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Approximation by Szász-integral type operators

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Abstract. We consider a summation-integral type operators and establish a quantitative Voronovskaja type theorem and weighted approximation for these operators. Finally, we calculate the rate of convergence for absolutely continuous functions whose derivative is equivalent to a function with bounded variation.

1. Introduction

Miheşan [18] constructed a significant extension of the prominent Szász operators based on $\tau \in \mathbb{R}$ as follows:

$$\mathcal{P}_n^{(\tau)}(\Lambda; z) = \sum_{j=0}^{\infty} v_{n,j}^{(\tau)}(z)\Lambda\left(\frac{j}{n}\right), \qquad z \in [0, \infty), \tag{1}$$

where $v_{n,j}^{(\tau)}(z) = \dfrac{(\tau)_j}{j!} \dfrac{\left(\frac{nz}{\tau}\right)^j}{\left(1 + \frac{nz}{\tau}\right)^{\tau+j}}$, $(\tau)_j = \tau(\tau+1)\cdots(\tau+j-1), (\tau)_0 = 1$, and $\tau + nz > 0$. The operator $\mathcal{P}_n^{(\tau)}$ preserves the linear polynomials, and for specific values of τ, several well-known operators can be derived. Păltănea [20] constructed a generalization of the Phillips operators by considering the modified basis functions under integration based on parameter. Gupta and Rassias [11] introduced several approximation properties, such as weighted approximation, asymptotic formula, and error estimate in terms of modulus of smoothness, by proposing a integral variant of certain Szász type operators. In 2015, Acar [1] constructed the general Szász-Mirakyan operators and compute quantitative Grüss type Voronovskaya theorems by with the help of weighted modulus of smoothness. Gupta [9] constructed a sequence of hybrid operators with weights of the Păltănea basis function and studied some approximation properties of these operators. Kajla and Agrawal [14] established a Voronovskaja type asymptotic theorem, weighted approximation, and statistical convergence by proposing a integral variant of Szász operators depending on Charlier polynomials. Acu and Gupta [2] proposed mixed hybrid operators involving two parameters. They proved the Korovkin type approximation theorem and the order of convergence for unbounded functions with derivatives of bounded variation. Kajla et al. [15] introduced the Baskakov-Szász type operators involving inverse Pólya-Eggenberger distribution and investigated their direct results. In the literature, many researchers

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have studied the approximation properties of various hybrid operators (cf. [3–5, 7, 8, 10, 13, 16, 17, 19, 22–30] etc.) and reference therein.

Suppose that $\theta > 0$, $\vartheta > 0$ and $\Lambda \in C_\vartheta[0,\infty) := \{\Lambda \in C[0,\infty) : \Lambda(w) = O(w^\vartheta)$, as $w \to \infty\}$, we derive integral type generalization of the operators defined by (1) as follows:

$$\mathcal{K}_{n,\tau}^{\theta}(\Lambda;z) = \sum_{j=1}^{\infty} v_{n,j}^{(\tau)}(z) \int_0^\infty l_{n,j}^{\theta}(w)\Lambda(w)dw + v_{n,0}^{(\tau)}(z)\Lambda(0), \tag{2}$$

where $l_{n,j}^{\theta}(w) = \dfrac{1}{B(j\theta, n\theta+1)} \dfrac{w^{j\theta-1}}{(1+w)^{j\theta+n\theta+1}}$ and $v_{n,j}^{(\tau)}(z)$ is given as above. It should be noted that this operators (2) preserves both constant and linear functions. The current article deals with some direct results of the operators $\mathcal{K}_{n,\tau}^{\theta}$.

2. Basic Results

Let $e_i(w) = w^i, i = \overline{0,6}$.

Lemma 2.1. *For the operators $\mathcal{K}_{n,\tau}^{\theta}(\Lambda;z)$, we have*

(i) $\mathcal{K}_{n,\tau}^{\theta}(e_0;z) = 1$,

(ii) $\mathcal{K}_{n,\tau}^{\theta}(e_1;z) = z$,

(iii) $\mathcal{K}_{n,\tau}^{\theta}(e_2;z) = \dfrac{z^2 n\theta(1+\tau)}{\tau(n\theta-1)} + \dfrac{z(1+\theta)}{(n\theta-1)}$,

(iv) $\mathcal{K}_{n,\tau}^{\theta}(e_3;z) = \dfrac{z^3 n^2\theta^2(1+\tau)(2+\tau)}{\tau^2(n\theta-1)(n\theta-2)} + \dfrac{3n\theta z^2(1+\tau)(1+\theta)}{\tau(n\theta-1)(n\theta-2)} + \dfrac{z(1+\theta)(2+\theta)}{(n\theta-1)(n\theta-2)}$,

(v) $$\mathcal{K}_{n,\tau}^{\theta}(e_4;z) = \frac{z^4 n^3\theta^3(1+\tau)(2+\tau)(3+\tau)}{\tau^3(n\theta-1)(n\theta-2)(n\theta-3)} + \frac{6n^2\theta^2 z^3(1+\tau)(2+\tau)(1+\theta)}{\tau^2(n\theta-1)(n\theta-2)(n\theta-3)} + \frac{n\theta z^2(1+\tau)(1+\theta)(11+7\theta)}{\tau(n\theta-1)(n\theta-2)(n\theta-3)} + \frac{z(1+\theta)(2+\theta)(3+\theta)}{(n\theta-1)(n\theta-2)(n\theta-3)},$$

