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The φ-mixed affine surface areas

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Abstract. In the paper, our main aim is to introduce a new φ-mixed affine surface area $\Omega_{\varphi,p}(K,L)$ of convex bodies, which obeys classical basic properties. The new affine geometric quantity in special case yields the classical L_p-affine surface area $\Omega_p(K)$, L_p-mixed affine surface area $\Omega_p(K,L)$ and the newly established L_{pq}-mixed affine surface area $\Omega_{p,q}(K,L)$, respectively. As an application, we establish a φ-Minkowski inequality for the φ-mixed affine surface area, which follows the classical Minkowski inequality for mixed affine surface area $\Omega_{-1}(K,L)$, L_p-Minkowski inequality for L_p-affine surface area and L_{pq}-Minkowski inequality for L_{pq}-mixed affine surface area, respectively.

1. Introduction

A body in $\mathbb{R}^n$ is a compact set equal to the closure of its interior. A set K is called a convex body if it is compact and convex subset with non-empty interiors. Let $\mathcal{K}^n$ denote the class of convex bodies in $\mathbb{R}^n$. Let $\mathcal{K}_o^n$ denote the class of convex bodies containing the origin in their interiors in $\mathbb{R}^n$. A convex body K was said to have a positive curvature function $f(K,\cdot):S^{n-1}\to[0,\infty)$,, if its surface area measure $S(K,\cdot)$, is absolutely continuous with respect to spherical Lebesgue measure, S, and (see [1])

$$\frac{dS(K,\cdot)}{dS}=f(K,\cdot), \tag{1.1}$$

almost everywhere with respect to S. A convex body K was said to have a positive curvature function $f_p(K,\cdot):S^{n-1}\to[0,\infty)$, and $p\geq 1$, if $S_p(K,\cdot)$, is absolutely continuous with respect to spherical Lebesgue measure, S, and (see e.g. [2])

$$\frac{dS_p(K,\cdot)}{dS}=f_p(K,\cdot), \tag{1.2}$$

almost everywhere with respect to S, and where $S_p(K,\cdot)$ denotes the positive Borel measure on S^{n-1}. The subset of $\mathcal{K}^n$ consisting of convex bodies which have a positive continuous curvature function will be denoted by $\mathcal{F}^n$. The subset of $\mathcal{K}_o^n$ consisting of convex bodies which have a positive continuous curvature function will be denoted by $\mathcal{F}_o^n$. The class of the origin-symmetric convex bodies with positive and

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continuous curvature function in $\mathbb{R}^n$ will be denoted by $\mathcal{F}_s^n$. Lutwak [2] introduced the L_p-affine surface areas: For $p \geq 1$, the L_p-affine surface area of $K \in \mathcal{F}_o^n$, denoted by $\Omega_p(K)$, defined by

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K,u)^{n/(n+p)} dS(u). \tag{1.3}$$

When $p = 1$, $\Omega_p(K)$ becomes the classical affine surface area $\Omega(K)$. Moreover, the mixed affine surface area of convex bodies was introduced in [3]. The classical L_p-Blaschke addition of convex bodies $K, L \in \mathcal{F}_s^n$, denoted by $K \check{+} L$, defined by (see [4])

$$dS_p(K +_p L, \cdot) = dS_p(K, \cdot) + dS_p(L, \cdot). \tag{1.4}$$

In the paper, we consider convex and strictly increasing function $\varphi : \mathbb{R} \to [0, \infty)$ with $\varphi(0) = 0$. Let Φ be the class of convex and strictly increasing functions $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$. Our main aim is to introduce a new concept call it φ-mixed affine surface area $\Omega_{\varphi,p}(K, L)$ of convex bodies K and L, which obeys classical properties, including continuity, bounded nature and affine invariance. The φ-mixed affine surface area $\Omega_{\varphi,p}(K, L)$ in special case yields the classical L_p-affine surface area $\Omega_p(K)$, L_p-mixed affine surface area $\Omega_p(K, L)$, and the newly established L_{pq}-mixed affine surface area $\Omega_{pq}(K, L)$, respectively. We establish a φ-Minkowski inequality for the φ-mixed affine surface areas, which follows the classical Minkowski inequality for mixed affine surface area $\Omega_{-1}(K, L)$, L_p-Minkowski inequality for L_p-mixed affine surface area and L_{pq}-Minkowski inequality for L_{pq}-mixed affine surface area, respectively. As applications, some generalized φ-Minkowski type inequalities are also derived.

