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Nonsmooth pseudo-linear vector optimization and vector variational inequalities on Hadamard manifolds

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Abstract. In this paper, we present geodesic pseudo-linear function and pseudo-affine operator defined on Hadamard manifolds involving Clarke sub-differential and study some characterizations of the function mentioned above. We consider a nonsmooth pseudo-linear vector optimization problem and generalized vector variational inequality problems on Hadamard manifolds and establish relationships between solutions to these considered problems. Moreover, a set-valued gap function investigates for a nonsmooth pseudo-linear vector optimization problem. A numerical example is constructed to evaluate the result of this paper.

1. Introduction

In the optimization theory, the convexity and its generalizations play a vital role to get an optimal solution. Mangasarian [20] has generalized the convex and concave functions, which are known as pseudo-convex and pseudo-concave functions, respectively. Kortanek and Evans [15] investigated the properties of the functions, namely pseudo-linear functions, which are both pseudo-convex and pseudo-concave. While from the last two decades, the convexity and its generalizations have studied on Riemannian or Hadamard manifolds. A manifold is not a linear space in this setting the linear space and line segment replaced by a Riemannian manifold and a geodesic, respectively. Rapcsak [8] and Udriste [24] presented the concept of geodesic convexity on Riemannian manifolds. Barnani [2] proposed the geodesic convexity in terms of Clarke sub-differentiable on Hadamard manifolds.

The variational inequality works as a powerful tool in the study of the optimization problem. It provides the necessary and sufficient conditions for a solution to the optimization problem in the presence of convexity. Firstly, it presented by Hartman and Stampacchia [10] in their seminal paper. Since variational inequality has various applications in basic sciences, economics and management sciences so that it becomes much popular among researchers. Giannessi [9] extended the classical Stampacchia variational inequality for vector-valued functions. Thence lots of research works happened in this area by numerous authors (see, for example, [14, 16–18, 23, 25]) and references therein.

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Last few years, variational inequalities and its generalization have been studied from Euclidean spaces to Riemannian or Hadamard manifolds. Nemeth [21] considered the variational inequalities and derived the existence and uniqueness results for their solutions on Hadamard manifolds. Zhou and Huang [26] obtained the relationships between a solution of a vector variational inequality and a vector optimization problem on Hadamard manifolds under differentiable geodesic convex function. Moreover, various authors established the research works related to the variational inequalities on Hadamard manifolds for details refer to [1, 5, 6, 13, 19].

Inspire from the above-described works the main consequence of this paper is to study the nonsmooth pseudo-linear vector optimization problem on Hadamard manifolds. The work plan for this paper is organized as follows. Section 2 contains the basic concepts, notations, known definitions and results, which helps to investigate the results of this paper. We introduce the notion of geodesic pseudo-linearity and pseudo-affiness on Hadamard manifolds in terms of Clarke sub-differential and study the properties for a class of pseudo-linear function and pseudo-affine operator in Section 3. Further, Section 4 contains two subsections, in the first subsection, we consider a nonsmooth pseudo-linear vector optimization problem and generalized vector variational inequality problems on Hadamard manifolds. And we also establish relationships among considered problems. On the other subsection, we investigate a set-valued gap function for a nonsmooth pseudo-linear vector optimization problem. Ultimately, in Section 5, we conclude the results of this paper.

2. Notations and preliminaries

We start with some basic concepts, notations about Riemannian manifolds which will use to prove the results of this paper. Let $\mathcal{M}$ be an m-dimensional Riemannian manifold endowed with Riemannian metric g. Assume that $T_x\mathcal{M}$ is the m-dimensional tangent space at $x \in \mathcal{M}$. The collection of all tangent space is called the tangent bundle of $\mathcal{M}$, i.e., $T\mathcal{M} = \bigcup_{x\in\mathcal{M}} T_x\mathcal{M}$. Let $\langle .,. \rangle$ be a scalar product on $T_x\mathcal{M}$ and the corresponding norm denoted by $\|.\|$. We denote the length of piecewise smooth curve $\Omega_{x,z} : [a,b] \to \mathcal{M}$ connecting two points x to z by $L(\Omega_{x,z}) = \int_a^b \|\dot{\Omega}_{x,z}(t)\|dt$. The Riemannian distance $d(x,z) = \inf\{L(\Omega_{x,z}) :$ $\Omega_{x,z}$ is a piecewise smooth curve joining the points x to $z\}$. We define the open ball with radius $\epsilon > 0$ centred at the point z as, $B(z,\epsilon) = \{x \in \mathcal{M} : d(x,z) < \epsilon\}$. Let ∇ be the Levi-Civita connection associated to $\mathcal{M}$. For any two points $x, z \in \mathcal{M}$, a curve $\Omega_{x,z} : [0,1] \to \mathcal{M}$ joining x to z is said to be a geodesic if $\Omega_{x,z}(0) = x$, $\Omega_{x,z}(1) = z$ and $\nabla_{\dot{\Omega}_{x,z}}\dot{\Omega}_{x,z} = 0$ on $[0,1]$. The exponential map $exp_x : T_x\mathcal{M} \to \mathcal{M}$ at x is defined by $exp_x(w) = \Omega_w(1,x)$ for each $w \in T_x\mathcal{M}$, where $\Omega(\cdot) = \Omega_w(\cdot,x)$ is the geodesic starting x with velocity w, that is, $\Omega(0) = x$ and $\dot{\Omega}(0) = w$. It is easy to see that $exp_x(tw) = \Omega_w(t,x)$ for each real number t. The exponential map has inverse $exp_x^{-1} : \mathcal{M} \to T_x\mathcal{M}$, i.e., $w = exp_x^{-1}z$.

A Hadamard manifold $\mathcal{M}$ is a complete simply connected Riemannian manifold with non-positive sectional curvature. If $\mathcal{M}$ is a Hadamard manifold, then $exp_x : T_x\mathcal{M} \to \mathcal{M}$ is a diffeomorphism for every $x \in \mathcal{M}$ and if $x, z \in \mathcal{M}$, then there exists a unique minimal geodesic joining x to z. In the present paper, throughout we assume that $\mathcal{M}$ is a Hadamard manifold.
For $x, z \in \mathbb{R}^q$, the following ordering for vectors in $\mathbb{R}^q$ will be adopted:

(i) $x \leq_q z \Leftrightarrow x - z \in -\mathbb{R}^q_+\backslash\{0\}$,

(ii) $x \nleq_q z \Leftrightarrow x - z \notin -\mathbb{R}^q_+\backslash\{0\}$,

where $\mathbb{R}^q_+ = \{x \in \mathbb{R}^q | x_i \geq 0, i = 1, \ldots, q\}$.