(vi) $$\mathcal{K}_{n,\tau}^{\theta}(e_5;z) = \frac{z^5 n^4\theta^4(1+\tau)(2+\tau)(3+\tau)(4+\tau)}{\tau^4(n\theta-1)(n\theta-2)(n\theta-3)(n\theta-4)} + \frac{10n^3\theta^3 z^4(1+\tau)(2+\tau)(3+\tau)(1+\theta)}{\tau^3(n\theta-1)(n\theta-2)(n\theta-3)(n\theta-4)} + \frac{5n^2\theta^2 z^3(1+\tau)(2+\tau)(1+\theta)(7+5\theta)}{\tau^2(n\theta-1)(n\theta-2)(n\theta-3)(n\theta-4)} + \frac{5n\theta z^2(1+\tau)(1+\theta)(2+\theta)(5+3\theta)}{\tau(n\theta-1)(n\theta-2)(n\theta-3)(n\theta-4)} + \frac{z(1+\theta)(2+\theta)(3+\theta)(4+\theta)}{(n\theta-1)(n\theta-2)(n\theta-3)(n\theta-4)},$$

(vii) $$\mathcal{K}_{n,\tau}^{\theta}(e_6;z) = \frac{z^6 n^5\theta^5(1+\tau)(2+\tau)(3+\tau)(4+\tau)(5+\tau)}{\tau^5(n\theta-1)(n\theta-2)(n\theta-3)(n\theta-4)(n\theta-5)} + \frac{15n^4\theta^4 z^5(1+\tau)(2+\tau)(3+\tau)(4+\tau)(1+\theta)}{\tau^4(n\theta-1)(n\theta-2)(n\theta-3)(n\theta-4)(n\theta-5)} + \frac{5n^3\theta^3 z^4(1+\tau)(2+\tau)(3+\tau)(1+\theta)(17+13\theta)}{\tau^3(n\theta-1)(n\theta-2)(n\theta-3)(n\theta-4)(n\theta-5)} + \frac{15n^2\theta^2 z^3(1+\tau)(2+\tau)(1+\theta)(2+\theta)(5+3\theta)}{\tau^2(n\theta-1)(n\theta-2)(n\theta-3)(n\theta-4)(n\theta-5)} + \frac{n\theta z^2(1+\tau)(1+\theta)(2+\theta)(137+\theta(132+31\theta))}{\tau(n\theta-1)(n\theta-2)(n\theta-3)(n\theta-4)(n\theta-5)} + \frac{z(1+\theta)(2+\theta)(3+\theta)(4+\theta)(5+\theta)}{(n\theta-1)(n\theta-2)(n\theta-3)(n\theta-4)(n\theta-5)}.$$

Lemma 2.2. *Using Lemma 2.1, we find that*

(i) $\mathcal{K}_{n,\tau}^{\theta}(w-z;z) = 0$,

(ii) $\mathcal{K}_{n,\tau}^{\theta}((w-z)^2;z) = \dfrac{z^2(\tau+n\theta)}{\tau(n\theta-1)} + \dfrac{z(1+\theta)}{(n\theta-1)}$,

(iii) $\mathcal{K}_{n,\tau}^{\theta}((w-z)^4;z) = \dfrac{3z^4\left(6\tau^3+n\tau^2(12+\tau)\theta+2n^2\tau(4+\tau)\theta^2+n^3(2+\tau)\theta^3\right)}{\tau^3(n\theta-1)(n\theta-2)(n\theta-3)}$

$$+\frac{6z^3(1+\theta)\left(6\tau^2+n\tau(6+\tau)\theta+n^2(2+\tau)\theta^2\right)}{\tau^2(n\theta-1)(n\theta-2)(n\theta-3)}+\frac{z^2(1+\theta)(n\theta(11+7\theta)+3\tau(8+\theta(4+n\theta+n)))}{\tau(n\theta-1)(n\theta-2)(n\theta-3)}$$
$$+\frac{z(1+\theta)(2+\theta)(3+\theta)}{(n\theta-1)(n\theta-2)(n\theta-3)}.$$

