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Korovkin type theorem for the functions defined in the Prism and the corresponding Meyer-König and Zeller operators

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Abstract. In this paper, we consider Meyer-König and Zeller (MKZ) operators defined in the prism. We prove a new Korovkin type theorem by using appropriate auxiliary test function and investigate the uniform approximation of these operators. We obtain the order of approximation in terms of the modulus of continuity and modified Lipschitz functions. Finally, we introduce the more general form of the operators and study their approximation properties by obtaining functional partial differential equation which help us to calculate the moments easily.

1. Introduction

After the famous theorem of Korovkin, which states that the approximation of three basic test functions determine the convergences in the whole space, the approximation by the linear positive operators has been the active area of research especially in the last three decades ([8],[15],[16],[17]). Among the linear positive operators Meyer-König and Zeller operators have particular interest of several authors. The operators

$$M_n(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n+k+1}\right)\binom{n+k}{k}x^k(1-x)^{n+1}, \quad x \in [0,1)$$

are known as the Meyer-König and Zeller operators [28]. In 1964, Cheney and Sharma [10] gave a slightly modified form of these operator by replacing the nodes $\frac{k}{n+k+1}$ by $\frac{k}{n+k}$, which they called them as Bernstein power series:

$$M_n^*(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k}x^k(1-x)^{n+1}, \quad x \in [0,1)\,.$$

Different variants of these two operators and their approximation properties have been an area of intensive research during the last five decades (see [1],[4],[12],[18],[19],[32][11], [22],[23]). In 2005, Altın, Doğru and Taşdelen [7] considered a generating function extension of the Bernstein power series and proved a Korovkin type approximation theorem by using the test functions $\widetilde{f}_i(t) = \left(\frac{t}{1-t}\right)^i$ instead of using the

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classical test functions $f_i(t) = t^i$ for $i = 0, 1, 2$. It is an important detail that it is not easy to calculate the second moment while investigating the approximation properties of $M_n(f; x)$ operators by using the usual Korovkin theorem (where the test functions are $f_i(t) = t^i$ $(i = 0, 1, 2)$). It was Alkemade [6], who derived an explicit expression for the second moment in terms of hypergeometric series, but the result was not sufficiently useful. For the estimation of the higher order moments, we refer [2],[9] and [20]. We also refer the very recent papers [14] and [3] concerning the second and higher order moment of Meyer–König and Zeller operators.

In [31], Taşdelen and Erençin introduced the bivariate tensor type generalization of the Bernstein power series by means of the generating functions. They proved a Korovkin type approximation theorem by introducing the test functions $f_0(s,t) = 1, f_1(s,t) = \frac{s}{1-s}, \ f_2(s,t) = \frac{t}{1-t}$, and $f_3(s,t) = \left(\frac{s}{1-s}\right)^2 + \left(\frac{t}{1-t}\right)^2$.

In [29], Özarslan introduced the non-tensor two variable Meyer-König and Zeller operators. He proved Korovkin type approximation theorem by presenting then test functions ($f_0(s,t) = 1, f_1(s,t) = \frac{s}{1-s-t}, \ f_2(s,t) = \frac{t}{1-s-t}$, and $f_3(s,t) = \left(\frac{s}{1-s-t}\right)^2 + \left(\frac{t}{1-s-t}\right)^2$). Furthermore, he computed the rate of convergence of these operators by means of the modulus of continuity and the elements of modified Lipschitz class functions. Moreover, he gave functional partial differential equations for this class and, using their corresponding equations, he calculated the first few moments of the non-tensor MKZ operators and investigated their approximation properties. For more studies on the new bivariate Meyer-König-Zeller operators, we refer to the papers [26], [24], [25], [25] and [33].

Now let

$$S_A := \{\mathbf{x} = (x,y,z) : 0 \le x \le A < 1, \ 0 \le y \le A - x, 0 \le z \le A < 1\}.$$

In the present paper, we introduce the following three variable Meyer-König and Zeller operators which provides an approximation to a functions defined on the prism S_A

$$L_{n,m}(f;x,y,z) = (1-z)^{m+1}(1-x-y)^{n+1} \times \sum_{k,l,p=0}^{\infty} f\left(\frac{k}{n+k+l+1}, \frac{l}{n+k+l+1}, \frac{p}{m+p+1}\right) P_m^n(k,l,p) x^k y^l z^p \tag{1}$$

where $P_m^n(k,l,p) = \dfrac{(n+k+l)!}{n!k!l!}\binom{m+p}{p}$ and $\frac{k}{n+k+l+1}, \frac{l}{n+k+l+1}, \frac{p}{m+p+1} \in S_A, (0 < A < 1)$.

The paper is organized as follows. In section 2, we prove a Korovkin type approximation theorem by introducing the test functions $\varphi_0(s,t,u) = 1, \ \varphi_1(s,t,u) = \frac{s}{1-s-t}, \varphi_2(s,t,u) = \frac{t}{1-s-t}, \ \varphi_3(s,t,u) = \frac{u}{1-u}, \varphi_4(s,t,u) = \left(\frac{s}{1-s-t}\right)^2 + \left(\frac{t}{1-s-t}\right)^2 + \left(\frac{u}{1-u}\right)^2$. In section 3, we give the functional partial differential equations satisfied by the new operator (1), and using these equations, we calculate the first few moments of these operators and investigate their approximation properties. In section 4, we compute the order of convergence of these operators by means of modulus of continuity and the elements of modified Lipschitz class. Finally, we introduce a generalized form of the operators $L_{n,m}$ and obtain functional partial differential equations for this class. Using the corresponding equations, we calculate the first few moments and investigate their approximation properties of the generalised operators.