For $K, L \in \mathcal{F}_s^n$, The φ-mixed affine surface area of K and L, is denoted by $\Omega_{\varphi,p}(K, L)$, is defined by (see Section 3 for definition)

$$\Omega_{\varphi,p}(K, L) = \inf\left\{\lambda > 0 : \int_{S^{n-1}} \varphi\left(\frac{f_p(K,u)}{\lambda f_p(L,u)}\right) d\Omega_p(L,u) \leq 1\right\}, \tag{1.5}$$

where $p \geq 1$, $d\Omega_p(L, u)$ denotes affine surface area probability measure of L, and (see [3])

$$d\Omega_p(L,u) = \frac{1}{\Omega_p(L)} f_p(L,u)^{n/(n+p)} dS(u).$$

Remark 1.1 With $\varphi = \varphi_1(t) = t^p$ and $p = 1$, (1.5) turns out that

$$\Omega_{\varphi_1,1}(K, L) = \frac{\Omega_{-1}(K, L)}{\Omega(L)}, \tag{1.6}$$

where $\Omega_{-1}(K, L)$ is the mixed affine surface area of K and L, and (see [5])

$$\Omega_{-1}(K, L) = \int_{S^{n-1}} f(K,u) f(L,u)^{-1/(n+1)} dS(u).$$

With $\varphi = \varphi_q(t) = t^q$, and $q \geq 1$, (1.5) yields that

$$\Omega_{\varphi_q,p}(K, L) = \left(\frac{\Omega_{p,q}(L, K)}{\Omega_p(L)}\right)^{1/q}, \tag{1.7}$$

where $\Omega_{p,q}(L, K)$ is the L_{pq}-mixed affine surface area of K and L, and (see [6])

$$\Omega_{p,q}(K, L) = \int_{S^{n-1}} \left(\frac{f_p(K,u)}{f_p(L,u)}\right)^q f_p(L,u)^{n/(n+p)} dS(u). \tag{1.8}$$

When $q = 1$, (1.8) becomes the following result.

$$\Omega_{\varphi_1,p}(K, L) = \frac{\Omega_{-p}(K, L)}{\Omega_p(L)}, \tag{1.9}$$

where $\Omega_{-p}(K,L)$ is the L_p-mixed affine surface area of K and L, and (see [7])

$$\Omega_{-p}(K,L)=\int_{S^{n-1}} f_p(K,u)f_p(L,u)^{-p/(n+p)}dS(u). \tag{1.10}$$

In Section 4, we establish the following φ-Minkowski inequality for the φ-mixed affine surface areas $\Omega_{\varphi,p}(K,L)$ of convex bodies K and L.

The φ-Minkowski inequality *If $K,L\in\mathcal{F}_s^n$, $p\geq 1$, $\varphi\in\Phi$ and $\varphi(c_\varphi)=1$, then*

$$c_\varphi\Omega_{\varphi,p}(K,L)\geq \Omega_p(K)^{(n+p)/n}\Omega_p(L)^{-(n+p)/n}. \tag{1.11}$$

If φ is strictly convex, equality holds if and only if K and L are homothetic.

Remark 1.2 When $\varphi=\varphi_1(t)=t^p$ and $p=1$, (1.11) becomes the following Minkowski inequality established by Lutwak [5]. If $K,L\in\mathcal{F}_s^n$, then

$$\Omega_{-1}(K,L)\geq \Omega(K)^{(n+1)/n}\Omega(L)^{-1/n}, \tag{1.12}$$

with equality if and only if K and L are homothetic.

