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On a version of the Korovkin theorem

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Abstract. In this article, some of the main results of the paper"*An abstract version of the Korovkin theorem via $\mathcal{A}$-summation proces*" [2] are re-stated and proved. A Korovkin type theorem is given on a compact Hausdorff space.

1. Introduction

One of the most important theorems in Approximation Theory is no doubt *Korovkin's Theorem*, which one version can be stated as follows: Let $C([0,1])$ be the Banach space of continuous functions from the interval $[0,1]$ to $\mathbb{R}$ with pointwise algebraic operations and the norm under usual

$$\|f\| = \sup_{x\in[a,b]} |f(x)|,$$

$f_i : [0,1] \to \mathbb{R}$ be continuous functions defined by

$$f_i(x) = x^i,$$

where $i = 1,2,3$ and (T_n) be a sequence of positive operators from $C([0,1])$ into $C([0,1])$ satisfying for each $i = 1,2,3$,

$$\|T_n(f_i) - f_i\| \to 0 \; (n \longrightarrow \infty).$$

Then for each $f \in C([0,1])$ one has

$$\|T_n(f) - f\| \to 0 \; (n \longrightarrow \infty),$$

for all $f \in C([0,1])$.

Recall that a map from $C([0,1])$ into $C([0,1])$ is called a *positive operator* if it is linear and $T(f) \geq 0$ whenever $f \geq 0$ (the latter meaning that $f(x) \geq 0$ for all $x \in [0,1]$). It can be easily observed that a proof of Korovkin's Theorem follows from the following fact: for each $\epsilon > 0$ there exist constants C_1, C_2, $C_3 \geq 0$ such that

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$$\|T_n(f) - f\| \le \epsilon + C_1\|T_n(f_0) - f_0\| + C_2\|T_n(f_1) - f_1\| + C_3\|T_n(f_2) - f_2\|$$

holds for all $f \in C([0,1])$. A proof of this inequality can be found in [1].

For a topological space X, as usual, $C(X)$ denotes the vector space of all continuous functions from X into $\mathbb{R}$ with the pointwise algebraic operations. The vector subspace $C_b(X)$, the space of all bounded real-valued continuous functions of $C(X)$ is a Banach space under the norm

$$\|f\| = \sup_{x\in X} \|f(x)\|.$$

2. On the Statement of the Paper [2]

One could be interested in generalizations of the Korovkin's theorem for the space $C_b(X)$. The best way for this is to start with $C(X)$, where X is compact. This is well done in [2], but unfortunately in the statement of [2] all lemmas and theorems involve the term $\mathcal{A}$-summation process on $C(X)$ for a sequence of positive operators (L_j), which is not necessary. Those were given more or less, in the following form:

$$\text{“}\forall f \in C(X), \lim_k \|B_k^{(n)}(f) - f\| = 0 \to \lim_k \|B_k^{(n)}(g) - f\| = 0 \text{ for some } g \in C(X)\text{”}.$$

The reason for this, we believe, is that the authors of [2] are ignored the fact that if

$$\|B_k^{(n)}(f) - f\| \to 0 \ (k \to \infty)$$

for all $f \in C(X)$ and n, then $B_k^{(n)}(f) \in C(X)$ and the operator $B_k^{(n)}$ is positive operator into $C(X)$. It should also be noted that Lemma 4 is incorrect. Indeed, for each n, let $T_n : C([0,1]) \to C([0,1])$ be defined by

$$T_n(f) = \frac{1}{n} f(0) + f.$$

Then T_n is a positive operator and $T_n(f) \to f$ for all $f \in C(X)$. For each $x \in [0,1] \setminus \{\frac{1}{n} : n \in \mathbb{N}\}$, let $f_x = 0$ and for $n \in \mathbb{N}$, let $f_{\frac{1}{n}} \in C([0,1])$ such that

$$f_{\frac{1}{n}}(0) = n \text{ and } f_{\frac{1}{n}}(\tfrac{1}{n}) = 0.$$

Now it is obvious that

$$1 \le \sup_{x\in[0,1]} |T_n(f_x)(x)|.$$

Also in the proof of this Lemma 4 of [2] m and M defined depend on $x \in X$, contrary to the situation in the proof which assumes their independence from x. Despite the above-mentioned facts, using similar lines of thought of the paper, we can restate the lemmas and theorems with simpler and more natural proofs.

3. Restatement of Theorems of [2]

We will restate and reprove in this section results of [2] using similar ideas. Let

$$f_1, f_2, g_1, g_2 \in C(X)$$

with the following properties:
i) $P(x, y) = 0$ if and only if $x = y$,
ii) $P(x, y) \ge 0$
for all $x, y \in X$, where $P \in C(X \times X)$ defined by,

$$P(x, y) = g_1(x) f_1(y) + g_2(x) f_2(y).$$

In particular for each $x \in X$ we define $P_x \in C(X)$ by

$$P_x(y) = P(x, y).$$

Throughout the paper $s, t \in X$ with $s \neq t$ and we let

$$Q = P_s + P_t.$$

Lemma 3.1. *Let X be a compact Hausdorff space and let $f \in C(X)$ be given. For each $\epsilon > 0$ there exists $K > 0$ such that*

$$|f - f(x)| \leq \epsilon + KP_x \ (x \in X).$$

In particular, if for $x \in X$, $h_x \in C(X)$ with

$$h_x(x) = 0 \text{ and } \sup_{x \in X} \|h_x\| < \infty$$

then there exists $K > 0$ such that for all x,

$$|h_x| \leq \epsilon + KP_x,$$

where K is independent of $x \in X$.