2. Korovkin Type Theorem

To prove the main theorem of this section, we introduce the following test functions

$$\varphi_0(s,t,u) = 1, \ \varphi_1(s,t,u) = \frac{s}{1-s-t},$$
$$\varphi_2(s,t,u) = \frac{t}{1-s-t}, \varphi_3(s,t,u) = \frac{u}{1-u},$$
$$\varphi_4(s,t,u) = \left(\frac{s}{1-s-t}\right)^2 + \left(\frac{t}{1-s-t}\right)^2 + \left(\frac{u}{1-u}\right)^2.$$

It's not hard to see that

$$\varphi_1\left(\frac{k}{n+k+l+1},\frac{l}{n+k+l+1},\frac{p}{m+p+1}\right)=\frac{k}{n+1}, \tag{2}$$

$$\varphi_2\left(\frac{k}{n+k+l+1},\frac{l}{n+k+l+1},\frac{p}{m+p+1}\right)=\frac{l}{n+1}, \tag{3}$$

$$\varphi_3\left(\frac{k}{n+k+l+1},\frac{l}{n+k+l+1},\frac{p}{m+p+1}\right)=\frac{p}{m+1}, \tag{4}$$

$$\varphi_4\left(\frac{k}{n+k+l+1},\frac{l}{n+k+l+1},\frac{p}{m+p+1}\right) \tag{5}$$
$$=\left(\frac{k}{n+1}\right)^2+\left(\frac{l}{n+1}\right)^2+\left(\frac{p}{m+1}\right)^2.$$

In this paper, we consider the following function space

$$\mathcal{H}_\omega(S_A):=\Big\{f\in C(S_A):\big|f(s,t,u)-f(x,y,z)\big| \tag{6}$$
$$\leq\Big\{\omega\Big(f,\big|(\varphi_1(s,t,u),\varphi_2(s,t,u),\varphi_3(s,t,u))-(\varphi_1(x,y,z),\varphi_2(x,y,z),\varphi_3(x,y,z))\big|\Big)\Big\}\Big\}$$

where $C(S_A)$ denotes the space of continuous functions defined on S_A,

$$\big|(\varphi_1(s,t,u),\varphi_2(s,t,u),\varphi_3(s,t,u))-(\varphi_1(x,y,z),\varphi_2(x,y,z),\varphi_3(x,y,z))\big|$$
$$:=\sqrt{\sum_{i=1}^{3}(\varphi_i(s,t,u)-\varphi_i(x,y,z))^2}$$

and

$$\omega(f,\delta)=\sup\Big\{\big|f(s,t,u)-f(x,y,z)\big|:(s,t,u),(x,y,z)\in S_A,$$
$$\sqrt{(s-x)^2+(t-y)^2+(u-z)^2}\leq\delta\Big\}$$

is the modulus of continuity of f satisfying the following properties:
(a) ω is non-negative and increasing function of δ,
(b) $\omega(f,\delta_1+\delta_2)\leq\omega(f,\delta_1)+\omega(f,\delta_2)$,
(c) $\lim_{\delta\to 0^+}\omega(f,\delta)=0$.

Clearly, for each $(s,t,u),(x,y,z)\in S_A$ and for all $f\in\mathcal{H}_\omega(S_A)$, we have

$$\big|f(s,t,u)-f(x,y,z)\big|\leq\omega(f,\delta)\left(1+\frac{\sqrt{\sum_{i=1}^{3}(\varphi_i(s,t,u)-\varphi_i(x,y,z))^2}}{\delta}\right). \tag{7}$$

In the following, we prove new type of Korovkin theorem which helps to investigate the approximation properties of the new Meyer-König-Zeller operators defined in the prism.

Theorem 1. *Let $T_{n,m} : \mathcal{H}_\omega(S_A) \to C(S_A)$ be a sequence of linear positive operators satisfying*

$$\lim_{n,m\to\infty} \left\|T_{n,m}(\varphi_i;\cdot,\cdot) - \varphi_i\right\|_{C(S_A)} = 0, \qquad (i = 0,1,2,3,4) \tag{8}$$

where $\|\cdot\|_{C(S_A)}$ denotes the usual supremum norm on $C(S_A)$. Then for all $f \in \mathcal{H}_\omega(S_A)$, we have

$$\lim_{n,m\to\infty} \left\|T_{n,m}(f;\cdot,\cdot) - f\right\|_{C(S_A)} = 0.$$

Proof. Let $f \in \mathcal{H}_\omega(S_A)$ be given. For all $\epsilon > 0$, we have from property (c) that

$$\left|f(s,t,u) - f(x,y,z)\right| < \epsilon$$

for $(s,t,u), (x,y,z) \in S_A$, satisfying

$$\sqrt{\left(\frac{s}{1-s-t} - \frac{x}{1-x-y}\right)^2 + \left(\frac{t}{1-s-t} - \frac{y}{1-x-y}\right)^2 + \left(\frac{u}{1-u} - \frac{z}{1-z}\right)^2} < \delta$$

with some $\delta > 0$. On the other hand, since $f \in C(S_A)$, for $(s,t,u), (x,y,z) \in S_A$ with

$$\sqrt{\left(\frac{s}{1-s-t} - \frac{x}{1-x-y}\right)^2 + \left(\frac{t}{1-s-t} - \frac{y}{1-x-y}\right)^2 + \left(\frac{u}{1-u} - \frac{z}{1-z}\right)^2} \geq \delta,$$

we have

$$\begin{aligned}
&\left|f(s,t,u) - f(x,y,z)\right| < \frac{2M}{\delta^2} \\
&\times \left\{\left(\frac{s}{1-s-t} - \frac{x}{1-x-y}\right)^2 + \left(\frac{t}{1-s-t} - \frac{y}{1-x-y}\right)^2 + \left(\frac{u}{1-u} - \frac{z}{1-z}\right)^2\right\},
\end{aligned}$$

where M is the bound of f. Combining the above inequalities, we get for all $(s,t,u), (x,y,z) \in S_A$ and $f \in \mathcal{H}_\omega(S_A)$ that

$$\begin{aligned}
&\left|f(s,t,u) - f(x,y,z)\right| < \epsilon + \frac{2M}{\delta^2} \\
&\times \left\{\left(\frac{s}{1-s-t} - \frac{x}{1-x-y}\right)^2 + \left(\frac{t}{1-s-t} - \frac{y}{1-x-y}\right)^2 + \left(\frac{u}{1-u} - \frac{z}{1-z}\right)^2\right\}.
\end{aligned} \tag{9}$$

By linearity and positivity of the operators $T_{n,m}$, we can write

$$\begin{aligned}
&\left|T_{n,m}(f;x,y,z) - f(x,y,z)\right| \\
&\leq T_{n,m}\left(\left|f(s,t,u) - f(x,y,z)\right|;x,y,z\right) + \left|f(x,y,z)\right|\left|T_{n,m}(\varphi_0;x,y,z) - \varphi_0(x,y,z)\right|.
\end{aligned}$$

Using the inequality (9), we get

$$\begin{aligned}
&\left|T_{n,m}(f;x,y,z) - f(x,y,z)\right| \\
&< \epsilon + \left(\epsilon + M + \frac{2M}{\delta^2}B(A)\right)\left|T_{n,m}(\varphi_0;x,y,z) - \varphi_0(x,y,z)\right| \\
&+ \frac{4M}{\delta^2}B(A)\left\{\left|T_{n,m}(\varphi_1;x,y) - \varphi_1(x,y)\right| + \left|T_{n,m}(\varphi_2;x,y) - \varphi_2(x,y)\right|\right. \\
&\left.+ \left|T_{n,m}(\varphi_3;x,y) - \varphi_3(x,y)\right|\right\} \\
&+ \frac{2M}{\delta^2}\left|T_{n,m}(\varphi_4;x,y) - \varphi_4(x,y)\right|
\end{aligned}$$

where

$$B(A)=\max_{(x,y,z)\in S_A}\left\{\frac{x^2+y^2}{(1-x-y)^2}+\frac{z^2}{(1-z)^2},\frac{x}{1-x-y},\frac{y}{1-x-y},\frac{z}{1-z}\right\}.$$

Taking into account (8), the proof is completed. □

3. Approximation of $L_{n,m}(f;x,y)$ in $\mathcal{H}_\omega(S_A)$

In this section, we prove the convergence of the operators $L_{n,m}(f;x,y,z)$ in the space $\mathcal{H}_\omega(S_A)$. For the calculation of the moments, we need the following Lemma.