When $\varphi=\varphi_1(t)=t^q$ and $q\geq 1$, (1.11) becomes the following L_{pq}-Minkowski inequality established in [6]. If $K,L\in\mathcal{F}_s^n$ and $p,q\geq 1$, then

$$\Omega_{p,q}(K,L)^{\frac{n}{n+p}}\geq \Omega_p(K)^{\frac{n}{n+p}-q}\Omega_p(L)^q, \tag{1.13}$$

with equality if and only if K and L are homothetic.

When $q=1$, (1.13) becomes the following well-known L_p-Minkowski inequality. If $K,L\in\mathcal{F}_s^n$ and $p\geq 1$, then (see [7])

$$\Omega_{-p}(K,L)\geq \Omega(K)^{(n+p)/n}\Omega(L)^{-p/n}, \tag{1.14}$$

with equality if and only if K and L are homothetic.

We establish also the following generalized φ-Brunn-Minkowski inequality for three convex bodies K, K' and L.

The φ-Brunn-Minkowski type inequality. *If $K,K',L\in\mathcal{F}_s^n$, $p\geq 1$ and $\varphi(c_\varphi)=1$, then*

$$\left(\Omega_{\varphi,p}(K,L)+\Omega_{\varphi,p}(K',L)\right)^{n/(n+p)}\geq \frac{1}{c_\varphi^{n/(n+p)}}\left(\frac{\Omega_p(K+_pK')}{\Omega_p(L)}\right). \tag{1.15}$$

If φ is strictly convex, equality holds if and only if K, L and K' are homothetic.

2 Notations and preliminaries

2.1 Basics regarding convex bodies

For $\phi\in\mathrm{GL}(n)$ write ϕ^t for the transpose of ϕ and ϕ^{-t} for the inverse of the transpose of ϕ. Write $|\phi|$ for the absolute value of the determinant of ϕ. Observe that from the definition of the support function it follows immediately that for $\phi\in\mathrm{GL}(n)$ the support function of the image $\phi K=\{\phi y: y\in K\}$ is given by (see [8])

$$h(\phi K,x)=h(K,\phi^t x), \tag{2.1}$$

Let d denote the Hausdorff metric on $\mathcal{K}^n$ (see [9]), i.e., for $K,L\in\mathcal{K}^n$,

$$d(K,L)=|h(K,u)-h(L,u)|_\infty,$$

where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$.

Let Φ be the class of convex and strictly increasing functions $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$. We say that the sequence $\{\varphi_i\}$, where the $\varphi_i \in \Phi$, is such that $\varphi_i \to \varphi_0 \in \Phi$ provided

$$|\varphi_i - \varphi_0|_I := \max_{t\in I} |\varphi_i(t) - \varphi_0(t)| \to 0,$$

for every compact interval $I \subset \mathbb{R}$.

For $K \in \mathcal{K}_o^n$, r_K and R_K are defined by

$$r_K = \min_{u\in S^{n-1}} f_p(K,u), \ \ R_K = \max_{u\in S^{n-1}} f_p(K,u). \tag{2.2}$$

2.2 L_{pq}-mixed affine surface areas

The L_{pq}-Blaschke addition of convex bodies $K, L \in \mathcal{F}_s^n$ denoted by $\check{+}_{pq}$, and is defined by (see [6])

$$f_p(K \mathbin{+}_{pq} L, u)^q = f_p(K,u)^q + f_p(L,u)^q, \tag{2.3}$$

for $u \in S^{n-1}$ and $p \geq 1$. Obviously, when $q = 1$, L_{pq}-Blaschke addition becomes L_p-Blaschke addition. The following result follows immediately form (2.3) with $p, q \geq 1$.

$$\frac{q(n+p)}{n} \lim_{\varepsilon\to 0^+} \frac{\Omega_p(K\check{+}_{pq}\varepsilon \cdot L) - \Omega_p(L)}{\varepsilon} = \int_{S^{n-1}} f_p(K,u)^{\frac{n}{n+p}-q} f_p(L,u)^q dS(u).$$

Definition 2.1 Let $K, L \in \mathcal{F}_s^n$ and $p, q \geq 1$, L_{pq}-mixed affine surface area of K and L, is denoted by $\Omega_{p,q}(K, L)$, is defined by (see [6])

$$\Omega_{p,q}(K,L) = \int_{S^{n-1}} f_p(K,u)^{\frac{n}{n+p}-q} f_p(L,u)^q dS(u). \tag{2.4}$$