Proof. Let $x \in X$ be given. Since $|f - f(x)|(x) = 0$, from the continuity of $f - f(x)$, there exists an open set U_x containing x such that

$$|f - f(x)|(y) \leq \epsilon$$

for all $y \in U_x$. Let

$$m_x = \inf_{y \in U_x} P(x, y) \text{ and } M = 2\|f\|.$$

Since P_x is continuous, $X \setminus U_x$ is compact and $P(x, y) > 0$ for all $y \in X \setminus U_x$ we have $m_x > 0$. Since X is compact and $(U_x)_{x \in X}$ is an open cover of X there exist $x_1, \ldots, x_n \in X$ such that

$$X = \bigcup_{i=1}^{n} U_{x_i}.$$

Let

$$m = \min\{m_{x_1}, ..., m_{x_n}\} > 0.$$

Set $K = \frac{M}{m}$. Now the required inequality is obvious.

Lemma 3.2. *Let X be a topological space and $T : C_b(X) \to C_b(X)$ be a linear map. Then there is $M \geq 0$ such that for each $x \in X$, one has*

$$\begin{aligned} |T(P_x)(x)| &\leq M[|T(f_1) - f_1|(x) + |T(f_2) - f_2|(x)] \\ &\leq M\left[\|T(f_1) - f_1\| + \|T(f_2) - f_2\|\right]. \end{aligned}$$

Proof. Let $M = \|g_1\| + \|g_2\|$. For each $x \in X$, it follows from the definition of P_x that $P_x(x) = 0$, and from the equality

$$T(P_x) = g_1(x)[T(f_1) - f_1(x)] + g_2(x)[T(f_2) - f_2(x)]$$

that we have the required inequality.

Corollary 3.3. *Let X be a compact Hausdorff space, (T_n) be a sequence of positive operators from $C(X)$ into $C(X)$ satisfying*

$$T_n(f_1) \to f_1 \text{ and } T_n(f_2) \to f_2 \, (n \to \infty).$$

Then

$$\sup_{x\in X} \|T_n(P_x)\| \to 0.$$

Proof. Follows immediately from Lemma 3.2.

Lemma 3.4. *Let X be a compact Hausdorff space, (T_n) is a sequence of positive operators from $C(X)$ into $C(X)$. If*

$$T_n(f_1) \to f_1 \text{ and } T_n(f_2) \to f_2 \ (n \to \infty)$$

then

i) $T_n(Q) \to Q \ (n \to \infty)$
ii) $\sup \|T_n(1)\| < \infty$.

Proof. i) This is obvious.
ii) Since for each $x \in X$, $Q(x) > 0$ and X is compact there exists $\epsilon > 0$ such that $\epsilon \leq Q$. Then $\epsilon T_n(1) \leq T_n(Q)$ and we have

$$\epsilon \|T_n(1)\| \leq \|T_n(Q)\| \to \|Q\|,$$

whence

$$\sup \|T_n(1)\| < \infty.$$

Lemma 3.5. *Let X be a compact Hausdorff space. For each $f \in C(X)$, $x \in X$ there exists $f_x \in C(X)$ with $f_x(x) = 0$ such that*

$$[T_n(f) - f](x) = T_n(h_x) + \frac{f(x)}{Q(x)}[T_n(Q) - Q](x)$$

for all $x \in X$.

Proof. It is enough to take

$$h_x = f - \tfrac{f(x)}{Q(x)} Q.$$

Theorem 3.6. *Let X be a compact Hausdorff space and (T_n) be a sequence of positive operators from $C(X)$ into $C(X)$. If*

$$T_n(f_i) \to f_i \ \ (i = 1, 2) \quad (n \to \infty)\,,$$

then

$$T_n(f) \to f \text{ for all } f \in C(X).$$

Proof. Let $f \in C(X)$ be given. By Lemma 3.5 for each $x \in X$ there exits $h_x \in C(X)$ such that

$$h_x(x) = 0, \sup_{x\in X} \|h_x\| < \infty$$

and

$$[T_n(f) - f](x) = T_n(h_x) + \tfrac{f(x)}{Q(x)}[T_n(Q) - Q].$$

This implies that

$$\|T_n(f) - f\| \leq \sup_{x\in X} \|T_n(h_x)\| + \|\tfrac{f}{Q}\|\|T_n(Q) - Q\|.$$

We know that

$$\sup_{x\in X} \|T_n(f_x)\| \leq \sup_{x\in X} \|T_n(P_x)\| \to 0 \text{ (Corollary 3.3)}$$

and

$$M\|T_n(Q) - Q\| \to 0 \text{ (Lemma 3.4)}$$

Hence we have

$$T_n(f) \to f.$$

This completes the proof.

References

[1] C. D. Aliprantis & O. Burkinshaw, *Principles of Real Analysis* , Academic Press, 1998.

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