Definition 2.1. [24] *A subset $\mathcal{S}$ of $\mathcal{M}$ is said to be geodesic convex set if for any points $x, z \in \mathcal{S}$, the geodesic joining x to z is contained in $\mathcal{S}$, that is, if $\Omega_{x,z} : [0,1] \to \mathcal{S}$ is a geodesic such that $\Omega_{x,z}(0) = x$ and $\Omega_{x,z}(1) = z$, then $\Omega_{x,z}(t) = exp_x t exp_x^{-1} z \in \mathcal{S}$, $\forall\, t \in [0,1]$.*

Definition 2.2. [3] *Let $\mathcal{S} \subset \mathcal{M}$ be an open geodesic convex set. A function $h : \mathcal{S} \to \mathbb{R}$ is said to be locally Lipschitz on $\mathcal{S}$ if for all $y \in \mathcal{S}$, there exists a $l \geq 0$ such that*

$$|h(x) - h(z)| \leq l d(x,z), \quad \forall\, x, z \in B(y,\epsilon),$$

where l is called the Lipschitz constant of h in the neighbourhood of y.

Definition 2.3. [3] *Let $\mathcal{S} \subset \mathcal{M}$ be an open geodesic convex set and $h : \mathcal{S} \to \mathbb{R}$ be a locally Lipschitz function on $\mathcal{S}$. The generalized directional derivative $h^o(z; v)$ of h at $z \in \mathcal{S}$ in the direction $v \in T_z\mathcal{M}$, is defined as*

$$h^o(z; v) = \lim_{x \to z} \sup_{t \downarrow 0} \frac{h(exp_x t(d\, exp_z)_{exp_z^{-1}x} v) - h(x)}{t},$$

where $(d\, exp_z)_{exp_z^{-1}x} : T_{exp_z^{-1}x}(T_z\mathcal{M}) \simeq T_z\mathcal{M} \to T_x\mathcal{M}$ is the differential of exponential mapping at $exp_z^{-1}x$.

Definition 2.4. [3] *Let $\mathcal{S} \subset \mathcal{M}$ be an open geodesic convex set and $h : \mathcal{S} \to \mathbb{R}$ be a locally Lipschitz function on $\mathcal{S}$. Then Clarke sub-differential (or generalized gradient) of h at $z \in \mathcal{S}$, denoted by $\partial_c h(z)$ is defined by*

$$\partial_c h(z) = \{\zeta \in T_z\mathcal{M} : h^o(z; v) \geq \langle \zeta,\ v\rangle,\ \ \forall\, v \in T_z\mathcal{M}\}.$$

Definition 2.5. [5] *Let $\mathcal{S} \subset \mathcal{M}$ be an open geodesic convex set and $h : \mathcal{S} \to \mathbb{R}$ be a locally Lipschitz function on $\mathcal{S}$. Then the function h is said to be geodesic pseudo-convex if for all $x, z \in \mathcal{M}$,*

$$\exists\ \mu \in \partial_c h(x) : \langle \mu, exp_x^{-1}z\rangle \geq 0 \Rightarrow h(z) \geq h(x),$$

equivalently,

$$h(z) < h(x) \Rightarrow \langle \mu, exp_x^{-1}z\rangle < 0,\ \forall\ \mu \in \partial_c h(x).$$

Definition 2.6. [5] *Let $\mathcal{S} \subset \mathcal{M}$ be an open geodesic convex set and $h : \mathcal{S} \to \mathbb{R}$ be a locally Lipschitz function on $\mathcal{S}$. Then the function h is said to be geodesic quasi-convex if for all $x, z \in \mathcal{M}$,*

$$h(z) \leq h(x) \Rightarrow \langle \mu, exp_x^{-1}z\rangle \leq 0,\ \forall\ \mu \in \partial_c h(x),$$

equivalently,

$$\exists\ \mu \in \partial_c h(x) : \langle \mu, exp_x^{-1}z\rangle > 0 \Rightarrow h(z) > h(x).$$

Lemma 2.7. [5] *Let $\mathcal{S} \subset \mathcal{M}$ be an open geodesic convex set. Let $h : \mathcal{S} \to \mathbb{R}$ be a locally Lipschitz and geodesic pseudo-convex function on $\mathcal{S}$. Then h is geodesic quasi-convex function on $\mathcal{S}$.*

Definition 2.8. [5] *Let $\mathcal{S} \subset \mathcal{M}$ be an open geodesic convex set and $h : \mathcal{S} \to \mathbb{R}$ be a locally Lipschitz function on $\mathcal{S}$. Then $\partial_c h$ is said to be pseudo-monotone on $\mathcal{S}$, if for all $x, z \in \mathcal{S}$ and $\mu \in \partial_c h(x)$,*

$$\langle \mu,\ exp_x^{-1}z\rangle \geq 0 \Rightarrow \langle v,\ exp_z^{-1}x\rangle \leq 0,\ \forall\ v \in \partial_c h(z).$$

Lemma 2.9. (Mean Value Theorem) [2] *Let $\mathcal{S} \subset \mathcal{M}$ be an open geodesic convex set and $h : \mathcal{S} \to \mathbb{R}$ be a locally Lipschitz function on $\mathcal{S}$. Then, for all $x, z \in \mathcal{S}$ there exist $t_0 \in (0, 1)$ and $\mu_0 \in \partial_c h(\Omega_{x,z}(t_0))$ such that*

$$f(x) - f(z) = \langle \mu_0,\ \dot{\Omega}_{x,z}(t_0)\rangle,$$

where $\Omega_{x,z}(t) = exp_x(t exp_x^{-1}z)$, for all $t \in [0, 1]$.

Proposition 2.10. [12] *Let $\mathcal{M}$ and $\mathcal{N}$ be two Hadamard manifold, $F : \mathcal{N} \to \mathcal{M}$ be continuously differentiable near x and $h : \mathcal{M} \to \mathbb{R}$ be Lipschitz near $F(x)$. Then $g = hoF$ is Lipschitz near x and we have*

$$\partial g(x) \subseteq dF(x)\ o\ \partial h(F(x)).$$

If $dF(x) : T_x\mathcal{N} \to T_{F(x)}\mathcal{M}$ is onto, then the equality holds.

3. Geodesic pseudo-linearity and pseudo-affiness

In this section, we define the pseudo-linear function and pseudo-affine operator in terms of Clarke sub-differential on Hadamard manifold and discuss some of its characterizations.

Definition 3.1. *Let $\mathcal{S} \subset \mathcal{M}$ be an open geodesic convex set and $h : \mathcal{S} \to \mathbb{R}$ be a locally Lipschitz function on $\mathcal{S}$. Then the function h is said to be geodesic pseudo-linear at $x \in \mathcal{S}$, if for all $z \in \mathcal{S}$,*

$$\exists\, \mu \in \partial_c h(x) : \langle \mu,\ exp_x^{-1} z\rangle \geq 0 \Rightarrow h(z) \geq h(x),$$

and

$$\exists\, \mu \in \partial_c h(x) : \langle \mu,\ exp_x^{-1} z\rangle \leq 0 \Rightarrow h(z) \leq h(x).$$

A function h is said to be geodesic pseudo-linear if it is both geodesic pseudo-convex and geodesic pseudo-concave.