Obviously, when $K = L$, the L_{pq}-mixed affine surface area $\Omega_{p,q}(K,K)$ becomes the L_p affine surface area $\Omega_p(K)$. A fundamental inequality for L_{pq}-mixed affine surface area is the following L_{pq}-Minkowski inequality: If $K, L \in \mathcal{F}_s^n$ and $p, q \geq 1$, then

$$\Omega_{p,q}(K,L)^{\frac{n}{n+p}} \geq \Omega_p(K)^{\frac{n}{n+p}-q} \Omega_p(L)^q, \tag{2.5}$$

with equality if and only if K and L are homothetic.

2.3 Orlicz mixed affine surface areas

Let us introduce Orlicz mixed affine surface areas convex bodies K and L.

Definition 2.2 For $K, L \in \mathcal{F}_s^n$, $\psi \in \Phi$ and $p \geq 1$, Orlicz mixed affine surface area of K and L, is denoted by $\Omega_{\psi,p}(K, L)$, is defined by (see [6])

$$\Omega_{\psi,p}(K,L) := \int_{S^{n-1}} \psi\left(\frac{f_p(L,u)}{f_p(K,u)}\right) \cdot f_p(K,u)^{\frac{n}{n+p}} dS(u). \tag{2.6}$$

Obviously, when $K = L$ and $p \geq 1$, the Orlicz-mixed affine surface area $\Omega_{\psi,p}(K, L)$ becomes the L_p-affine surface area $\Omega_p(K)$. When $\psi(t) = t^q$ and $q \geq 1$, the Orlicz L_ψ-mixed affine surface area $\Omega_{\psi,p}(K, L)$ becomes the L_{pq}-mixed affine surface area $\Omega_{p,q}(K, L)$.

A fundamental inequality for Orlicz mixed affine surface area is the following Orlicz Minkowski inequality for Orlicz-mixed affine surface area. If $K, L \in \mathcal{F}_s^n$, $p \geq 1$ and $\psi \in \Phi$, then (see [6])

$$\Omega_{\psi,p}(K,L) \geq \Omega_p(K) \cdot \psi\left(\left(\frac{\Omega_p(L)}{\Omega_p(K)}\right)^{\frac{n+p}{n}}\right). \tag{2.7}$$

If ψ is strictly convex, equality holds if and only if K and L are homothetic.

When $\psi(t) = t^q$ and $q \geq 1$, (2.7) becomes the L_{pq}-Minkowski inequality (1.13) stated in the introduction.

3 The φ-mixed affine surface areas

Definition 3.1 (*L_p-affine surface area measure*) Let $L \in \mathcal{F}_s^n$, $p \geq 1$, the L_p-affine surface area measure of L, is denoted by $d\Omega_p(L,u)$, is defined by

$$d\Omega_p(L,u) = \frac{1}{\Omega_p(L)} f_p(L,u)^{n/(n+p)} dS(u). \tag{3.1}$$

Next, we first give the definition of φ-mixed affine surface area of convex bodies K and L.

Definition 3.2 Let $K, L \in \mathcal{F}_s^n$, $p \geq 1$ and $\varphi \in \Phi$, the φ-mixed affine surface area of convex bodies K and L, is denoted by $\Omega_{\varphi,p}(K,L)$, is defined by

$$\Omega_{\varphi,p}(K,L) = \inf\left\{\lambda > 0 : \int_{S^{n-1}} \varphi\left(\frac{f_p(K,u)}{\lambda f_p(L,u)}\right) d\Omega_p(L,u) \leq 1\right\}, \tag{3.2}$$

Lemma 3.3 (see [10]) *If $K \in \mathcal{F}_o^n$, $p \geq 1$ and $A \in \mathrm{SL}(n)$, then*

$$f_p(AK,u) = f_p(K, A^t u), \tag{3.3}$$

for all $u \in S^{n-1}$.