Remark 2.3. *If $\tau=\tau(n)\to\infty$, as $n\to\infty$ and $\lim_{n\to\infty}\frac{n}{\tau(n)}=d\in\mathbb{R}$, then*

$$\begin{aligned}
\lim_{n\to\infty} n\,\mu_{n,\theta}^{\tau,1}(z) &= 0,\\
\lim_{n\to\infty} n\,\mu_{n,\theta}^{\tau,2}(z) &= \frac{z^2(1+d\theta)}{\theta}+\frac{z\,(1+\theta)}{\theta},\\
\lim_{n\to\infty} n^2\,\mu_{n,\theta}^{\tau,4}(z) &= \frac{3z^4(1+d\theta)^2}{\theta^2}+\frac{6z^3(1+\theta)(1+d\theta)}{\theta^2}+\frac{3z^2(1+\theta)^2}{\theta^2},\\
\lim_{n\to\infty} n^3\,\mu_{n,\theta}^{\tau,6}(z) &= \frac{15z^3(1+\theta)^3}{\theta^3}+\frac{45z^4\left(1+\theta(2+d+\theta+2d\theta+\theta^2)\right)}{\theta^3}+\frac{45z^5(1+\theta)(1+d\theta)^2}{\theta^3}\\
&\quad+\frac{15z^6\left(1+d\theta(3+3d\theta+\theta^2)\right)}{\theta^3},\ \text{where}\ \mu_{n,\theta}^{\tau,m}:=\mathcal{K}_{n,\tau}^{\theta}((w-z)^m;z), m=1,2,4,6.
\end{aligned}$$

3. Direct Results

Theorem 3.1. *Suppose that $\Lambda\in C_{\vartheta}[0,\infty)$ and $\tau=\tau(n)\to\infty$ as $n\to\infty$. Then $\lim_{n\to\infty}\mathcal{K}_{n,\tau}^{\theta}(\Lambda;z)=\Lambda(z)$, uniformly in any compact subset of $[0,\infty)$.*

Let $\eta_1\geq 0, \eta_2>0$ be fixed. We examine the following Lipschitz-type space (see [21]):

$$Lip_M^{(\eta_1,\eta_2)}(\beta):=\left\{\Lambda\in C[0,\infty):|\Lambda(w)-\Lambda(z)|\leq M\frac{|w-z|^\beta}{(w+\eta_1z^2+\eta_2z)^{\frac{\beta}{2}}};\quad z,w\in(0,\infty),0<\beta\leq 1\right\}.$$

Theorem 3.2. *Suppose that $\Lambda\in Lip_M^{(\eta_1,\eta_2)}(\beta)$ and $\beta\in(0,1]$. Then,*

$$|\,\mathcal{K}_{n,\tau}^{\theta}(\Lambda;z)-\Lambda(z)\,|\leq M\left(\frac{\mu_{n,\theta}^{\tau,2}(z)}{\eta_1z^2+\eta_2z}\right)^{\frac{\beta}{2}}, z\in(0,\infty).$$

Proof. Using Hölder's inequality with $p=\frac{2}{\beta}, q=\frac{2}{2-\beta}$, we obtain

$|\,\mathcal{K}_{n,\tau}^{\theta}(\Lambda;z)-\Lambda(z)\,|$

$$\begin{aligned}
&= \sum_{j=1}^{\infty} v_{n,j}^{(\tau)}(z)\int_0^\infty l_{n,j}^{\theta}(w)|\Lambda(w)-\Lambda(z)|dw+v_{n,0}^{(\tau)}(z)|\Lambda(0)-\Lambda(z)|\\
&\leq \sum_{j=1}^{\infty} v_{n,j}^{(\tau)}(z)\left(\int_0^\infty l_{n,j}^{\theta}(w)|\Lambda(w)-\Lambda(z)|^{\frac{2}{\beta}}dw\right)^{\frac{\beta}{2}}+v_{n,0}^{(\tau)}(z)|\Lambda(0)-\Lambda(z)|\\
&\leq \left\{\sum_{j=1}^{\infty} v_{n,j}^{(\tau)}(z)\int_0^\infty l_{n,j}^{\theta}(w)|\Lambda(w)-\Lambda(z)|^{\frac{2}{\beta}}dw+v_{n,0}^{(\tau)}(z)|\Lambda(0)-\Lambda(z)|^{\frac{2}{\beta}}\right\}^{\frac{\beta}{2}}\left(\sum_{j=0}^{\infty} v_{n,j}^{(\tau)}(z)\right)^{\frac{2-\beta}{2}}\\
&= \left\{\sum_{j=1}^{\infty} v_{n,j}^{(\tau)}\int_0^\infty l_{n,j}^{\theta}(w)|\Lambda(w)-\Lambda(z)|^{\frac{2}{\beta}}dw+v_{n,0}^{(\tau)}(z)|\Lambda(0)-\Lambda(z)|^{\frac{2}{\beta}}\right\}^{\frac{\beta}{2}}
\end{aligned}$$

$$\leq M\left(\sum_{j=1}^{\infty} v_{n,j}^{(\tau)}(z)\int_0^\infty l_{n,j}^{\theta}(w)\frac{(w-z)^2}{(w+\eta_1 z^2+\eta_2 z)}dw + v_{n,0}^{(\tau)}(z)\frac{z^2}{(\eta_1 z^2+\eta_2 z)}\right)^{\frac{\beta}{2}}$$

$$\leq \frac{M}{(\eta_1 z^2+\eta_2 z)^{\frac{\beta}{2}}}\left(\sum_{j=1}^{\infty} v_{n,j}^{(\tau)}(z)\int_0^\infty l_{n,j}^{\theta}(w)(w-z)^2 dw + z^2 v_{n,0}^{(\tau)}(z)\right)^{\frac{\beta}{2}}$$

$$= \frac{M\left(\mathcal{K}_{n,\tau}^{\theta}((w-z)^2;z)\right)^{\frac{\beta}{2}}}{(\eta_1 z^2+\eta_2 z)^{\frac{\beta}{2}}} = \frac{M}{(\eta_1 z^2+\eta_2 z)^{\frac{\beta}{2}}}(\mu_{n,\theta}^{\tau,2}(z))^{\frac{\beta}{2}}.$$