Example 3.2. *Let $\mathcal{M} = \mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ be endowed with the Riemannian metric defined by $g(x) = x^{-2}$. Clearly, $\mathcal{M}$ is a Hadamard manifold and the tangent space $T_x\mathcal{M}$ at $x \in \mathcal{M}$ is equal to $\mathbb{R}$. The geodesic curve $\Omega : \mathbb{R} \to \mathcal{M}$ satisfying $\Omega(0) = x \in \mathcal{M}$ and $\dot{\Omega}(0) = w \in T_x\mathcal{M}$ is given by $\Omega(t) = xe^{(w/x)t}$. Thus, $exp_x(tw) = xe^{(w/x)t}$ and is followed by $exp_x^{-1} z = x \ln\left(\frac{z}{x}\right)$. Let $\mathcal{S} = \{x | x = e^t, t \in [0, 1]\} \subset \mathcal{M}$ be a geodesic convex set. We consider the function $h(x) = |x - \sqrt{2}| + x - \sqrt{2}$ and obtain $\partial_c h(\sqrt{2}) = [0, 4]$. Evidently, the function h is a geodesic pseudo-linear function at $x = \sqrt{2}$ on S.*

Definition 3.3. *Let $\mathcal{S} \subset \mathcal{M}$ be an open geodesic convex set and $h : \mathcal{S} \to \mathbb{R}$ be a locally Lipschitz function on $\mathcal{S}$. Then $\partial_c h$ is said to be pseudo-affine on $\mathcal{S}$, if for all $x, z \in \mathcal{S}$ and $\mu \in \partial_c h(x)$,*

$$\langle \mu,\ exp_x^{-1} z\rangle \geq 0 \Rightarrow \langle \upsilon,\ exp_z^{-1} x\rangle \leq 0,\ \forall\, \upsilon \in \partial_c h(z),$$

and

$$\langle \mu,\ exp_x^{-1} z\rangle \leq 0 \Rightarrow \langle \upsilon,\ exp_z^{-1} x\rangle \geq 0,\ \forall\, \upsilon \in \partial_c h(z).$$

A set $\partial_c h$ is said to be a pseudo-affine if $\partial_c h$ and $-\partial_c h$ are both pseudo-monotone.

Theorem 3.4. *Let $\mathcal{S} \subset \mathcal{M}$ be an open geodesic convex set. If $h : \mathcal{S} \to \mathbb{R}$ is a locally Lipschitz and geodesic pseudo-linear function on $\mathcal{S}$, then $h(x) = h(z)$ if and only if there exists $\mu \in \partial_c h(x)$ such that $\langle \mu,\ exp_x^{-1} z\rangle = 0,\ \forall\, x, z \in \mathcal{S}$.*

Proof. Assume that for any $x, z \in \mathcal{S}$, there exists $\mu \in \partial_c h(x)$ such that

$$\langle \mu,\ exp_x^{-1} z\rangle = 0.$$

By the definition of geodesic pseudo-linearity of h, we obtain

$$h(x) = h(z).$$

Conversely, for any $x, z \in \mathcal{S}$, we assume that $h(x) = h(z)$. We have to prove that there exists $\mu \in \partial_c h(x)$ such that $\langle \mu,\ exp_x^{-1} z\rangle = 0$. Firstly, we will show that

$$h(\Omega_{x,z}(t)) = h(x),\ \forall\, t \in [0, 1].$$

It is obvious, when $t = 0$ and $t = 1$. Now, we will prove it for $t \in (0, 1)$. If $h(\Omega_{x,z}(t)) > h(x)$, then by a geodesic pseudo-convexity of h, we have

$$\langle \mu',\ exp^{-1}_{\Omega_{x,z}(t)} x\rangle < 0,\ \forall\, \mu' \in \partial_c h(\Omega_{x,z}(t)). \tag{1}$$

Since, $exp^{-1}_{\Omega_{x,z}(t)} z = -t\dot{\Omega}_{x,z}(t)$ and $exp^{-1}_{\Omega_{x,z}(t)} x = (1 - t)\dot{\Omega}_{x,z}(t)$, we obtain

$$exp^{-1}_{\Omega_{x,z}(t)} x = -\frac{(1 - t)}{t} exp^{-1}_{\Omega_{x,z}(t)} z. \tag{2}$$

By using the relation (2) in inequality (1), we get

$$\langle \mu',\ exp^{-1}_{\Omega_{x,z}(t)} z\rangle > 0,\ \forall\ \mu' \in \partial_c h(\Omega_{x,z}(t)).$$

From the geodesic pseudo-convexity of h and Lemma 2.7, we have

$$h(z) > h(\Omega_{x,z}(t)),$$

which contradicts the fact $h(\Omega_{x,z}(t)) > h(x) = h(z)$.
Again, if $h(\Omega_{x,z}(t)) < h(x)$, proceeding as above, we get

$$h(x) = h(z) < h(\Omega_{x,z}(t)).$$

Hence, for all $t \in [0,1]$, $h(\Omega_{x,z}(t)) = h(x)$.
Now, we consider a geodesic $\beta_{x,\Omega_{x,z}(t)} : [0,1] \to \mathcal{S}$ joining the points x and $\Omega_{x,z}(t)$, can be reparametrized in terms of a geodesic $\Omega_{x,z}$ as $\beta_{x,\Omega_{x,z}(t)}(s) = \Omega_{x,z}(st)$, $\forall\ s \in [0,1]$. By Lemma 2.9, there exist $\bar{t} \in (0,1)$ and $\mu_{\bar{t}} \in \partial_c h(\beta_{x,\Omega_{x,z}(t)}(\bar{t}))$ such that

$$0 = h(x) - h(\Omega_{x,z}(t)) = \langle \mu_{\bar{t}},\ \dot{\beta}_{x,\Omega_{x,z}(t)}(\bar{t})\rangle = t\langle \mu_{\bar{t}},\ \dot{\Omega}_{x,z}(b)\rangle,$$

where $b = \bar{t}t$ and $\dot{\beta}_{x,\Omega_{x,z}(t)}(\bar{t}) = t\dot{\Omega}_{x,z}(b)$.
Hence,

$$\langle \mu_{\bar{t}},\ \dot{\Omega}_{x,z}(b)\rangle = 0, \text{ for some } \mu_{\bar{t}} \in \partial_c h(\beta_{x,\Omega_{x,z}(t)}(\bar{t})). \tag{3}$$

Since $exp^{-1}_{\Omega_{x,z}(b)} z = -b\dot{\Omega}_{x,z}(b)$ and $exp^{-1}_{\Omega_{x,z}(b)} z = -b exp^{-1}_x z$, we obtain $\dot{\Omega}_{x,z}(b) = exp^{-1}_x z$ and thus from (3), we have

$$\langle \mu_{\bar{t}},\ exp^{-1}_x z\rangle = 0. \tag{4}$$

Without loss of generality, we may assume that $\lim_{\bar{t}\to 0} \mu_{\bar{t}} = \mu$, since $\beta_{x,\Omega_{x,z}(t)}(\bar{t}) \to x$ as $\bar{t} \to 0$, from (4), it follows that there exists $\mu \in \partial_c h(x)$ such that

$$\langle \mu,\ exp^{-1}_x z\rangle = 0.$$

This completes the proof. □

Theorem 3.5. *Let $\mathcal{S} \subset \mathcal{M}$ be an open geodesic convex set and $h : \mathcal{S} \to \mathbb{R}$ be a locally Lipschitz on $\mathcal{S}$. Then the function h is a geodesic pseudo-linear on $\mathcal{S}$, if and only if there exists a function $\mathcal{P} : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_{++}$ such that for all $x, z \in \mathcal{S}$ and for some $\mu \in \partial_c h(x)$,*

$$h(z) = h(x) + \mathcal{P}(x,z)\langle \mu,\ exp^{-1}_x z\rangle, \tag{5}$$

where $\mathbb{R}_{++}$ denotes the set of all positive real numbers.