Since $\varphi \in \Phi$, it follows that the function:

$$\lambda \to \int_{S^{n-1}} \varphi\left(\frac{f_p(K,u)}{\lambda f_p(L,u)}\right) d\Omega_p(L,u)$$

is also strictly decreasing in $(0,\infty)$. This yields that

Lemma 3.4 *If $K, L \in \mathcal{F}_s^n$, $p \geq 1$ and $\varphi \in \Phi$, then*

$$\int_{S^{n-1}} \varphi\left(\frac{f_p(K,u)}{\lambda_0 f_p(L,u)}\right) d\Omega_p(L,u) = 1$$

if and only if

$$\Omega_{\varphi,p}(K,L) = \lambda_o.$$

In the following, we prove that the φ-mixed affine surface area $\Omega_{\varphi,p}(K,L)$ is continuous.

Lemma 3.5 *If $K, L \in \mathcal{F}_s^n$, $p \geq 1$ and $\varphi \in \Phi$, then φ-mixed affine surface area $\Omega_{\varphi,p}(K,L) : \mathcal{F}_s^n \times \mathcal{F}_s^n \to [0,\infty)$ is continuous.*

Proof To see this, indeed, let $K, L \in \mathcal{F}_s^n$, $i \in \mathbb{N} \cup \{0\}$ be such that $K_i \to K$ and $L_i \to L$ as $i \to \infty$. Noting that

$$\begin{aligned}
\Omega_{\varphi,p}(K_i,L_i) &= \inf\left\{\lambda > 0 : \int_{S^{n-1}} \varphi\left(\frac{f_p(K_i,u)}{\lambda f_p(L_i,u)}\right) d\Omega_p(L_i,u) \leq 1\right\} \\
&= \inf\left\{\lambda > 0 : \frac{1}{\Omega_p(L_i)} \int_{S^{n-1}} \varphi\left(\frac{f_p(K_i,u)}{\lambda f_p(L_i,u)}\right) f_p(L_i,u)^{n/(n+p)} dS(u) \leq 1\right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
\lim_{i\to\infty} \Omega_{\varphi,p}(K_i,L_i) &= \inf\left\{\lambda > 0 : \int_{S^{n-1}} \varphi\left(\frac{f_p(K,u)}{\lambda f_p(L,u)}\right) d\Omega_p(L,u) \leq 1\right\} \\
&= \Omega_{\varphi,p}(K,L).
\end{aligned}$$

This shows that the φ-mixed affine surface area $\Omega_{\varphi,p}(K,L)$ is continuous. □

Lemma 3.6 *If $K, L \in \mathcal{F}_s^n$, $p \geq 1$ and $\varphi_i \in \Phi$, then*

$$\varphi_i \to \varphi \in \Phi \Rightarrow \Omega_{\varphi_i,p}(K,L) \to \Omega_{\varphi,p}(K,L). \tag{3.4}$$

Proof Noting that $\varphi_i \to \varphi \in \Phi$, implies that

$$\varphi_i\left(\frac{f_p(K,u)}{\lambda f_p(L,u)}\right) \to \varphi\left(\frac{f_p(K,u)}{\lambda f_p(L,u)}\right) \in \Phi.$$

Further

$$\int_{S^{n-1}} \varphi_i\left(\frac{f_p(K,u)}{\lambda f_p(L,u)}\right) d\Omega_p(L,u) \to \int_{S^{n-1}} \varphi\left(\frac{f_p(K,u)}{\lambda f_p(L,u)}\right) d\Omega_p(L,u).$$

Hence

$$\begin{aligned}\lim_{i\to\infty} \Omega_{\varphi_i,p}(K,L) &= \inf\left\{\lambda > 0 : \int_{S^{n-1}} \varphi\left(\frac{f_p(K,u)}{\lambda f_p(L,u)}\right) d\Omega_p(L,u) \leq 1\right\} \\ &= \Omega_{\varphi_i,p}(K,L).\end{aligned}$$

This completes the proof. □

Lemma 3.7 *If $K, L \in \mathcal{F}_s^n$, $p \geq 1$ and $\varphi \in \Phi$, then φ-mixed affine surface area $\Omega_{\varphi,p}(K,L)$ is bounded.*