Thus, the proof is completed. □

Define $H_\zeta[0,\infty)$ to represent the space of all real valued functions on $[0,\infty)$ that satisfy the constraint $|\Lambda(z)| \leq N_\Lambda \zeta(z)$, where N_Λ is a positive constant that depends only on Λ and $\zeta(z) = 1+z^2$ is a weight function. Let $C_\zeta[0,\infty)$ be the space of all continuous functions in $H_\zeta[0,\infty)$ endowed with the norm $\|\Lambda\|_\zeta := \sup_{z\in[0,\infty)} \frac{|\Lambda(z)|}{\zeta(z)}$ and $C_\zeta^0[0,\infty) := \left\{\Lambda \in C_\zeta[0,\infty) : \lim_{z\to\infty} \frac{|\Lambda(z)|}{\zeta(z)} < \infty\right\}$. The classical modulus of continuity of Λ on $[0,b]$ is given as $\omega_b(\Lambda,\delta) = \sup_{0<|w-z|\leq\delta}\ \sup_{z,w\in[0,b]} |\Lambda(w)-\Lambda(z)|$.

Theorem 3.3. *Suppose that $\Lambda \in C_\zeta[0,\infty)$. Then, show that*

$$|\mathcal{K}_{n,\tau}^{\theta}(\Lambda;z)-\Lambda(z)| \leq 4N_\Lambda(1+z^2)\mu_{n,\theta}^{\tau,2}(z) + 2\omega_{b+1}\left(\Lambda, \sqrt{\mu_{n,\theta}^{\tau,2}(z)}\right). \tag{3}$$

Proof. See [12], for $z \in [0,b]$ and $w \geq 0$, we may write

$$|\Lambda(w)-\Lambda(z)| \leq 4N_\Lambda(1+z^2)(w-z)^2 + \left(1+\frac{|w-z|}{\delta}\right)\omega_{b+1}(\Lambda,\delta),\quad \delta>0.$$

According to the Cauchy-Schwarz inequality, we get

$$\begin{aligned}|\mathcal{K}_{n,\tau}^{\theta}(\Lambda;z)-\Lambda(z)| &\leq 4N_\Lambda(1+z^2)\mathcal{K}_{n,\tau}^{\theta}((w-z)^2;z)+\omega_{b+1}(\Lambda,\delta)\left(1+\frac{1}{\delta}\mathcal{K}_{n,\tau}^{\theta}(|w-z|;z)\right)\\ &\leq 4N_\Lambda(1+z^2)\mu_{n,\theta}^{\tau,2}(z) + \omega_{b+1}(\Lambda,\delta)\left(1+\frac{1}{\delta}\sqrt{\mu_{n,\theta}^{\tau,2}(z)}\right).\end{aligned}$$

Now, choosing $\delta = \sqrt{\mu_{n,\theta}^{\tau,2}(z)}$, we get (3). □

4. Weighted approximation

Theorem 4.1. *Let $\Lambda \in C_\zeta^0[0,\infty)$ and $\tau = \tau(n) \to \infty$ as $n \to \infty$. Then, we have*

$$\lim_{n\to\infty} \|\mathcal{K}_{n,\tau}^{\theta}(\Lambda)-\Lambda\|_\zeta = 0. \tag{4}$$

Proof. It is sufficient to verify the following three relationships to prove this result (see [6])

$$\lim_{n\to\infty} \|\mathcal{K}_{n,\tau}^{\theta}(w^m;z)-z^m\|_\zeta = 0,\ \ m=0,1,2. \tag{5}$$

Because $\mathcal{K}_{n,\tau}^{\theta}(e_0;z)=1$, The criterion in (5) is valid for $m=0$.
Applying Lemma 2.1, $\|\mathcal{K}_{n,\tau}^{\theta}(w;z)-z\|_{\zeta}=\sup\limits_{z\geq 0}\dfrac{1}{1+z^2}|z-z|=0$.
Thus, $\lim\limits_{n\to\infty}\|\mathcal{K}_{n,\tau}^{\theta}(w;z)-z\|_{\zeta}=0$. Finally, we obtain

$$\|\mathcal{K}_{n,\tau}^{\theta}(w^2;z)-z^2\|_{\zeta}=\sup_{z\geq 0}\frac{1}{1+z^2}\left|\frac{z^2 n\theta(1+\tau)}{\tau(n\theta-1)}+\frac{z(1+\theta)}{(n\theta-1)}-z^2\right|\leq \sup_{z\geq 0}\frac{z^2}{1+z^2}\frac{(\tau+n\theta)}{\tau(n\theta-1)}+\sup_{z\geq 0}\frac{z}{1+z^2}\frac{(1+\theta)}{\tau(n\theta-1)},$$

which implies that $\lim\limits_{n\to\infty}\|\mathcal{K}_{n,\tau}^{\theta}(w^2;z)-z^2\|_{\zeta}=0$. □