Proof. Let h be a geodesic pseudo-linear function on $\mathcal{S}$. We construct a function $\mathcal{P} : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_{++}$ such that for all $x, z \in \mathcal{S}$ and for some $\mu \in \partial_c h(x)$,

$$h(z) = h(x) + \mathcal{P}(x,z)\langle \mu,\ exp^{-1}_x z\rangle.$$

If $\langle \mu,\ exp^{-1}_x z\rangle = 0$, for some $\mu \in \partial_c h(x)$ and for any $x, z \in \mathcal{S}$, we define $\mathcal{P}(x,z) = 1$. By Theorem 3.4, we get $h(z) = h(x)$ and thus, (5) holds.
Now, if $\langle \mu,\ exp^{-1}_x z\rangle \neq 0$, then again by Theorem 3.4, we obtain $h(z) \neq h(x)$ and we can define

$$\mathcal{P}(x,z) = \frac{h(z) - h(x)}{\langle \mu,\ exp^{-1}_x z\rangle}. \tag{6}$$

It remain to show $\mathcal{P}(x,z) > 0$, for all $x, z \in \mathcal{S}$.
If $h(z) > h(x)$, then by geodesic pseudo-concavity of h, we have

$$\langle \mu,\ exp_x^{-1}z\rangle > 0,\ \forall\ \mu \in \partial_c h(x).$$

Thus, from the equation (6), we get $\mathcal{P}(x,z) > 0$, for all $x, z \in \mathcal{S}$.
Similarly, if $h(z) < h(x)$, then by using the geodesic pseudo-convexity of h, we obtain the result.

Conversely, assume that for any $x, z \in \mathcal{S}$, there exists a function $\mathcal{P} : \mathcal{S}\times\mathcal{S} \to \mathbb{R}_{++}$ such that for all $x, z \in \mathcal{S}$, there exists $\mu \in \partial_c h(x)$ such that (5) holds.
If for some $\mu \in \partial_c h(x)$ and for any $x, z \in \mathcal{S}$, $\langle \mu,\ exp_x^{-1}z\rangle \geq 0$, from inequality (5), it follows that

$$h(z) - h(x) = \mathcal{P}(x,z)\langle \mu,\ exp_x^{-1}z\rangle \geq 0,$$

and hence, the function h is a geodesic pseudo-convex on $\mathcal{S}$.
Similarly, if $\langle \mu,\ exp_x^{-1}z\rangle \leq 0$, for some $\mu \in \partial_c h(x)$ and for any $x, z \in \mathcal{S}$, from inequality (5), it follows that function h is geodesic pseudo-concave on $\mathcal{S}$. □

Theorem 3.6. *Let $\mathcal{S} \subset \mathcal{M}$ be an open geodesic convex set. Let $h : \mathcal{S} \to \mathbb{R}$ be a locally Lipschitz and geodesic pseudo-linear function on $\mathcal{S}$. Suppose that $F : \mathbb{R} \to \mathbb{R}$ is a differential and onto function with $dF(\rho) > 0$ or $dF(\rho) < 0$, for all $\rho \in \mathbb{R}$. Then the composite function $g = Foh$ is also geodesic pseudo-linear on $\mathcal{S}$.*

Proof. Let $g(x) = (Foh)(x)$, for all $x \in \mathcal{S}$ and let $dF(\rho) > 0$, for all $\rho \in \mathbb{R}$. By Proposition 2.10, we get

$$g^o(x; exp_x^{-1}z) = dF(h(x))\ h^o(x; exp_x^{-1}z).$$

Thus for all $\zeta \in \partial_c g(x)$, we have

$$\langle \zeta,\ exp_x^{-1}z\rangle \geq 0 \Rightarrow \langle dF(h(x))\mu,\ exp_x^{-1}z\rangle \geq 0,\ \forall\ \mu \in \partial_c h(x),$$

$$\langle \zeta,\ exp_x^{-1}z\rangle \leq 0 \Rightarrow \langle dF(h(x))\mu,\ exp_x^{-1}z\rangle \leq 0,\ \forall\ \mu \in \partial_c h(x).$$

Since $dF(\rho) > 0$, for all $\rho \in \mathbb{R}$. Then,

$$\langle \zeta,\ exp_x^{-1}z\rangle \geq 0 \Rightarrow \langle \mu,\ exp_x^{-1}z\rangle \geq 0,\ \forall\ \mu \in \partial_c h(x),$$

$$\langle \zeta,\ exp_x^{-1}z\rangle \leq 0 \Rightarrow \langle \mu,\ exp_x^{-1}z\rangle \leq 0,\ \forall\ \mu \in \partial_c h(x).$$

Using the geodesic pseudo-linearity of h, it follows that

$$h(z) \geq h(x),$$

$$h(z) \leq h(x).$$

Since F is strictly increasing, we get

$$g(z) \geq g(x),$$

$$g(z) \leq g(x)).$$

Similarly, the result follows when $dF(\rho) < 0$, for all $\rho \in \mathbb{R}$.
Hence, g is a geodesic pseudo-linear function on $\mathcal{S}$. □

Proposition 3.7. *Let $\mathcal{S} \subset \mathcal{M}$ be an open geodesic convex set. Let $h : \mathcal{S} \to \mathbb{R}$ is a locally Lipschitz and geodesic pseudo-linear function on $\mathcal{S}$. Assume that $x \in \mathcal{S}$ be arbitrary. Then for any $\mu, \mu^* \in \partial_c h(x)$ there exists $\rho > 0$ such that $\mu^* = \rho\mu$.*

Proof. If $\mu = 0$, then for any $z \in \mathcal{S}$, we have

$$\langle \mu,\ exp_x^{-1} z\rangle = 0, \text{ hence } h(z) = h(x).$$

Consequently, h is a constant function on $\mathcal{S}$. It means $\partial_c h(x) = \{0\}$, so that $\mu = \mu^* = 0$ and we can take $\rho = 1$. The same argument is valid if $\mu^* = 0$.
Now, if $\mu \neq 0$ and $\mu^* \neq 0$. Assume that there does not exist any $\rho > 0$ such that $\mu^* = \rho\mu$. Then there exists $w \in T_x\mathcal{M}$ such that

$$\langle \mu,\ w\rangle \geq 0 \text{ and } \langle \mu^*,\ w\rangle \leq 0.$$

Since $\mathcal{S}$ is open, we can choose $t > 0$, such that

$$z_0 = exp_x tw \in \mathcal{S}.$$

Thus, $\langle \mu,\ exp_x^{-1} z_0\rangle \geq 0$ and $\langle \mu^*,\ exp_x^{-1} z_0\rangle \leq 0$. So, z_0 belongs to the open set

$$\mathcal{U} = \mathcal{S} \cap \{z \in \mathcal{S} : \langle \mu,\ exp_x^{-1} z\rangle \geq 0\} \cap \{z \in \mathcal{S} : \langle \mu^*,\ exp_x^{-1} z\rangle \leq 0\}.$$

