomen,* Compositio Math., **7** (1939), 89-95.