We use the weighted modulus of continuity $\Theta(\Lambda,\delta)$ defined on $[0,\infty)$ (see [13]). Suppose that

$$\Theta(\Lambda,\delta)=\sup_{|h|<\delta,z\in[0,\infty)}\frac{|\Lambda(z+h)-\Lambda(z)|}{(1+h^2)(1+z^2)}\quad \text{for each } C_{\zeta}^{0}[0,\infty).$$

Lemma 4.2. *[13] Let $\Lambda\in C_{\zeta}^{0}[0,\infty)$, then:*

i) $\Theta(\Lambda;\delta)$ *is a monotone increasing function of* δ*;*

ii) $\lim\limits_{\delta\to 0^+}\Theta(\Lambda;\delta)=0$*;*

iii) Every $m\in\mathbb{N},\Theta(\Lambda,m\delta)\leq m\Theta(\Lambda;\delta)$*;*

iv) Every $\lambda\in[0,\infty),\Theta(\Lambda;\lambda\delta)\leq(1+\lambda)\Theta(\Lambda;\delta)$.

In this part, we will examine at a quantitative Voronvoskaja type result for the $\mathcal{K}_{n,\tau}^{\theta}$ operators.

Theorem 4.3. *Let $\Lambda''\in C_{\zeta}^{0}[0,\infty)$ and $\theta,\tau,z>0$. Then,*

$$\left|\mathcal{K}_{n,\tau}^{\theta}(\Lambda;z)-\Lambda(z)-\frac{\Lambda''(z)}{2}\left(\frac{z^2(\tau+n\theta)+\tau z(1+\theta)}{\tau(n\theta-1)}\right)\right|\leq 8(1+z^2)O(n^{-1})\Theta\left(\Lambda'',\frac{1}{\sqrt{n}}\right).$$

Proof. Using Taylor's series, $\exists,\xi$ which lies between z and w such that

$$\Lambda(w)=\Lambda(z)+\Lambda'(z)(w-z)+\frac{1}{2}\Lambda''(z)(w-z)^2+\varepsilon(w,z)(w-z)^2, \tag{6}$$

where

$$\varepsilon(w,z)=\frac{\Lambda''(\xi)-\Lambda''(z)}{2},$$

is a continuous function that terminates at 0. Using the operators $\mathcal{K}_{n,\tau}^{\theta}$ on the above relation and the Lemma 4.2, we can write

$$\left|\mathcal{K}_{n,\tau}^{\theta}(\Lambda;z)-\Lambda(z)-\frac{\Lambda''(z)}{2}\left(\frac{z^2(\tau+n\theta)+\tau z(1+\theta)}{\tau(n\theta-1)}\right)\right|\leq \mathcal{K}_{n,\tau}^{\theta}(|\varepsilon(w,z)|(w-z)^2,z).$$

From Lemma 4.2 and the definition of $\Theta(\Lambda,\delta)$, we find that

$$|\Lambda(w)-\Lambda(z)|\leq 2(1+z^2)[1+(w-z)^2]\left(1+\frac{|w-z|}{\delta}\right)(1+\delta^2)\Theta(\Lambda,\delta), \tag{7}$$

for every $\Lambda\in C_{\zeta}^{0}[0,\infty)$ and $z,w\in[0,\infty)$. By (7) and the relation $|\xi-z|\leq|w-z|$, we have

$$\varepsilon(w,z)\leq(1+z^2)[1+(w-z)^2]\left(1+\frac{|w-z|}{\delta}\right)(1+\delta^2)\Theta(\Lambda'',\delta).$$

Also,

$$\varepsilon(w,z)=\begin{cases}2(1+z^2)(1+\delta^2)^2\Theta(\Lambda'',\delta), & |w-z|<\delta\\ (1+z^2)[1+(w-z)^2]\left(1+\frac{|w-z|}{\delta}\right)(1+\delta^2)\Theta(\Lambda'',\delta), & |w-z|\geq\delta.\end{cases}$$

Now taking $\delta < 1$, we find that

$$\varepsilon(w,z) \leq 2(1+z^2)\left(1+\frac{(w-z)^4}{\delta^4}\right)(1+\delta^2)^2\Theta(\Lambda'',\delta) \leq 8(1+z^2)\left(1+\frac{(w-z)^4}{\delta^4}\right)\Theta(\Lambda'',\delta).$$

By Lemma 4.2, we obtain

$$\begin{aligned}\mathcal{K}_{n,\tau}^{\theta}(|\varepsilon(w,z)|(w-z)^2,z) &= 8(1+z^2)\Theta(\Lambda'',\delta)\left(\mathcal{K}_{n,\tau}^{\theta}((w-z)^2,z)+\frac{1}{\delta^4}\mathcal{K}_{n,\tau}^{\theta}((w-z)^6,z)\right)\\ &= 8(1+z^2)\Theta(\Lambda'',\delta)\left(O(n^{-1})+\frac{1}{\delta^4}O(n^{-3})\right).\end{aligned}$$