Hence, $\mathcal{U}$ is non-empty. For all $z \in \mathcal{U}$, one has

$$\langle \mu,\ exp_x^{-1} z\rangle \geq 0 \Rightarrow h(z) \geq h(x),$$
$$\langle \mu^*,\ exp_x^{-1} z\rangle \leq 0 \Rightarrow h(z) \leq h(x).$$

Hence, h is constant on $\mathcal{U}$. But this means that $\partial_c h(z) = \{0\}$ for all $z \in \mathcal{U}$. Since $z \in \mathcal{S}$ and h is geodesic pseudo-linear function on $\mathcal{S}$, it follows that h is constant every where on $\mathcal{S}$. Hence, $\partial_c h(x) = \{0\}$, which is a contradiction to our assumption. □

Theorem 3.8. *Let $\mathcal{S} \subset \mathcal{M}$ be an open geodesic convex set and $h : \mathcal{S} \to \mathbb{R}$ be a locally Lipschitz function on $\mathcal{S}$. Then $\partial_c h$ is pseudo-affine on $\mathcal{S}$ if and only if, for all $x, z \in \mathcal{S}$, $\mu \in \partial_c h(x)$ and $\upsilon \in \partial_c h(z)$,*

$$\langle \mu,\ exp_x^{-1} z\rangle = 0 \Rightarrow \langle \upsilon,\ exp_z^{-1} x\rangle = 0. \tag{7}$$

Proof. Let $\partial_c h$ be a pseudo-affine. Assume that $\langle \mu,\ exp_x^{-1} z\rangle = 0$. Then pseudo-monotonicity of $\partial_c h$ and $-\partial_c h$, implies

$$\langle \upsilon,\ exp_z^{-1} x\rangle \leq 0 \text{ and } \langle -\upsilon,\ exp_z^{-1} x\rangle \leq 0,$$

and thus

$$\langle \upsilon,\ exp_z^{-1} x\rangle = 0.$$

Conversely, assume that $\partial_c h$ is not pseudo-monotone. Then there exist $x, z \in \mathcal{S}$, $\mu \in \partial_c h(x)$ and $\upsilon \in \partial_c h(z)$

$$\langle \mu,\ exp_x^{-1} z\rangle \geq 0 \text{ and } \langle \upsilon,\ exp_z^{-1} x\rangle > 0,$$

equivalently,

$$\langle \mu,\ \dot{\Omega}_{xz}(0)\rangle \geq 0 \text{ and } \langle \upsilon,\ \dot{\Omega}_{zx}(0)\rangle > 0.$$

Let $\psi : [0, 1] \to \mathbb{R}$ be a continuous function defined by $\psi(t) = \langle \bar{\mu},\ \dot{\Omega}_{xz}(t)\rangle$, where $\bar{\mu} \in \partial_c h(\Omega_{xz}(t))$ with $t \in [0, 1]$. Then, we have $\psi(0) \geq 0$ and $\psi(1) = \langle \upsilon,\ \dot{\Omega}_{xz}(1)\rangle = \langle \upsilon,\ -\dot{\Omega}_{zx}(0)\rangle < 0$ and by continuity there exists $t^* \in [0, 1]$ such that $\psi(t^*) = 0$, that is,

$$\langle \mu^*,\ \dot{\Omega}_{xz}(t^*)\rangle = 0, \text{ for some } \mu^* \in \partial_c h(\Omega_{xz}(t^*)).$$

Since $exp_{\Omega_{xz}(t^*)}^{-1} z = -t^* \dot{\Omega}_{xz}(t^*)$, we get

$$\langle \mu^*,\ exp_{\Omega_{xz}(t^*)}^{-1} z\rangle = 0, \text{ for some } \mu^* \in \partial_c h(\Omega_{xz}(t^*)),$$

which in view of relation (7), implies that

$$\langle \upsilon,\ exp_z^{-1} \Omega_{xz}(t^*)\rangle = 0, \text{ for some } \upsilon \in \partial_c h(z).$$

Since $exp_z^{-1} \Omega_{xz}(t^*) = (1 - t^*) exp_z^{-1} x$, we have

$$\langle \upsilon,\ exp_z^{-1} x\rangle = 0, \text{ for some } \upsilon \in \partial_c h(z),$$

which is a contradiction. Thus $\partial_c h$ is pseudo-monotone. In a similar manner, we can be established that $-\partial_c h$ is pseudo-monotone. □

4. Nonsmooth pseudo-linear vector optimization problem

In this section, we consider a nonsmooth pseudo-linear vector optimization problem (NPVOP) on Hadamard manifold:

$$\textbf{(NPVOP)} \qquad \begin{aligned} &\text{Min } \; h(x) = (h_1(x), \ldots, h_q(x)), \\ &\text{s.t. } \; x \in \mathcal{S}, \end{aligned}$$

where $h_i : \mathcal{S} \to \mathbb{R}, i \in \mathcal{I} = \{1, \ldots, q\}$ are locally Lipschitz and geodesic pseudo-linear functions on an open geodesic convex set $\mathcal{S}$ then there exist proportional functions $\mathcal{P}_i : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_{++}, i \in \mathcal{I}$.

Remark 4.1. (i) *In virtue of Theorem 3.5, for some $\mu_i \in \partial_c h_i(x), i \in \mathcal{I}$*

$$\begin{aligned} & h_i(z) - h_i(x) = \mathcal{P}_i(x,z)\langle \mu_i,\ exp_x^{-1} z\rangle, \;\; \forall\, i \in \mathcal{I}, \\ \text{or,} \quad & h(z) - h(x) = \left(\langle \mathcal{P}_1(x,z)\mu_1,\ exp_x^{-1} z\rangle, \ldots, \langle \mathcal{P}_q(x,z)\mu_q,\ exp_x^{-1} z\rangle\right). \end{aligned} \tag{8}$$

(ii) *A set of Clarke sub-differential of h at any point $x \in \mathcal{S}$ is $\partial_c h(x) = \partial_c h_1(x) \times \ldots \times \partial_c h_q(x)$.*

Definition 4.2. *A point $x \in \mathcal{S}$ is said to be an efficient solution to the problem (NPVOP), if there exists no $z \in \mathcal{S}$ such that $h(z) \leq_q h(x)$.*

4.1. Generalized nonsmooth vector variational inequalities

We present the following generalized nonsmooth Stampacchia and Minty vector variational inequalities (GNSVVI) and (GNMVVI), respectively on Hadamard manifold:

(GNSVVI): Find $x \in \mathcal{S}$ such that for all $z \in \mathcal{S}$, there exists $\mu_i \in \partial_c h_i(x), i \in \mathcal{I}$ such that

$$\left(\langle \mathcal{P}_1(x,z)\mu_1,\ exp_x^{-1} z\rangle, \ldots, \langle \mathcal{P}_q(x,z)\mu_q,\ exp_x^{-1} z\rangle\right) \nleq_q 0.$$

(GNMVVI): Find $x \in \mathcal{S}$ such that for all $z \in \mathcal{S}$ and $\upsilon_i \in \partial_c h_i(z), i \in \mathcal{I}$,

$$\left(\langle \mathcal{P}_1(z,x)\upsilon_1,\ exp_z^{-1} x\rangle, \ldots, \langle \mathcal{P}_q(z,x)\upsilon_q,\ exp_z^{-1} x\rangle\right) \ngeq_q 0.$$

Remark 4.3. (i) *If $\mathcal{P}_i(x,z) = 1, i \in \mathcal{I}$, then* (GNSVVI) *and* (GNMVVI) *reduce to* (SVVI) *and* (MVVI), *respectively considered by Chen and Fang* [5].
(ii) *If $\mathcal{M} = \mathbb{R}^n$, then $exp_x^{-1} y = y - x$ and function is differential, we obtain* (GNSVVI) *becomes vector variational inequality presented by Yang* [25].
(iii) *If $\mathcal{M} = \mathbb{R}^n$, then $exp_x^{-1} y = y - x$ and $\mathcal{P}_i(x,z) = 1, i \in \mathcal{I}$. In this particular situation,* (GNSVVI) *and* (GNMVVI) *converts in vector variational inequalities studied by Lee* [18].