Lemma 2. *Let $(x,y,z)\in S_A$, $f\in C(S_A)$. Then $L_{n,m}(f;x,y,z)$ satisfy the following functional partial differential equations*

$$\begin{aligned}&x\frac{\partial}{\partial x}L_{n,m}(f;x,y,z) \\ &=\frac{-x(n+1)}{1-x-y}L_{n,m}(f;x,y,z)+(n+1)L_{n,m}(\varphi_1 f;x,y,z),\end{aligned}\tag{10}$$

$$\begin{aligned}&y\frac{\partial}{\partial y}L_{n,m}(f;x,y,z) \\ &=\frac{-y(n+1)}{1-x-y}L_{n,m}(f;x,y,z)+(n+1)L_{n,m}(\varphi_2 f;x,y,z),\end{aligned}\tag{11}$$

$$\begin{aligned}&z\frac{\partial}{\partial z}L_{n,m}(f;x,y,z) \\ &=\frac{-z(m+1)}{1-z}L_{n,m}(f;x,y,z)+(m+1)L_{n,m}(\varphi_3 f;x,y,z),\end{aligned}\tag{12}$$

$$\begin{aligned}&\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)L_{n,m}(f;x,y,z) \\ &=\left\{\frac{-(x+y)(n+1)}{1-x-y}-\frac{z(m+1)}{1-z}\right\}L_{n,m}(f;x,y,z) \\ &+(n+1)L_{n,m}(hf;x,y,z)+(m+1)L_{n,m}(gf;x,y,z)\end{aligned}\tag{13}$$

where

$$h(s,t)=\frac{s}{1-s-t}+\frac{t}{1-s-t},g(u)=\frac{u}{1-u}.$$

Proof. The general version of the proof for the more general operators including $L_{n,m}$ has been given in Theorem 9. □

In order the give an alternative calculation of the moments, we have the following lemma.

Lemma 3. *Given any $n,m\in\mathbb{N}$, the following recurrence formula holds*

$$\text{(i) } (k+1)P_m^n(k+1,l,p)=(n+1)P_m^{n+1}(k,l,p),\tag{14}$$

$$\text{(ii) } (l+1)P_m^{n+1}(k,l+1,p)=(n+1)P_m^{n+1}(k,l,p),\tag{15}$$

$$\text{(iii) } (m+1)P_{m+1}^n(k,l,p)=(p+1)P_m^n(k,l,p+1).\tag{16}$$

Proof. Using (1), we get

$$\text{(i) } (n+1)P_m^{n+1}(k,l,p) = (n+1)\frac{(n+k+l+1)!}{(n+1)!k!l!}\binom{m+p}{p}$$
$$= (k+1)P_m^{n}(k+1,l,p),$$

$$\text{(ii) } (n+1)P_m^{n+1}(k,l,p) = (n+1)\frac{(n+k+l+1)!}{(n+1)!k!l!}\binom{m+p}{p}$$
$$= (l+1)P_m^{n+1}(k,l+1,p)$$

and

$$\text{(iii) } (p+1)P_m^{n}(k,l,p+1) = \frac{(n+k+l)!}{n!k!l!}\binom{m+p+1}{p+1}$$
$$= (m+1)P_{m+1}^{n}(k,l,p).$$

□

Lemma 4. *For the operators $L_{n,m}$ defined by (1), we have*
(a)$L_{n,m}(\varphi_0;x,y,z) = \varphi_0(x,y,z)$,
(b)$L_{n,m}(\varphi_1;x,y,z) = \varphi_1(x,y,z)$,
(c)$L_{n,m}(\varphi_2;x,y,z) = \varphi_2(x,y,z)$,
(d) $L_{n,m}(\varphi_3;x,y,z) = \varphi_3(x,y,z)$,
(e)$L_{n,m}(\varphi_4;x,y,z) = \frac{(n+2)}{(n+1)}\varphi_4(x,y,z) + \left\{\frac{(m+2)}{(m+1)} - \frac{(n+2)}{(n+1)}\right\}\varphi_3^2(x,y,z)$
$+\frac{1}{(n+1)}\{\varphi_1(x,y,z)+\varphi_2(x,y,z)+\varphi_3(x,y,z)\} + \left\{\frac{1}{(m+1)} - \frac{1}{(n+1)}\right\}\varphi_3(x,y,z).$

Proof. It is obvious from (1) that $L_{n,m}(1;x,y,z) = 1$. Taking $f = \varphi_0 = 1$ in (10), (11) and (12), we get

$$L_{n,m}(\varphi_1;x,y,z) = \varphi_1(x,y,z),$$
$$L_{n,m}(\varphi_2;x,y,z) = \varphi_2(x,y,z),$$
$$L_{n,m}(\varphi_3;x,y,z) = \varphi_3(x,y,z).$$

Now set $h_1(x,y,z) = \left(\frac{x}{1-x-y}\right)^2$, $h_2(x,y,z) = \left(\frac{y}{1-x-y}\right)^2$ and $h_3(x,y,z) = \left(\frac{z}{1-z}\right)^2$. Choosing $f = \varphi_1$in (10), $f = \varphi_2$ in (11) and $f = \varphi_3$ in (12), we have

$$L_{n,m}(h_1;x,y,z) = \frac{(n+2)}{(n+1)}h_1(x,y,z) + \frac{1}{(n+1)}\varphi_1(x,y,z),$$
$$L_{n,m}(h_2;x,y,z) = \frac{(n+2)}{(n+1)}h_2(x,y,z) + \frac{1}{(n+1)}\varphi_2(x,y,z)$$

and

$$L_{n,m}(h_3;x,y,z) = \frac{(m+2)}{(m+1)}h_3(x,y,z) + \frac{1}{(m+1)}\varphi_3(x,y,z).$$

Since $\varphi_4(x,y,z) = h_1(x,y,z)+h_2(x,y,z)+h_3(x,y,z)$, the proof is completed from the above equalities and the linearity of the operators $L_{n,m}$.