Taking $\delta = \frac{1}{\sqrt{n}}$, we have

$$\mathcal{K}_{n,\tau}^{\theta}(|\varepsilon(w,z)|(w-z)^2,z) \leq 8(1+z^2)O(n^{-1})\Theta\left(\Lambda'',\frac{1}{\sqrt{n}}\right).$$

Hence, we get the desired result. □

5. Rate of convergence

Define $DBV[0,\infty)$ be the class of all functions $\Lambda \in H_{\zeta}[0,\infty)$, having a derivative of bounded variation on every finite subinterval of $[0,\infty)$. We observe that $\Lambda \in DBV[0,\infty)$ possess a representation $\Lambda(z) = \int_0^z g(w) + \Lambda(0)$, where g is of bounded variation on every finite subinterval of $[0,\infty)$. In order to study the convergence of the operators $\mathcal{K}_{n,\tau}^{\theta}$ for functions having a derivative of bounded variation. The operators (2) should be rewritten as follows

$$\mathcal{K}_{n,\tau}^{\theta}(\Lambda;z) = \int_0^{\infty}\mathcal{J}_{n,\tau}^{\theta}(z,w)\Lambda(w)dw, \tag{8}$$

$$\mathcal{J}_{n,\tau}^{\theta}(z,w) = \sum_{j=1}^{\infty} v_{n,j}^{(\tau)}(z)l_{n,j}^{\theta}(w) + v_{n,0}^{(\tau)}(z)\delta(w).$$

The Dirac-delta function is represented by $\delta(w)$.

Lemma 5.1. *Let $\tau = \tau(n) \to \infty$, as $n \to \infty$ and $\lim_{n\to\infty}\frac{n}{\tau(n)} = d \in \mathbb{R}$. For all $z \in (0,\infty)$ and sufficiently large n, we have*

i) $\lambda_{n,\theta}^{(\tau)}(z,w) = \int_0^w \mathcal{J}_{n,\tau}^{\theta}(z,v)dv \leq \frac{M(\theta,d)}{(z-w)^2}\frac{1+z^2}{n},\ 0 \leq w < z,$

ii) $1-\lambda_{n,\theta}^{(\tau)}(z,w) = \int_w^{\infty} \mathcal{J}_{n,\tau}^{\theta}(z,v)dv \leq \frac{M(\theta,d)}{(w-z)^2}\frac{1+z^2}{n},\ z \leq w < \infty,$

where $M(\theta,d)$ is a positive constant that varies with θ and d.

Proof. For sufficiently large n, it follows from Remark 2.3 that

$$\mathcal{K}_{n,\tau}^{\theta}((v-z)^2;z) < M(\theta,d)\frac{1+z^2}{n}. \tag{9}$$

Applying Lemma 2.2, we have

$$\lambda_{n,\theta}^{(\tau)}(z,w) = \int_0^w \mathcal{J}_{n,\tau}^{\theta}(z,v)dv \leq \int_0^w \left(\frac{z-v}{z-w}\right)^2\mathcal{J}_{n,\tau}^{\theta}(z,v)dv \leq \frac{1}{(z-w)^2}\mathcal{K}_{n,\tau}^{\theta}\left((v-z)^2;z\right) \leq \frac{M(\theta,d)}{(z-w)^2}\frac{1+z^2}{n}.$$

The proof for ii) is similar, but the details are left out. □

Theorem 5.2. *Let $\Lambda \in DBV[0,\infty)$, $\tau=\tau(n)\to\infty$, as $n\to\infty$ and $\lim_{n\to\infty}\frac{n}{\tau(n)}=d\in\mathbb{R}$. Then for sufficiently large n, we have*

$$\begin{aligned}|\mathcal{K}^{\theta}_{n,\tau}(\Lambda;z)-\Lambda(z)| \leq\ & \sqrt{M(\theta,d)\frac{1+z^2}{n}}\left|\frac{\Lambda'(z+)-\Lambda'(z-))}{2}\right|+M(\theta,d)\frac{1+z^2}{nz}\sum_{j=1}^{[\sqrt{n}]}\left(\bigvee_{z-\frac{z}{j}}^{z}\Lambda'_z\right)\\ &+\frac{z}{\sqrt{n}}\left(\bigvee_{z-\frac{z}{\sqrt{n}}}^{z}\Lambda'_z\right)+\left(4N_\Lambda+\frac{N_\Lambda+|\Lambda(z)|}{z^2}\right)M(\theta,d)\frac{1+z^2}{n}+|\Lambda'(z+)|\sqrt{M(\theta,d)\frac{1+z^2}{n}}\\ &+M(\theta,d)\frac{1+z^2}{nz^2}|\Lambda(2z)-\Lambda(z)-xf'(z+)|+\frac{z}{\sqrt{n}}\bigvee_{z}^{z+\frac{z}{\sqrt{n}}}\Lambda'_z+M(\theta,d)\frac{1+z^2}{nz}\sum_{j=1}^{[\sqrt{n}]}\bigvee_{z}^{z+\frac{z}{j}}\Lambda'_z,\end{aligned}$$

where $M(\theta,d)$ is a positive constant that varies with θ and d, $\bigvee_a^b\Lambda$ denotes the total variation of Λ on $[a,b]$ and Λ'_z is defined by

$$\Lambda'_z(w)=\begin{cases}\Lambda'(w)-\Lambda'(z-), & 0\leq w<z,\\ 0, & w=z,\\ \Lambda'(w)-\Lambda'(z+), & z<w<\infty.\end{cases}\tag{10}$$