Proof For $\varphi \in \Phi$, there must be a real number $0 < c_\varphi < \infty$ such that $\varphi(c_\varphi) = 1$, and let

$$\Omega_{\varphi,p}(K,L) = \lambda_0.$$

Hence

$$\begin{aligned}1 &= \varphi(c_\varphi) \\ &= \int_{S^{n-1}} \varphi\left(\frac{f_p(K,u)}{\lambda_0 f_p(L,u)}\right) d\Omega_p(L,u) \\ &\geq \varphi\left(\int_{S^{n-1}} \left(\frac{f_p(K,u)}{\lambda_0 f_p(L,u)}\right) d\Omega_p(L,u)\right) \\ &\geq \varphi\left(\int_{S^{n-1}} \frac{r_K}{\lambda_0 R_L} d\Omega_p(L,u)\right) \\ &= \varphi\left(\frac{r_K}{\lambda_0 R_L}\right).\end{aligned}$$

Since φ is monotone increasing on $[0,\infty)$, from this we obtain the lower bound,

$$\lambda_0 \geq \frac{r_L}{c_\varphi R_K}.$$

In a similar approach, we can obtain upper bound for $\Omega_{\varphi,p}(K,L)$,

$$\lambda_0 \leq \frac{R_L}{c_\varphi r_K}.$$

This completes the proof. □

We easy find that the φ-mixed affine surface area $\Omega_{\varphi,p}(K,L)$ is invariant under simultaneous unimodular centro-affine transformation.

Lemma 3.8 *If $K, L \in \mathcal{F}_s^n$, $p \geq 1$, $A \in \mathrm{SL}(n)$ and $\varphi \in \Phi$, then*

$$\Omega_{\varphi,p}(AK,AL) = \Omega_{\varphi,p}(K,L). \tag{3.5}$$

Proof From (3.2) and (3.3), we obtain

$$\begin{aligned}\Omega_{\varphi,p}(K,AL) &= \inf\left\{\lambda>0:\int_{S^{n-1}}\varphi\left(\frac{f_p(K,u)}{\lambda f_p(AL,u)}\right)d\Omega_p(AL,u)\le 1\right\}\\ &= \inf\left\{\lambda>0:\frac{1}{\Omega_p(L)}\int_{S^{n-1}}\varphi\left(\frac{f_p(K,u)}{\lambda f_p(L,A^tu)}\right)f_p(L,A^tu)^{n/(n+p)}dS(u)\le 1\right\}\\ &= \inf\left\{\lambda>0:\frac{1}{\Omega_p(L)}\int_{S^{n-1}}\varphi\left(\frac{f_p(K,A^{-t}u)}{\lambda f_p(L,u)}\right)f_p(L,u)^{n/(n+p)}dS(A^{-t}u)\le 1\right\}\\ &= \inf\left\{\lambda>0:\int_{S^{n-1}}\varphi\left(\frac{f_p(A^{-1}K,u)}{\lambda f_p(L,u)}\right)d\Omega_p(L,u)\le 1\right\}\\ &= \Omega_{\varphi,p}(A^{-1}K,L).\end{aligned}$$

Hence

$$\Omega_{\varphi,p}(AK,AL)=\Omega_{\varphi,p}(K,L).$$

This completes the proof. □

4 The φ-Minkowski inequality for φ-mixed affine surface areas

Lemma 4.1 (Jensen's inequality) *Let μ be a probability measure on a space X and $g: X\to I\subset\mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. If $\psi: I\to\mathbb{R}$ is a convex function, then*

$$\int_X\psi(g(x))d\mu(x)\ge\psi\left(\int_X g(x)d\mu(x)\right). \tag{4.1}$$

If ψ is strictly convex, equality holds if and only if $g(x)$ is constant for μ-almost all $x\in X$ (see [11, p.165]).

Lemma 4.2 *Let $K,L\in\mathcal{F}_s^n$ and $p\ge 1$.*

(1) If K and L are homothetic, then K and $K+_pL$ are homothetic.

(2) If K and $K+_pL$ are homothetic, then K and L are homothetic.