Now, we shall present the equivalence between the solution of (NPVOP) and (GNSVVI) as well as the relationship between the solution of (NPVOP) and (GNMVVI).

Theorem 4.4. *Let $h_i : \mathcal{S} \to \mathbb{R}, i \in \mathcal{I}$ be locally Lipschitz and geodesic pseudo-linear functions on an open geodesic convex set $\mathcal{S}$. Assume that for any $\mu_i, \mu_i^* \in \partial_c h_i(x^0)$ there exists $\rho_i > 0, i \in \mathcal{I}$. Then $x^0 \in \mathcal{S}$ is an efficient solution to the problem* (NPVOP) *if and only if $x^0 \in \mathcal{S}$ is a solution to the problem* (GNSVVI).

Proof. Let $x^0 \in \mathcal{S}$ be an efficient solution to the problem (NPVOP), but it is not a solution to the problem (GNSVVI), then there exist $z \in \mathcal{S}$ and $\mu_i \in \partial_c h_i(x^0), i \in \mathcal{I}$ such that

$$\left(\langle \mathcal{P}_1(x^0,z)\mu_1,\ exp_{x^0}^{-1} z\rangle, \ldots, \langle \mathcal{P}_q(x^0,z)\mu_q,\ exp_{x^0}^{-1} z\rangle\right) \leq_q 0. \tag{9}$$

Since each $h_i, i \in \mathcal{I}$ is geodesic pseudo-linear function, by Theorem 3.5 there exists proportional function $\mathcal{P}_i(x^0, z)$ such that for some $\mu_i \in \partial_c h_i(x^0)$, we have

$$h(z) - h(x^0) = \left(\langle \mathcal{P}_1(x^0,z)\mu_1,\ exp_{x^0}^{-1} z\rangle, \ldots, \langle \mathcal{P}_q(x^0,z)\mu_q,\ exp_{x^0}^{-1} z\rangle\right),$$

the above inequality along with inequality (9), yields

$$h(z) - h(x^0) \leq_q 0,$$

which contradicts the fact that $x^0 \in \mathcal{S}$ is an efficient solution to the problem (NVOP).

Conversely, suppose that x^0 is a solution to the problem (GNSVVI), but it is not an efficient solution to the problem (NPVOP). Then there exists $z \in \mathcal{S}$ such that

$$h(z) - h(x^0) \leq_q 0.$$

Since each $h_i, i \in \mathcal{I}$ is geodesic pseudo-linear function, by Theorem 3.5 there exists proportional function $\mathcal{P}_i(x^0, z)$ such that for some $\mu_i^* \in \partial_c h_i(x^0)$, we get

$$\left(\langle \mathcal{P}_1(x^0,z)\mu_1^*,\ exp_{x^0}^{-1}z\rangle, \ldots, \langle \mathcal{P}_q(x^0,z)\mu_q^*,\ exp_{x^0}^{-1}z\rangle\right) \leq_q 0.$$

By using Proposition 3.7, for $\mu_i \in \partial_c h_i(x^0), i \in \mathcal{I}$, there exists $\rho_i > 0, i \in \mathcal{I}$ such that $\mu_i^* = \rho_i\mu_i, i \in \mathcal{I}$. Thus, we obtain

$$\left(\langle \mathcal{P}_1(x^0,z)\rho_1\mu_1,\ exp_{x^0}^{-1}z\rangle, \ldots, \langle \mathcal{P}_q(x^0,z)\rho_q\mu_q,\ exp_{x^0}^{-1}z\rangle\right) \leq_q 0.$$

Since $\rho_i > 0, i \in \mathcal{I}$, it follows that for $\mu_i \in \partial_c h_i(x^0), i \in \mathcal{I}$, one has

$$\left(\langle \mathcal{P}_1(x^0,z)\mu_1,\ exp_{x^0}^{-1}z\rangle, \ldots, \langle \mathcal{P}_q(x^0,z)\mu_q,\ exp_{x^0}^{-1}z\rangle\right) \leq_q 0,$$

which contradicts our assumption that $x^0 \in \mathcal{S}$ is a solution to the problem (GNSVVI). This completes the proof. □

Now, we present a numerical example to illustrate Theorem 4.4.

Example 4.5. *Let $\mathcal{M} = \mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ be endowed with the Riemannian metric defined by $g(x) = x^{-2}$. Clearly, $\mathcal{M}$ is a Hadamard manifold and the tangent space $T_x\mathcal{M}$ at $x \in \mathcal{M}$ is equal to $\mathbb{R}$. The geodesic curve $\Omega : \mathbb{R} \to \mathcal{M}$ satisfying $\Omega(0) = x \in \mathcal{M}$ and $\dot{\Omega}(0) = w \in T_x\mathcal{M}$ is given by $\Omega(t) = xe^{(w/x)t}$. Thus, $exp_x(tw) = xe^{(w/x)t}$ and is followed by $exp_x^{-1}z = x\ln\left(\frac{z}{x}\right)$.*
Consider the following a nonsmooth pseudo-linear vector optimization problem:

$$\textbf{(NPVOP1)} \quad Min\ \ h(x) = (h_1(x), h_2(x)),$$
$$s.t.\ \ x \in \mathcal{M},\ \ where$$

$$h_1(x) = \begin{cases} \frac{-4(x-1)}{x}, & if\, 0 < x < 1, \\ \frac{4(x-1)}{x}, & if\, x \geq 1 \end{cases} \qquad and \qquad h_2(x) = x^3 - x.$$

It is manifest that h is geodesic pseudo-linear function with respect to proportional function $\mathcal{P}(x,z) = (\mathcal{P}_1(x,z),\ \mathcal{P}_2(x,z)) = \left(\frac{4|z-x|}{xz\ln(z)},\ \frac{z^3-x^3-z+x}{x\ln(z)}\right)$ with $\ln(z) \neq 0$.
Moreover, we have $\partial_c h_1(1) = [-4,4]$ and $\partial_c h_2(1) = \{2\}$. Clearly, $(\mu_1^*, \mu_1) = (2,1) \in \partial_c h_1(1)$ and $(\mu_2^*, \mu_2) = (2,2) \in \partial_c h_2(1)$, there exists $(\rho_1, \rho_2) = (2,1) > 0$, it satisfies the condition $\mu_i^* = \rho_i\mu_i,\ i = 1,2$.
Then, obviously $x^0 = 1$ is an efficient solution to the problem (NPVOP1). Further, for $x^0 = 1$ and for all $z \in \mathcal{M}$, there exist $\mu_1 = 1 \in \partial_c h_1(1)$ and $\mu_2 = 2 \in \partial_c h_2(1)$ such that

$$\textbf{(GNSVVI1)} \quad (\left\langle \mathcal{P}_1(x^0,z)\mu_1,\ exp_{x^0}^{-1}z\right\rangle,\ \langle \mathcal{P}_2(x^0,z)\mu_2,\ exp_{x^0}^{-1}z\rangle\big)$$

$$= \left(\frac{4|z-1|}{z},\ 2(z^3 - z)\right) \nleq_2 0.$$

Thus, $x^0 = 1$ is a solution to the problem (GNSVVI1).