Using (2) and considering (14), we have

$$
\begin{aligned}
L_{n,m}(\varphi_1; x, y, z) &= (1-z)^{m+1}(1-x-y)^{n+1} \sum_{k,l,p=0}^{\infty} \frac{k}{n+1} P_m^n(k,l,p) x^k y^l z^p \\
&= \frac{x}{1-x-y}(1-z)^{m+1}(1-x-y)^{n+2} \sum_{k,l,p=0}^{\infty} P_m^{n+1}(k,l,p) x^k y^l z^p.
\end{aligned}
$$

By (1), we get $L_{n,m}(\varphi_1; x, y) = \varphi_1(x, y)$. In a similar manner, using (3) and (4), then considering (15) and (16), we get (c) and (d), respectively.

Finally, by (5), we get

$$
\begin{aligned}
&L_{n,m}(\varphi_4; x, y, z) = (1-z)^{m+1}(1-x-y)^{n+1} \\
&\times \sum_{k,l,p=0}^{\infty} \left\{ \left(\frac{k}{n+1}\right)^2 + \left(\frac{l}{n+1}\right)^2 + \left(\frac{p}{m+1}\right)^2 \right\} P_m^n(k,l,p) x^k y^l z^p \\
&= (1-z)^{m+1}(1-x-y)^{n+1} \sum_{k,l,p=0}^{\infty} \left(\frac{k}{n+1}\right)^2 P_m^n(k,l,p) x^k y^l z^p \\
&+ (1-z)^{m+1}(1-x-y)^{n+1} \sum_{k,l,p=0}^{\infty} \left(\frac{l}{n+1}\right)^2 P_m^n(k,l,p) x^k y^l z^p \\
&+ (1-z)^{m+1}(1-x-y)^{n+1} \sum_{k,l,p=0}^{\infty} \left(\frac{p}{m+1}\right)^2 P_m^n(k,l,p) x^k y^l z^p.
\end{aligned}
$$

By the recurrence relations (14)-(16), we obtain

$$\begin{aligned}
&L_{n,m}(\varphi_4; x, y, z)\\
&= \frac{x^2(n+2)}{(n+1)(1-x-y)^2}(1-z)^{m+1}(1-x-y)^{n+3}\sum_{k,l,p=0}^{\infty} P_m^{n+2}(k,l,p)x^k y^l z^p\\
&+ \frac{x}{(n+1)(1-x-y)}(1-z)^{m+1}(1-x-y)^{n+2}\sum_{k,l,p=0}^{\infty} P_m^{n+1}(k,l,p)x^k y^l z^p\\
&+ \frac{y^2(n+2)}{(n+1)(1-x-y)^2}(1-z)^{m+1}(1-x-y)^{n+3}\sum_{k,l,p=0}^{\infty} P_m^{n+2}(k,l,p)x^k y^l z^p\\
&+ \frac{y}{(n+1)(1-x-y)}(1-z)^{m+1}(1-x-y)^{n+2}\sum_{k,l,p=0}^{\infty} P_m^{n+1}(k,l,p)x^k y^l z^p\\
&+ \frac{z^2(m+2)}{(m+1)(1-z)^2}(1-z)^{m+3}(1-x-y)^{n+1}\sum_{k,l,p=0}^{\infty} P_{m+2}^{n}(k,l,p)x^k y^l z^p\\
&+ \frac{z}{(n+1)(1-z)}(1-z)^{m+2}(1-x-y)^{n+2}\sum_{k,l,p=0}^{\infty} P_{m+1}^{n}(k,l,p)x^k y^l z^p\\
&= \frac{x^2(n+2)}{(n+1)(1-x-y)^2} + \frac{x}{(n+1)(1-x-y)}\\
&+ \frac{y^2(n+2)}{(n+1)(1-x-y)^2} + \frac{y}{(n+1)(1-x-y)}\\
&+ \frac{z^2(m+2)}{(m+1)(1-z)^2} + \frac{z}{(n+1)(1-z)}.
\end{aligned}$$

Therefore we get (e). □

By Theorem 1 and Lemma 4, we have the following approximation theorem.

Theorem 5. *Let $L_{n,m} : \mathcal{H}_\omega(S_A) \to C(S_A)$ be a sequence of linear positive operators defined by (1). Then for all $f \in \mathcal{H}_\omega(S_A)$, we have*

$$\lim_{n,m\to\infty} \left\| L_{n,m}(f;\cdot,\cdot) - f \right\|_{C(S_A)} = 0.$$

Proof. According to Theorem 1, we should prove that

$$\lim_{n,m\to\infty} \left\| L_{n,m}(\varphi_i;\cdot,\cdot) - \varphi_i \right\|_{C(S_A)} = 0, \qquad (i = 0, 1, 2, 3, 4).$$

By (1), $L_{n,m}(\varphi_0; x, y) = \varphi_0(x, y)$. From Lemma 4 (b),(c) and (d), we obtain

$$\lim_{n,m\to\infty} \left\| L_{n,m}(\varphi_i;\cdot,\cdot) - \varphi_i \right\|_{C(S_A)} = 0, \quad (i = 1, 2, 3).$$

Finally, by Lemma 4 (e), we get

$$\begin{aligned}
&L_{n,m}\left(\varphi_4;x,y,z\right)-\varphi_4(x,y,z)\\
&=\left[\frac{(n+2)}{(n+1)}-1\right]\varphi_4(x,y,z)+\left\{\frac{(m+2)}{(m+1)}-\frac{(n+2)}{(n+1)}\right\}\varphi_3^2(x,y,z)\\
&+\frac{1}{(n+1)}\left\{\varphi_1(x,y,z)+\varphi_2(x,y,z)+\varphi_3(x,y,z)\right\}\\
&+\left\{\frac{1}{(m+1)}-\frac{1}{(n+1)}\right\}\varphi_3(x,y,z)
\end{aligned}$$

and hence

$$\begin{aligned}
&\left|L_n\left(\varphi_4;x,y,z\right)-\varphi_3(x,y,z)\right|\\
&\leq\left|\frac{(n+2)}{(n+1)}-1\right|\varphi_4(x,y,z)+\left|\frac{(m+2)}{(m+1)}-\frac{(n+2)}{(n+1)}\right|\varphi_3^2(x,y,z)\\
&+\frac{1}{(n+1)}\left\{\varphi_1(x,y,z)+\varphi_2(x,y,z)+\varphi_3(x,y,z)\right\}\\
&+\left|\frac{1}{(m+1)}-\frac{1}{(n+1)}\right|\varphi_3(x,y,z).
\end{aligned}$$