Proof Suppose exist a constant $\delta>0$ such that $L=\delta K$, for $p\ge 1$, we have

$$f_p(K+_pL,u)=(1+\delta^{n-p})f_p(K,u).$$

On the other hand, the exist unique constant $\eta>0$ such that

$$f_p(\eta K,u)=(1+\delta^{n-p})f_p(K,u),$$

where η satisfies that

$$\eta=[(1+\delta^{n-p})]^{1/(n-p)}.$$

This shows that $K+_pL=\eta K$.

For $p\ge 1$, suppose exist a constant $\delta>0$ such that $K+_pL=\delta K$. Then

$$\frac{f_p(L,u)}{f_p(K,u)}=\delta^{n-p}-1.$$

This shows that K and L are homothetic.

This completes the proof. □

Lemma 4.3 *If $K,K',L\in\mathcal{F}_s^n$, $p\ge 1$ and $\varphi\in\Phi$, then*

$$\Omega_{\varphi,p}(K+_pK',L)\le\Omega_{\varphi,p}(K,L)+\Omega_{\varphi,p}(K',L). \tag{4.2}$$

If φ is strictly convex, equality holds if and only if $K+_pK'$ and L are homothetic.

Proof Let $\Omega_{\varphi,p}(K,L)=\lambda_1$ and $\Omega_{\varphi,p}(K',L)=\lambda_2$, then

$$\int_{S^{n-1}}\varphi\left(\frac{f_p(K,u)}{\lambda_1 f_p(L,u)}\right)d\Omega_p(L,u)=1,$$

and

$$\int_{S^{n-1}}\varphi\left(\frac{f_p(K',u)}{\lambda_1 f_p(L,u)}\right)d\Omega_p(L,u)=1,$$

Combining the convexity of the function, we obtain

$$\begin{aligned}1 &= \frac{\lambda_1}{\lambda_1+\lambda_2}\int_{S^{n-1}}\varphi\left(\frac{f_p(K,u)}{\lambda_1 f_p(L,u)}\right)d\Omega_p(L,u)\\ &+ \frac{\lambda_2}{\lambda_1+\lambda_2}\int_{S^{n-1}}\varphi\left(\frac{f_p(K',u)}{\lambda_2 f_p(L,u)}\right)d\Omega_p(L,u)\\ &\geq \int_{S^{n-1}}\varphi\left(\frac{f_p(K,u)+f_p(K',u)}{(\lambda_1+\lambda_2)h(L,u)}\right)d\Omega_p(L,u)\\ &= \int_{S^{n-1}}\varphi\left(\frac{f_p(K\,\boldsymbol{+}_p\,K',u)}{(\lambda_1+\lambda_2)h(L,u)}\right)d\Omega_p(L,u)\end{aligned}$$

Hence

$$\begin{aligned}\Omega_{\varphi,p}(K\,\boldsymbol{+}_p\,K',L) &\leq \lambda_1+\lambda_2\\ &= \Omega_{\varphi,p}(K,L)+\Omega_{\varphi,p}(K',L).\end{aligned}$$

If φ is strictly convex, from the equality of Jensen's inequality, it follows that the equality in (4.2) holds if and only if $K\,\boldsymbol{+}_p\,K'$ and L are homothetic

This completes the proof. □

Theorem 4.4 (φ-Minkowski inequality for φ-mixed affine surface area) *If $K,L\in\mathcal{F}_s^n$, $p\geq 1$, $\varphi\in\Phi$ and $\varphi(c_\varphi)=1$, then*

$$\Omega_{\varphi,p}(K,L)\geq\frac{1}{c_\varphi}\Omega_p(K)^{(n+p)/n}\Omega_p(L)^{-(n+p)/n}. \tag{4.3}$$

If φ is strictly convex, equality holds if and only if K and L are homothetic.