Proof. For any $\Lambda\in DBV[0,\infty)$, from (10) we find that

$$\begin{aligned}\Lambda'(v)=&\frac{1}{2}(\Lambda'(z+)+\Lambda'(z-))+\Lambda'_z(v)+\frac{1}{2}(\Lambda'(z+)-\Lambda'(z-))\,sgn(v-z)\\ &+\delta_z(v)\left(\Lambda'(v)-\frac{1}{2}(\Lambda'(z+)+\Lambda'(z-))\right),\end{aligned}\tag{11}$$

where

$$\delta_z(v)=\begin{cases}1, & v=z,\\ 0, & v\neq z.\end{cases}$$

Since $\mathcal{K}^{\theta}_{n,\tau}(e_0;z)=1$, using (8), for every $z\in(0,\infty)$ we get

$$\begin{aligned}\mathcal{K}^{\theta}_{n,\tau}(\Lambda;z)-\Lambda(z) &= \int_0^\infty \mathcal{J}^{\theta}_{n,\tau}(z,w)(\Lambda(w)-\Lambda(z))dw=\int_0^\infty\mathcal{J}^{\theta}_{n,\tau}(z,w)\int_z^w\Lambda'(v)dvdw\\ &= -\int_0^z\left(\int_w^x f'(v)dv\right)\mathcal{J}^{\theta}_{n,\tau}(z,w)dw+\int_z^\infty\left(\int_z^t f'(v)dv\right)\mathcal{J}^{\theta}_{n,\tau}(z,w)dw.\end{aligned}$$

Denote

$$I_1:=\int_0^z\left(\int_w^z f'(v)dv\right)\mathcal{J}^{\theta}_{n,\tau}(z,w)dw,\quad I_2:=\int_z^\infty\left(\int_z^t f'(v)dv\right)\mathcal{J}^{\theta}_{n,\tau}(z,w)dw.$$

Since $\int_z^w\delta_z(v)dv=0$, and from relation (11), we may write

$$\begin{aligned}I_1=&\int_0^z\left\{\int_w^z\frac{1}{2}(\Lambda'(z+)+\Lambda'(z-))+\Lambda'_z(v)+\frac{1}{2}(\Lambda'(z+)-\Lambda'(z-))\,sgn(v-z)dv\right\}\mathcal{J}^{\theta}_{n,\tau}(z,w)dw\\ =&\frac{1}{2}(\Lambda'(z+)+\Lambda'(z-))\int_0^z(z-w)\mathcal{J}^{\theta}_{n,\tau}(z,w)dw+\int_0^z\left(\int_w^z\Lambda'_z(v)dv\right)\mathcal{J}^{\theta}_{n,\tau}(z,w)dw\\ &-\frac{1}{2}(\Lambda'(z+)-\Lambda'(z-))\int_0^z(z-w)\mathcal{J}^{\theta}_{n,\tau}(z,w)dw.\end{aligned}\tag{12}$$

Similarly, we find

$$
\begin{aligned}
I_2 &= \int_z^{\infty}\left\{\int_z^{w}\frac{1}{2}\left(\Lambda'(z+)+\Lambda'(z-)\right)+\Lambda'_z(v)+\frac{1}{2}\left(\Lambda'(z+)-\Lambda'(z-)\right)sgn(v-z)dv\right\}\mathcal{J}^{\theta}_{n,\tau}(z,w)dw\\
&=\frac{1}{2}\left(\Lambda'(z+)+\Lambda'(z-)\right)\int_z^{\infty}(w-z)\mathcal{J}^{\theta}_{n,\tau}(z,w)dw+\int_z^{\infty}\left(\int_z^{w}\Lambda'_z(v)dv\right)\mathcal{J}^{\theta}_{n,\tau}(z,w)dw\\
&+\frac{1}{2}\left(\Lambda'(z+)-\Lambda'(z-)\right)\int_z^{\infty}(w-z)\mathcal{J}^{\theta}_{n,\tau}(z,w)dw. \qquad (13)
\end{aligned}
$$