Taking supremum over S_A and letting

$$B(A)=\sup_{(x,y,z)\in S_A}\left\{\varphi_1(x,y,z),\varphi_2(x,y,z),\varphi_3(x,y,z),\varphi_3^2(x,y,z),\varphi_4(x,y,z)\right\},$$

we get

$$\begin{aligned}
&\left\|L_{n,m}\left(\varphi_4;x,y,z\right)-\varphi_4(x,y,z)\right\|_{C(S_A)}\\
&\leq B(A)\left[\frac{2}{(n+1)}+\left|\frac{(m+2)}{(m+1)}-\frac{(n+2)}{(n+1)}\right|+\left|\frac{1}{(m+1)}-\frac{1}{(n+1)}\right|\right].
\end{aligned}$$

Passing to limit as $n,m\to\infty$, the proof is completed. □

4. The Order of Approximation

In this section, we compute the order of approximation of $L_{n,m}\left(f;x,y,z\right)$ to $f(x,y,z)$ in terms of the modulus of continuity and the modified Lipschitz class functionals. We start with the following lemma.

Lemma 6. *Let $L_{n,m}$ be a sequence of linear positive operators defined by (1). Then the following estimate*

$$\begin{aligned}
&L_{n,m}\left(\sqrt{\sum_{i=1}^{3}\left(\varphi_i(s,t,u)-\varphi_i(x,y,z)\right)^2};x,y,z\right)\\
&\leq\sqrt{B(A)\left[\frac{2}{(n+1)}+\left|\frac{(m+2)}{(m+1)}-\frac{(n+2)}{(n+1)}\right|+\left|\frac{1}{(m+1)}-\frac{1}{(n+1)}\right|\right]},
\end{aligned}$$

holds true, where

$$B(A)=\sup_{(x,y,z)\in S_A}\left\{\varphi_1(x,y,z),\varphi_2(x,y,z),\varphi_3(x,y,z),\varphi_3^2(x,y,z),\varphi_4(x,y,z)\right\}.$$

Proof. Using Cauchy-Schwarz inequality and noting that $L_n(1; x, y) = 1$, we get

$$L_{n,m}\left(\sqrt{\sum_{i=1}^{3}(\varphi_i(s,t,u)-\varphi_i(x,y,z))^2}; x,y,z\right) \leq \sqrt{L_{n,m}\left(\left(\sum_{i=1}^{3}(\varphi_i(s,t,u)-\varphi_i(x,y,z))^2\right); x,y,z\right)}.$$

On the other hand by Lemma 4, we can write

$$\begin{aligned}
&L_{n,m}\left((\varphi_1(s,t,u)-\varphi_1(x,y,z))^2+(\varphi_2(s,t,u)-\varphi_2(x,y,z))^2; x,y,z\right)\\
&\leq \left|L_{n,m}(\varphi_4; x,y)-\varphi_4(x,y)\right| + \frac{2z}{1-z}\left|L_{n,m}(\varphi_3; x,y,z)-\varphi_3(x,y,z)\right|\\
&+\frac{2y}{1-x-y}\left|L_n(\varphi_2; x,y,z)-\varphi_2(x,y,z)\right| + \frac{2x}{1-x-y}\left|L_n(\varphi_1; x,y,z)-\varphi_1(x,y,z)\right|\\
&\leq B(A)\left[\frac{2}{(n+1)}+\left|\frac{(m+2)}{(m+1)}-\frac{(n+2)}{(n+1)}\right|+\left|\frac{1}{(m+1)}-\frac{1}{(n+1)}\right|\right].
\end{aligned}$$

Whence the result. □

Theorem 7. *Let $L_{n,m}$ be a sequence of linear positive operators defined by (1). Then for all $f \in \mathcal{H}_\omega(S_A)$, we have*

$$\left\|L_{n,m}(f;\cdot,\cdot)-f\right\|_{C(S_A)} \leq 2\omega\left(f, \sqrt{B(A)\left[\frac{2}{(n+1)}+\left|\frac{(m+2)}{(m+1)}-\frac{(n+2)}{(n+1)}\right|+\left|\frac{1}{(m+1)}-\frac{1}{(n+1)}\right|\right]}\right)$$

where $B(A)$ is the same as in Lemma 6.

Proof. Because of linearity and monotonicity of the operators $L_{n,m}$, we get by (6) and (7) that

$$\begin{aligned}
&\left|L_{n,m}(f; x,y,z)-f(x,y,z)\right|\\
&\leq L_{n,m}\left(\left|f(s,t,u)-f(x,y,z)\right|; x,y,z\right)\\
&\leq L_{n,m}\left(\omega\left(f, \sqrt{\sum_{i=1}^{3}(\varphi_i(s,t,u)-\varphi_i(x,y,z))^2}\right); x,y,z\right)\\
&\leq \omega(f,\delta_n)\left[1+\frac{L_{n,m}\left(\sqrt{\sum_{i=1}^{3}(\varphi_i(s,t,u)-\varphi_i(x,y,z))^2}; x,y,z\right)}{\delta_n}\right].
\end{aligned}$$

Now using Lemma 6 and choosing

$$\delta_n = \sqrt{B(A)\left[\frac{2}{(n+1)}+\left|\frac{(m+2)}{(m+1)}-\frac{(n+2)}{(n+1)}\right|+\left|\frac{1}{(m+1)}-\frac{1}{(n+1)}\right|\right]},$$

we get the result after taking supremum over S_A on both sides of the inequality. □

Now, we are aimed to compute the order of convergence of the operators in terms of the modified Lipschitz class functionals. Note that in recent years modified versions of Lipschitz class functionals have severe attraction in approximation theory (see for instance [30]). In order to coorporate nicely with the MKZ operators, in the present paper, we introduce the modified Lipschitz class functions by

$$\begin{aligned} &Lip^*_M(\alpha) \\ &:= \left\{ f \in C(S_A) : \left| f(s,t,u) - f(x,y,z) \right| \le M \left[\left(\frac{s}{1-s-t} - \frac{x}{1-x-y} \right)^2 \right.\right. \\ &\left.\left. + \left(\frac{t}{1-s-t} - \frac{y}{1-x-y} \right)^2 + \left(\frac{u}{1-u} - \frac{z}{1-z} \right)^2 \right]^{\alpha/2} ; (t,s,u), (x,y,z) \in S_A \right\} \end{aligned}$$

where M is a positive constant depending on f and $\alpha \in (0,1]$.