Theorem 4.6. *Let $h_i : \mathcal{S} \to \mathbb{R}, i \in \mathcal{I}$ be locally Lipschitz and geodesic pseudo-linear functions on an open geodesic convex set $\mathcal{S}$. If $x^0 \in \mathcal{S}$ is an efficient solution to the problem* (NPVOP)*, then it is also a solution to the problem* (GNMVVI).

Proof. Let $x^0 \in \mathcal{S}$ be an efficient solution to the problem (NPVOP) but not a solution to the problem (GNMVVI). Then there exist $z \in S$ and $v_i \in \partial_c h_i(z), i \in \mathcal{I}$ such that

$$\left(\langle \mathcal{P}_1(z, x^0)v_1,\ exp_z^{-1}x^0\rangle, \ldots, \langle \mathcal{P}_q(z, x^0)v_q,\ exp_z^{-1}x^0\rangle\right) \geq_q 0. \tag{10}$$

Since each $h_i, i \in \mathcal{I}$ is geodesic pseudo-linear functions, by Theorem 3.5 there exists proportional function $\mathcal{P}_i(z, x^0)$ such that for some $v_i \in \partial_c h_i(z)$, we get

$$h(x^0) - h(z) = \left(\langle \mathcal{P}_1(z, x^0)v_1,\ exp_z^{-1}x^0\rangle, \ldots, \langle \mathcal{P}_q(z, x^0)v_q,\ exp_z^{-1}x^0\rangle\right),$$

the above inequality along with inequality (10), yields

$$h(x^0) - h(z) \geq_q 0,$$

which leads to contradiction that $x^0 \in \mathcal{S}$ is an efficient solution to the problem (NPVOP). This completes the proof. □

Now, we take an assumption if the proportional function $\mathcal{P}_i(x^0, z), i \in \mathcal{I}$ is independent of the index i and have the same value $\bar{\mathcal{P}}(x^0, z)$, then we shall show that the following equivalence relation between a solution to the problem (NPVOP) and (SVVI) considered in [5] .

Theorem 4.7. *Let $h_i : \mathcal{S} \to \mathbb{R}, i \in \mathcal{I}$ be locally Lipschitz and geodesic pseudo-linear functions on an open geodesic convex set $\mathcal{S}$. Assume that for any $\mu_i, \mu_i^* \in \partial_c h_i(x^0)$ there exists $\rho_i > 0, i \in \mathcal{I}$. Then $x^0 \in \mathcal{S}$ is an efficient solution to the problem* (NPVOP) *if and only if $x^0 \in \mathcal{S}$ is a solution to the problem* (SVVI).

Proof. Assume that $x^0 \in \mathcal{S}$ is an efficient solution to the problem (NVOP) but it is not a solution to the problem (SVVI). Then there exist $z \in \mathcal{S}$ and $\mu_i \in \partial_c h_i(x^0), i \in \mathcal{I}$ such that

$$\left(\langle \mu_1,\ exp_{x^0}^{-1}z\rangle, \ldots, \langle \mu_q,\ exp_{x^0}^{-1}z\rangle\right) \leq_q 0.$$

On multiplying the above inequality by $\bar{\mathcal{P}}(x^0, z) > 0$, it follows that there exists $\mu_i \in \partial_c h_i(x^0), i \in \mathcal{I}$ such that

$$\left(\langle \bar{\mathcal{P}}(x^0, z)\mu_1,\ exp_{x^0}^{-1}z\rangle, \ldots, \langle \bar{\mathcal{P}}(x^0, z)\mu_q,\ exp_{x^0}^{-1}z\rangle\right) \leq_q 0.$$

Applying the assumption in proportional functions $\mathcal{P}_i(x^0, z), i \in \mathcal{I}$ have the same value $\bar{\mathcal{P}}(x^0, z)$ in equality (8), it follows that

$$h(z) - h(x^0) \leq_q 0,$$

which shows that x^0 is not an efficient solution to the problem (NVOP).

Conversely, suppose that $x^0 \in \mathcal{S}$ is a solution to the problem but it is not an efficient solution to the problem (NVOP). Then there exists $z \in \mathcal{S}$ such that

$$h(z) - h(x^0) \leq_q 0. \tag{11}$$

Since each $h_i, i \in \mathcal{I}$ is geodesic pseudo-linear function, by Theorem 3.5 there exists proportional function $\mathcal{P}_i(x^0, z)$ such that for some $\mu_i^* \in \partial_c h_i(x^0)$, we get

$$h(z) - h(x^0) = \left(\langle \mathcal{P}_1(x^0, z)\mu_1^*,\ exp_{x^0}^{-1}z\rangle, \ldots, \langle \mathcal{P}_q(x^0, z)\mu_q^*,\ exp_{x^0}^{-1}z\rangle\right). \tag{12}$$

By combining (11) and (12), we obtain

$$\left(\langle \mathcal{P}_1(x^0, z)\mu_1^*,\ exp_{x^0}^{-1}z\rangle, \ldots, \langle \mathcal{P}_q(x^0, z)\mu_q^*,\ exp_{x^0}^{-1}z\rangle\right) \leq_q 0.$$

Using Proposition 3.7, there exists $\rho_i > 0$, $\mu_i \in \partial_c h_i(x^0), i \in \mathcal{I}$ such that $\mu_i^* = \rho_i \mu_i, i \in \mathcal{I}$, we get

$$\left(\langle \mathcal{P}_1(x^0, z)\rho_1\mu_1,\ exp_{x^0}^{-1}z\rangle, \ldots, \langle \mathcal{P}_q(x^0, z)\rho_q\mu_q,\ exp_{x^0}^{-1}z\rangle\right) \leq_q 0.$$

Since $\mathcal{P}_i(x^0, z) > 0$ and $\rho_i > 0, i \in \mathcal{I}$, it follows that there exists $\mu_i \in \partial_c h_i(x^0), i \in \mathcal{I}$ such that

$$\left(\langle \mu_1,\ exp_{x^0}^{-1}z\rangle, \ldots, \langle \mu_q,\ exp_{x^0}^{-1}z\rangle\right) \leq_q 0,$$

which shows that x^0 is not a solution to the problem (SVVI). This contradiction leads to the result. □

4.2. Gap function for nonsmooth pseudo-linear vector optimization problem

The gap function concept for convex programming problem discussed by Hearn [11] and later studied by Soleimani-damaneh [22] for nonsmooth convex optimization problems. Chen et. al [7] and Caristi et. al [4] investigated the set-valued gap function for smooth and nonsmooth convex multiobjective optimization problem, respectively. Along the lines of him, we have defined the gap function for nonsmooth pseudo-linear vector optimization problem (NPVOP) on Hadamard manifold.