Proof For $\varphi\in\Phi$, let

$$\Omega_{\varphi,p}(K,L)=\lambda. \tag{4.4}$$

Then

$$\int_{S^{n-1}}\varphi\left(\frac{f_p(K,u)}{\lambda f_p(L,u)}\right)d\Omega_p(L,u)=1$$

By using Jensen's inequality and L_p-Minkowski inequality (1.14), we obtain

$$\begin{aligned}1 &= \varphi(c_\varphi)\\ &= \int_{S^{n-1}}\varphi\left(\frac{f_p(K,u)}{\lambda f_p(L,u)}\right)d\Omega_p(L,u)\\ &\geq \varphi\left(\frac{1}{\lambda\Omega_p(L)}\int_{S^{n-1}}f_p(K,u)f_p(L,u)^{-p/(n+p)}dS(u)\right)\\ &= \varphi\left(\frac{1}{\lambda}\cdot\frac{\Omega_{-p}(K,L)}{\Omega_{p(L)}}\right)\\ &\geq \varphi\left(\frac{1}{\lambda}\cdot\frac{\Omega_p(K)^{(n+p)/n}\Omega_p(L)^{-p/n}}{\Omega_p(L)}\right).\end{aligned}$$

Hence

$$\Omega_{\varphi,p}(K,L) \geq \frac{1}{c_\varphi}\Omega_p(K)^{(n+p)/n}\Omega_p(L)^{-(n+p)/n}. \tag{4.5}$$

If φ is strictly convex, from equalities of Jensen's inequality and L_p-Minkowski inequality (1.14), it yields the equality in (4.5) holds if and only if K and L are homothetic.

This completes the proof. □

We establish also the following φ-Brunn-Minkowski inequality for three convex bodies K, K' and L.

Theorem 4.5 (The φ-Brunn-Minkowski inequality for φ-mixed affine surface areas) *If $K, K', L \in \mathcal{F}_s^n$, $\varphi \in \Phi$, $p \geq 1$ and $\varphi(c_\varphi) = 1$, then*

$$\Omega_{\varphi,p}(K,L) + \Omega_{\varphi,p}(K',L) \geq \frac{1}{c_\varphi}\left(\frac{\Omega_p(K +_p K')}{\Omega_p(L)}\right)^{(n+p)/n}. \tag{4.6}$$

If φ is strictly convex, equality holds if and only if K, L and K' are homothetic.

Proof This follows immediately from Theorem 4.4 and Lemmas 4.2-4.3, □

Corollary 4.6 (The Brunn-Minkowski type inequality for mixed affine surface area) *If $K, K', L \in \mathcal{F}_s^n$, then*

$$\Omega_{-1}(K,L) + \Omega_{-1}(K',L) \geq \Omega(K + K')^{(n+1)/n}\Omega(L)^{-1/n}. \tag{4.7}$$

with equality if and only if K, L and K' are homothetic.

Proof This follows immediately from (1.6) and Theorem 4.5 with $p = 1$. □

When $K' = K$, (4.7) becomes the following well-known Minkowski inequality for mixed affine surface area, which was established by Lutwak [5]. If $K, L \in \mathcal{F}_s^n$, then

$$\Omega_{-1}(K,L)^n \geq \Omega(K)^{n+1}\Omega(L)^{-1}, \tag{4.8}$$

with equality if and only if K and L are homothetic.

Corollary 4.7 (The L_{pq}-Brunn-Minkowski type inequality for φ-mixed affine surface areas) *If $K, K', L \in \mathcal{F}_s^n$ and $p, q \geq 1$, then*

$$\Omega_{p,q}(L,K)^{1/q} + \Omega_{p,q}(L,K')^{1/q} \geq \Omega_p(K +_p K')^{(n+p)/n}\Omega_p(L)^{(n-q(n+p))/(nq)}. \tag{4.9}$$

If φ is strictly convex, equality holds if and only if K, L and K' are homothetic.

Proof This follows immediately from (1.7) and Theorem 4.5 □

When $K' = K$, (4.9) becomes the following L_{pq}-Minkowski inequality, which was established in [6]. If $K, L \in \mathcal{F}_s^n$ and $p, q \geq 1$, then

$$\Omega_{p,q}(K,L)^{\frac{n}{n+p}} \geq \Omega_p(K)^{\frac{n}{n+p}-q}\Omega_p(L)^q, \tag{4.10}$$

with equality if and only if K and L are homothetic

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