Theorem 8. *Let $L_{n,m}$ be a sequence of linear positive operators defined by 1. Then for all $f \in Lip^*_M(\alpha)$, we have*

$$\begin{aligned} &\left\| L_{n,m}(f;\cdot,\cdot) - f \right\|_{C(S_A)} \\ &\le M \left[B(A) \left[\frac{2}{(n+1)} + \left| \frac{(m+2)}{(m+1)} - \frac{(n+2)}{(n+1)} \right| + \left| \frac{1}{(m+1)} - \frac{1}{(n+1)} \right| \right] \right]^{\alpha/2}, \end{aligned}$$

where $B(A)$ is the same as in Lemma 6.

Proof. Using linearity and monotonicity properties of the operators $L_{n,m}$ and considering that $f \in Lip^*_M(\alpha)$, we get

$$\begin{aligned} &\left| L_{n,m}(f;x,y,z) - f(x,y,z) \right| \\ &\le L_{n,m} \left(\left| f(s,t,u) - f(x,y,z) \right| ; x,y,z \right) \\ &\le M L_{n,m} \left(\left[\left(\frac{s}{1-s-t} - \frac{x}{1-x-y} \right)^2 + \left(\frac{t}{1-s-t} - \frac{y}{1-x-y} \right)^2 \right.\right. \\ &\left.\left. + \left(\frac{u}{1-u} - \frac{z}{1-z} \right)^2 \right]^{\alpha/2} ; x,y,z \right). \end{aligned}$$

Applying the Hölder inequality with $p = \dfrac{1}{\alpha}$ and $q = \dfrac{1}{1-\alpha}$ and taking into account that $L_n(1;x,y) = 1$, we obtain

$$\begin{aligned} &\left| L_{n,m}(f;x,y) - f(x,y) \right| \\ &\le M \left\{ L_{n,m} \left(\sqrt{\sum_{i=1}^{3} \left(\varphi_i(s,t,u) - \varphi_i(x,y,z) \right)^2} ; x,y,z \right) \right\}^{\alpha}. \end{aligned}$$

Taking supremum over S_A on both sides of the above inequality and considering Lemma 6, the proof is completed. □

5. Generalized form of $L_{n,m}$

Let $P^{n,m}_{k,l,p}(\beta,\alpha,\gamma) \ge 0$ and $\{\Omega_{n,m}(s,t,u;x,y,z)\}_{n,m\in\mathbb{N}}$ is the generating function for the triple indexed sequence $\left\{ P^{n,m}_{k,l,p}(\beta,\alpha,\gamma) \right\}_{k,l,p\in\mathbb{N}_0}$ given in the form

$$\Omega_{n,m}(\alpha,\gamma,\beta;x,y,z) = \sum_{k,l,p=0}^{\infty} P^{n,m}_{k,l,p}(\beta,\alpha,\gamma) x^k y^l z^p \tag{17}$$

where the series on the righthand side is uniformly convergent for all $(x, y, z) \in S_A$.

We define the generalized form of the non-tensor three variable Meyer-König and Zeller operators;

$$L^*_{n,m}(f; x, y, z) = \frac{1}{\Omega_{n,m}(\alpha, \gamma, \beta; x, y, z)} \times \sum_{k,l,p=0}^{\infty} f\left(\frac{k}{k+l+b_n}, \frac{l}{k+l+b_n}, \frac{p}{p+d_m}\right) P^{n,m}_{k,l,p}(\beta, \alpha, \gamma) x^k y^l z^p \tag{18}$$

where $\Omega_{n,m}(\beta, \alpha, \gamma; x, y, z)$ is given by (17), $\{b_n\}_{n\in\mathbb{N}}$ and $\{d_m\}_{m\in\mathbb{N}}$ are number sequences.

In this section, we obtain a functional partial differantial equation satisfied by $L^*_{n,m}(f; x, y, z)$.

Theorem 9. *Let $(x, y, z) \in S_A$, $f \in C(S_A)$, $L^*_{n,m}(f; x, y, z)$ be defined by (18) and (17). Assume that*

$$\frac{\partial}{\partial x}\left(\Omega_{n,m}(\beta, \alpha, \gamma; x, y, z)\right) = A_n(x, y, z)\Omega_{n,m}(\alpha, \gamma, \beta; x, y, z), \tag{19}$$

$$\frac{\partial}{\partial y}\left(\Omega_{n,m}(\beta, \alpha, \gamma; x, y, z)\right) = B_n(x, y, z)\Omega_{n,m}(\alpha, \gamma, \beta; x, y, z), \tag{20}$$

$$\frac{\partial}{\partial z}\left(\Omega_{n,m}(\beta, \alpha, \gamma; x, y, z)\right) = D_m(x, y, z)\Omega_{n,m}(\alpha, \gamma, \beta; x, y, z). \tag{21}$$

Then ${}^*L^{(\beta,\alpha,\gamma)}_{n,m}(f; x, y, z)$ satisfy the following functional partial differential equations

$$x\frac{\partial}{\partial x}L^*_{n,m}(f; x, y, z) = -xA_n(x, y, z)L^*_{n,m}(f; x, y, z) + b_n L^*_{n,m}(\varphi_1 f; x, y, z) \tag{22}$$

$$y\frac{\partial}{\partial y}L^*_{n,m}(f; x, y, z) = -yB_n(x, y, z)L^*_{n,m}(f; x, y, z) + b_n L^*_{n,m}(\varphi_2 f; x, y, z) \tag{23}$$

$$z\frac{\partial}{\partial z}L^*_{n,m}(f; x, y, z) = -zD_m(x, y, z)L^*_{n,m}(f; x, y, z) + d_m L^*_{n,m}(\varphi_3 f; x, y, z) \tag{24}$$

$$\begin{aligned}&\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)L^*_{n,m}(f; x, y, z)\\ &= \{-xA_n(x, y, z) - yB_n(x, y, z) - zD_m(x, y, z)\}\, L^*_{n,m}(f; x, y, z)\\ &+ b_n L^*_{n,m}(hf; x, y, z) + d_m L^*_{n,m}(gf; x, y, z)\end{aligned} \tag{25}$$

where

$$h(s, t) = \frac{s}{1-s-t} + \frac{t}{1-s-t}, \quad g(u) = \frac{u}{1-u}.$$

Proof. Since $f \in C(S_A)$, it is clear by (17) that the series (18) converges uniformly for all $(x, y, z) \in S_A$. Therefore, term by term differentiation is permissible in S_A. Differentiating both sides of (18) with respect to x and considering (19), we get

$$\frac{\partial}{\partial x}L^*_{n,m}(f; x, y, z) = -A_n(x, y, z)L^*_{n,m}(f; x, y, z) + \frac{1}{\Omega_{n,m}(\beta, \alpha, \gamma; x, y, z)} \times \sum_{k=1,l,p=0}^{\infty} kf\left(\frac{k}{k+l+b_n}, \frac{l}{k+l+b_n}, \frac{p}{p+d_m}\right) P^{n,m}_{k,l,p}(\beta, \alpha, \gamma) x^{k-1} y^l z^p.$$