Definition 4.8. *The gap function for problem (NPVOP) is a set-valued function $\Psi : \mathcal{S} \to 2^{\mathbb{R}^q}$ defined as follows:*

$$\Psi(x) = \max_{z\in\mathcal{S}} \{\langle \mathcal{P}_i(x, z)\mu_i,\ exp_x^{-1}z\rangle,\ \ \mu_i \in \partial_c h_i(x),\ \forall\ i \in \mathcal{I}\}.$$

Theorem 4.9. *Let $\Psi(x) \nleq_q 0$, for all $x \in \mathcal{S}$. Further, $0 \in \Psi(x^0)$ if and only if $x^0 \in \mathcal{S}$ solves* (NPVOP).

Proof. Let $z = x$, then $\langle \mathcal{P}_i(x, z)\mu_i,\ exp_x^{-1}z\rangle = 0,\ \ \mu_i \in \partial_c h_i(x),\ \forall\ i \in \mathcal{I}$.

$$\Rightarrow \quad \max_{z\in\mathcal{S}} \{\langle \mathcal{P}_i(x, z)\mu_i,\ exp_x^{-1}z\rangle,\ \ \mu_i \in \partial_c h_i(x),\ \forall\ i \in \mathcal{I}\} = \Psi(x) \nleq_q 0.$$

Now, let us assume that $0 \in \Psi(x^0)$. Then,

$$\max_{z\in\mathcal{S}} \{\langle \mathcal{P}_i(x^0, z)\mu_i,\ exp_{x^0}^{-1}z\rangle,\ \ \mu_i \in \partial_c h_i(x),\ \forall\ i \in \mathcal{I}\} = 0.$$

Hence, there does not exist $z \in \mathcal{S}$, satisfying

$$\left(\langle \mathcal{P}_1(x^0, z)\mu_1,\ exp_{x^0}^{-1}z\rangle, \ldots, \langle \mathcal{P}_q(x^0, z)\mu_q,\ exp_{x^0}^{-1}z\rangle\right) \leq_q 0.$$

Since each $h_i, i \in \mathcal{I}$ is geodesic pseudo-linear function, by Theorem 3.5 there exists proportional function $\mathcal{P}_i(x^0, z)$ such that for some $\mu_i \in \partial_c h_i(x^0)$, we get

$$h(z) - h(x^0) = \left(\langle \mathcal{P}_1(x^0, z)\mu_1,\ exp_{x^0}^{-1}z\rangle, \ldots, \langle \mathcal{P}_q(x^0, z)\mu_q,\ exp_{x^0}^{-1}z\rangle\right).$$

It follows that there does not exist $z \in \mathcal{S}$ such that

$$h(z) \leq_q h(x^0).$$

Thus, x^0 is an efficient solution to the problem (NPVOP).

Conversely, we proceed by contradiction, let x^0 does not solve (NPVOP), then there exists $z \in \mathcal{S}$ such that

$$h(z) \leq_q h(x^0),$$

that is,

$$h(z) - h(x^0) \leq_q 0.$$

Since each $h_i, i \in \mathcal{I}$ is geodesic pseudo-linear function, by Theorem 3.5 there exists proportional function $\mathcal{P}_i(x^0, z)$ and for any $\mu_i \in \partial_c h_i(x^0)$, we get

$$h(z) - h(x^0) = \left(\langle \mathcal{P}_1(x^0, z)\mu_1,\ exp_{x^0}^{-1}z\rangle, \ldots, \langle \mathcal{P}_q(x^0, z)\mu_q,\ exp_{x^0}^{-1}z\rangle\right).$$

Thus, $\left(\langle \mathcal{P}_1(x^0, z)\mu_1,\ exp_{x^0}^{-1}z\rangle, \ldots, \langle \mathcal{P}_q(x^0, z)\mu_q,\ exp_{x^0}^{-1}z\rangle\right) \leq_q 0$, implies that

$$0 \notin \Psi(x^0).$$

Hence, the proof is complete. □

5. Conclusions

This paper have concerned with a geodesic pseudo-linear function on Hadamard manifolds. We derived the several properties of a geodesic pseudo-linear function and also defined the pseudo-affiness operator. Besides, we have considered a nonsmooth pseudo-linear vector optimization problem and generalized vector variational inequality problems on Hadamard manifolds and investigated the relations between solutions of these problems. Additionally, we have studied the gap function for a nonsmooth pseudo-linear vector optimization problem. In future, the author will study bi-function vector variational inequality under geodesic pseudo-linear function on Hadamard manifolds.

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Appendix

Let $\alpha(s)$ be a geodesic joining $\Omega_{xz}(t)$ and z satisfying $\alpha(0) = \Omega_{xz}(t)$ and $\alpha(1) = z$.

$$\text{That is,} \qquad \alpha(s) = \Omega_{xz}(t - st), \ \forall \ s \in [0, 1]$$

$$\dot{\alpha}(s) = -t\dot{\Omega}_{xz}(t - st).$$

In particular, $\dot{\alpha}(0) = -t\dot{\Omega}_{xz}(t)$ (a)

It is well know that the geodesic starting at $\Omega_{xz}(t)$ and join with z can also be expressed as

$$\alpha(s) = exp_{\Omega_{xz}(t)}(s\, exp^{-1}_{\Omega_{xz}(t)} z), \quad \forall\, s \in [0,1]$$

$$\text{and} \quad \dot{\alpha}(0) = exp^{-1}_{\Omega_{xz}(t)} z \qquad (b)$$

From relation (a) and (b), we obtain

$$exp^{-1}_{\Omega_{xz}(t)} z = -t\dot{\Omega}_{xz}(t)$$

Similarly, assume that $\beta(s)$ is a geodesic joining $\Omega_{xz}(t)$ and x satisfying $\alpha(0) = \Omega_{xz}(t)$ and $\alpha(1) = x$.

$$\text{That is,} \qquad \beta(s) = \Omega_{xz}(t + s(1-t)), \ \forall\, s \in [0,1]$$

$$\dot{\beta}(s) = (1-t)\dot{\Omega}_{xz}(t + s(1-t)).$$

In particular, $\dot{\beta}(0) = (1-t)\dot{\Omega}_{xz}(t)$ (c)

It is well know that the geodesic starting at $\Omega_{xz}(t)$ and join with x can also be expressed as

$$\beta(s) = exp_{\Omega_{xz}(t)}(s\, exp^{-1}_{\Omega_{xz}(t)} x), \quad \forall\, s \in [0,1]$$

$$\text{and} \quad \dot{\beta}(0) = exp^{-1}_{\Omega_{xz}(t)} x \qquad (d)$$

From relation (c) and (d), we obtain

$$exp^{-1}_{\Omega_{xz}(t)} x = (1-t)\dot{\Omega}_{xz}(t)$$