Muliplying both sides by x, we have

$$x\frac{\partial}{\partial x}L^*_{n,m}(f;x,y,z) = -xA_n(x,y,z)L^*_{n,m}(f;x,y,z) + \frac{b_n}{\Omega_{n,m}(\beta,\alpha,\gamma;x,y,z)}$$
$$\times \sum_{k=0,l,p=0}^{\infty} \frac{k}{b_n} f\left(\frac{k}{k+l+b_n}, \frac{l}{k+l+b_n}, \frac{p}{p+d_m}\right) P^{n,m}_{k,l,p}(\beta,\alpha,\gamma)x^k y^l z^p$$
$$= -xA_n(x,y,z)L^*_{n,m}(f;x,y,z) + b_n L^*_{n,m}(\varphi_1 f;x,y,z)$$

which gives (22). Now, differentiating both sides of (18) with respect to y and taking into account (20), we get

$$\frac{\partial}{\partial y}L^*_{n,m}(f;x,y,z) = -B_n(x,y,z)L^*_{n,m}(f;x,y,z) + \frac{1}{\Omega_{n,m}(\beta,\alpha,\gamma;x,y,z)}$$
$$\times \sum_{k=0,l=1,p=0}^{\infty} l f\left(\frac{k}{k+l+b_n}, \frac{l}{k+l+b_n}, \frac{p}{p+d_m}\right) P^{n,m}_{k,l,p}(\beta,\alpha,\gamma)x^k y^{l-1} z^p.$$

Muliplying both sides by y, we have

$$y\frac{\partial}{\partial y}L^*_{n,m}(f;x,y,z) = -yB_n(x,y,z)L^*_{n,m}(f;x,y,z) + \frac{b_n}{\Omega_{n,m}(\alpha,\gamma,\beta;x,y,z)}$$
$$\times \sum_{k=0,l,p=0}^{\infty} \frac{l}{b_n} f\left(\frac{k}{k+l+b_n}, \frac{l}{k+l+b_n}, \frac{p}{p+d_m}\right) P^{n,m}_{k,l,p}(\beta,\alpha,\gamma)x^k y^l z^p$$
$$= -yB_n(x,y,z)L^*_{n,m}(f;x,y,z) + b_n L^*_{n,m}(\varphi_2 f;x,y,z).$$

Finally, differentiating both sides of (18) with respect to z and taking into account (21), we get

$$\frac{\partial}{\partial z}L^*_{n,m}(f;x,y,z) = -D_m(x,y,z)L^*_{n,m}(f;x,y,z) + \frac{1}{\Omega_{n,m}(\beta,\alpha,\gamma;x,y,z)}$$
$$\times \sum_{k=0,l=0,p=1}^{\infty} p f\left(\frac{k}{k+l+b_n}, \frac{l}{k+l+b_n}, \frac{p}{p+d_m}\right) P^{n,m}_{k,l,p}(\beta,\alpha,\gamma)x^k y^l z^{p-1}.$$

Muliplying both sides by z, we have

$$z\frac{\partial}{\partial y}L^*_{n,m}(f;x,y,z) = -zD_m(x,y,z)L^*_{n,m}(f;x,y,z) + \frac{d_m}{\Omega_{n,m}(\beta,\alpha,\gamma;x,y,z)}$$
$$\times \sum_{k=0,l,p=0}^{\infty} \frac{p}{d_m} f\left(\frac{k}{k+l+b_n}, \frac{l}{k+l+b_n}, \frac{p}{p+d_m}\right) P^{n,m}_{k,l,p}(\beta,\alpha,\gamma)x^k y^l z^p$$
$$= -zD_m(x,y,z)L^*_{n,m}(f;x,y,z) + d_m L^*_{n,m}(\varphi_3 f;x,y,z).$$

This proves (24). Adding (22),(23) and (24), then taking into account that $L^*_{n,m}$ is linear, we get (25). □

Remark 10. *Letting* $\frac{1}{\Omega_{n,m}(\alpha,\gamma,\beta;x,y,z)} = (1-z)^{m+1}(1-x-y)^{n+1}$, $d_m = m+1$ *and* $b_n = n+1$ *in (18), we get three variable Meyer-König and Zeller operators given by (1):*

$$L_{n,m}(f;x,y,z) = (1-z)^{m+1}(1-x-y)^{n+1}$$
$$\times \sum_{k,l,p=0}^{\infty} f\left(\frac{k}{n+k+l+1}, \frac{l}{n+k+l+1}, \frac{p}{m+p+1}\right) P^n_m(k,l,p)x^k y^l z^p.$$

Remark 11. *Let $\beta, \alpha, \gamma \geq 0$ be fixed, and take $\frac{1}{\Omega_{n,m}(\beta,\alpha,\gamma;x,y,z)} = (1-z)^{\beta+m+1}(1-x-y)^{\alpha+\gamma+n+1}$, we define the generalized form of the nontensor three variable Meyer-König and Zeller operators:*

$$A_{n,m}^{(\beta,\alpha,\gamma)}(f;x,y,z) = (1-z)^{d_m}(1-x-y)^{b_n} \times \sum_{k,l,p=0}^{\infty} f\left(\frac{k}{k+l+b_n}, \frac{l}{k+l+b_n}, \frac{p}{p+d_m}\right) P_{k,l,p}^{n,m}(\beta,\alpha,\gamma)x^k y^l z^p$$

where $b_n = n+\alpha+\gamma+1$, $d_m = m+\beta+1$ and

$$P_{k,l,p}^{n,m}(\beta,\alpha,\gamma) = \frac{(n+\alpha+\gamma+k+l)!}{k!l!}\binom{m+\beta+p}{p} = \frac{\Gamma(n+\alpha+\gamma+k+l+1)}{\Gamma(n+\alpha+\gamma+1)\,k!l!}.\frac{\Gamma(m+\beta+p+1)}{\Gamma(p+1)\,\Gamma(m+\beta+1)}.$$

Clearly, $A_{n,m}^{(0,0,0)}(f;x,y,z) = L_{n,m}(f;x,y,z)$.

Corollary 12. *Let $(x,y,z) \in S_A, f \in C(S_A)$. Then $A_{n,m}^{(\beta,\alpha,\gamma)}(f;x,y,z)$ satisfy the following functional partial differential equations*

$$x\frac{\partial}{\partial x}A_{n,m}^{(\beta,\alpha,\gamma)}(f;x,y,z) = \frac{-x(n+\alpha+\gamma+1)}{(1-x-y)}A_{n,m}^{(\beta,\alpha,\gamma)}(f;x,y,z) + (n+\alpha+\gamma+1)\,A_{n,m}^{(\beta,\alpha,\gamma)}(\varphi_1 f;x,y,z) \tag{26}$$

$$y\frac{\partial}{\partial y}A_{n,m}^{(\beta,\alpha,\gamma)}(f;x,y,z) = \frac{-y(n+\alpha+\gamma+1)}{(1-x-y)}A_{n,m}^{(\beta,\alpha,\gamma)}(f;x,y,z) + (n+\alpha+\gamma+1)\,A_{n,m}^{(\beta,\alpha,\gamma)}(\varphi_2 f;x,y,z) \tag{27}$$

$$z\frac{\partial}{\partial z}A_{n,m}^{(\beta,\alpha,\gamma)}(f;x,y,z) = \frac{-z(m+\beta+1)}{(1-z)}A_{n,m}^{(\beta,\alpha,\gamma)}(f;x,y,z) + (m+\beta+1)\,A_{n,m}^{(\beta,\alpha,\gamma)}(\varphi_3 f;x,y,z) \tag{28}$$

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)A_{n,m}^{(\beta,\alpha,\gamma)}(f;x,y,z) = \left\{\frac{-(n+\alpha+\gamma+1)(x+y)}{(1-x-y)} - \frac{z(m+\beta+1)}{(1-z)}\right\}A_{n,m}^{(\beta,\alpha,\gamma)}(f;x,y,z) + (n+\alpha+\gamma+1)\,A_{n,m}^{(\beta,\alpha,\gamma)}(hf;x,y,z) + (m+\beta+1)\,A_{n,m}^{(\beta,\alpha,\gamma)}(gf;x,y,z) \tag{29}$$

where

$$h(s,t) = \frac{s}{1-s-t} + \frac{t}{1-s-t}, g(u) = \frac{u}{1-u}.$$

Using the same technique as in Lemma 4, we obtain

$$
\begin{aligned}
A_{n,m}^{(\beta,\alpha,\gamma)}(\varphi_0;x,y,z) &= \varphi_0(x,y,z)\\
A_{n,m}^{(\beta,\alpha,\gamma)}(\varphi_1;x,y,z) &= \varphi_1(x,y,z)\\
A_{n,m}^{(\beta,\alpha,\gamma)}(\varphi_2;x,y,z) &= \varphi_2(x,y,z)\\
A_{n,m}^{(\beta,\alpha,\gamma)}(\varphi_3;x,y,z) &= \varphi_3(x,y,z)\\
A_{n,m}^{(\beta,\alpha,\gamma)}(\varphi_4;x,y,z) &= \frac{(n+\alpha+\gamma+2)}{(n+\alpha+\gamma+1)}\varphi_4(x,y,z)\\
&\quad+\left|\frac{(m+\beta+2)}{(m+\beta+1)}-\frac{(n+\alpha+\gamma+2)}{(n+\alpha+\gamma+1)}\right|\varphi_3^2(x,y,z)\\
&\quad+\frac{1}{(n+\alpha+\gamma+1)}\left\{\varphi_1(x,y,z)+\varphi_2(x,y,z)+\varphi_3(x,y,z)\right\}\\
&\quad+\left|\frac{1}{(m+\beta+1)}-\frac{1}{(n+\alpha+\gamma+1)}\right|\varphi_3(x,y,z).
\end{aligned}
$$

As a consequence of Theorem 1, we can state the following corollary.

Corollary 13. *For all $f\in\mathcal{H}_\omega(S_A)$, we have*

$$\lim_{n\to\infty}\left\|A_{n,m}^{(\beta,\alpha,\gamma)}(f;.,.,.)-f\right\|_{C(S_A)}=0.$$

Following the similar procedures as in the proofs of Theorem 7 and Theorem 8, we can state the following corollaries, which gives the order of approximation by the generalised form of the Meyer-König and Zeller operator in terms of modulus of continuity and Lipshitz class functions, respectively.

Corollary 14. *For all $f\in\mathcal{H}_\omega(S_A)$, we have*

$$
\begin{aligned}
&\left\|A_{n,m}^{(\beta,\alpha,\gamma)}(f;.,.,.)-f\right\|_{C(S_A)}\\
&\leq 2\omega\left(f,\sqrt{B(A)\left[\frac{2}{(n+\alpha+\gamma+1)}+\left|\frac{(m+\beta+2)}{(m+\beta+1)}-\frac{(n+\alpha+\gamma+2)}{(n+\alpha+\gamma+1)}\right|+\left|\frac{1}{(m+\beta+1)}-\frac{1}{(n+\alpha+\gamma+1)}\right|\right]}\right)
\end{aligned}
$$

where

$$B(A)=\sup_{(x,y,z)\in S_A}\left\{\varphi_1(x,y,z),\varphi_2(x,y,z),\varphi_3(x,y,z),\varphi_3^2(x,y,z),\varphi_4(x,y,z)\right\}.$$

Corollary 15. *Let $\alpha\in(0,1]$. For all $f\in Lip_M^*(\alpha)$, we have*

$$
\begin{aligned}
\left\|A_{n,m}^{(\beta,\alpha,\gamma)}(f;.,.,.)-f\right\|_{C(S_A)} \leq M\Bigg[B(A)\Bigg(&\frac{2}{(n+\alpha+\gamma+1)}+\left|\frac{(m+\beta+2)}{(m+\beta+1)}-\frac{(n+\alpha+\gamma+2)}{(n+\alpha+\gamma+1)}\right|\\
&\left|+\frac{1}{(m+\beta+1)}-\frac{1}{(n+\alpha+\gamma+1)}\right|\Bigg)\Bigg]^{\frac{\alpha}{2}}
\end{aligned}
$$

where

$$B(A)=\sup_{(x,y,z)\in S_A}\left\{\varphi_1(x,y,z),\varphi_2(x,y,z),\varphi_3(x,y,z),\varphi_3^2(x,y,z),\varphi_4(x,y,z)\right\}.$$

Data Availibility

No data were used to support this study.

Declarations

The authors declare no competing financial interests

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