Combining the relations (12)-(13), we get

$$
\begin{aligned}
\mathcal{K}^{\theta}_{n,\tau}(\Lambda;z)-\Lambda(z) =&\frac{1}{2}\left(\Lambda'(z+)+\Lambda'(z-)\right)\int_0^{\infty}(w-z)\mathcal{J}^{\theta}_{n,\tau}(z,w)dw\\
&+\frac{1}{2}\left(\Lambda'(z+)-\Lambda'(z-)\right)\int_0^{\infty}|w-z|\mathcal{J}^{\theta}_{n,\tau}(z,w)dw\\
&-\int_0^{z}\left(\int_w^{z}f'_z(v)dv\right)\mathcal{J}^{\theta}_{n,\tau}(z,w)dw+\int_z^{\infty}\left(\int_z^{t}f'_z(v)dv\right)\mathcal{J}^{\theta}_{n,\tau}(z,w)dw.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left|\mathcal{K}^{\theta}_{n,\tau}(\Lambda;z)-\Lambda(z)\right| &= \left|\frac{\Lambda'(z+)+\Lambda'(z-)}{2}\right|\left|\mathcal{K}^{\theta}_{n,\tau}(w-z;z)\right|+\left|\frac{\Lambda'(z+)-\Lambda'(z-)}{2}\right|\mathcal{K}^{\theta}_{n,\tau}(|w-z|;z)\\
&+\left|\int_0^{z}\left(\int_w^{z}f'_z(v)dv\right)\mathcal{J}^{\theta}_{n,\tau}(z,w)dw\right|+\left|\int_z^{\infty}\left(\int_z^{t}f'_z(v)dv\right)\mathcal{J}^{\theta}_{n,\tau}(z,w)dw\right|. \qquad (14)
\end{aligned}
$$

Now, let $E^{(\tau)}_{n,\theta}(\Lambda'_z,z)=\int_0^{z}\left(\int_w^{z}f'_z(v)dv\right)\mathcal{J}^{\theta}_{n,\tau}(z,w)dw$, and $F^{(\tau)}_{n,\theta}(\Lambda'_z,z)=\int_z^{\infty}\left(\int_z^{t}f'_z(v)dv\right)\mathcal{J}^{\theta}_{n,\tau}(z,w)dw$.
Our problem is solved by obtaining the estimates of the terms $E^{(\tau)}_{n,\theta}(\Lambda'_z,z)$ and $F^{(\tau)}_{n,\theta}(\Lambda'_z,z)$. From the definition of $\lambda^{(\tau)}_{n,\theta}$ given in Lemma 5.1, we may write using the integration by parts

$$
E^{(\tau)}_{n,\theta}(\Lambda'_z,z)=\int_0^{z}\left(\int_w^{z}\Lambda'_z(v)dv\right)\frac{\partial}{\partial w}\lambda^{(\tau)}_{n,\theta}(z,w)dw=\int_0^{z}f'_z(w)\lambda^{(\tau)}_{n,\theta}(z,w)dw.
$$

Thus,

$$
\left|E^{(\tau)}_{n,\theta}(\Lambda'_z,z)\right|\leq\int_0^{z}|\Lambda'_z(w)|\lambda^{(\tau)}_{n,\theta}(z,w)dw\leq\int_0^{z-\frac{z}{\sqrt{n}}}|\Lambda'_z(w)|\lambda^{(\tau)}_{n,\theta}(z,w)dw+\int_{z-\frac{z}{\sqrt{n}}}^{z}|\Lambda'_z(w)|\lambda^{(\tau)}_{n,\theta}(z,w)dw.
$$

Since $\Lambda'_z(z)=0$ and $\lambda^{(\tau)}_{n,\theta}(z,w)\leq 1$, we get

$$
\begin{aligned}
\int_{z-\frac{z}{\sqrt{n}}}^{z}|\Lambda'_z(w)|\lambda^{(\tau)}_{n,\theta}(z,w)dw &= \int_{z-\frac{z}{\sqrt{n}}}^{z}|\Lambda'_z(w)-\Lambda'_z(z)|\lambda^{(\tau)}_{n,\theta}(z,w)dw\\
&\leq \int_{z-\frac{z}{\sqrt{n}}}^{z}\bigvee_{w}^{z}\Lambda'_z dw\leq\bigvee_{z-\frac{z}{\sqrt{n}}}^{z}\Lambda'_z\int_{z-\frac{z}{\sqrt{n}}}^{z}dw=\frac{z}{\sqrt{n}}\bigvee_{z-\frac{z}{\sqrt{n}}}^{z}\Lambda'_z.
\end{aligned}
$$

From Lemma 5.1 and considering $w=z-\frac{z}{v}$, we may write

$$
\begin{aligned}
\int_0^{z-\frac{z}{\sqrt{n}}}|\Lambda'_z(w)|\lambda^{(\tau)}_{n,\theta}(z,w)dw &\leq M(\theta,d)\frac{1+z^2}{n}\int_0^{z-\frac{z}{\sqrt{n}}}|\Lambda'_z(w)|\frac{dw}{(z-w)^2}\leq M(\theta,d)\frac{1+z^2}{n}\int_0^{z-\frac{z}{\sqrt{n}}}\left(\bigvee_{w}^{z}\Lambda'_z\right)\frac{dw}{(z-w)^2}\\
&= M(\theta,d)\frac{1+z^2}{nz}\int_1^{\sqrt{n}}\left(\bigvee_{z-\frac{z}{v}}^{z}\Lambda'_z\right)dv\leq M(\theta,d)\frac{1+z^2}{nz}\sum_{j=1}^{[\sqrt{n}]}\left(\bigvee_{z-\frac{z}{j}}^{z}\Lambda'_z\right).
\end{aligned}
$$

Therefore,

$$|E_{n,\theta}^{(\tau)}(\Lambda'_z, z)| \leq M(\theta, d)\frac{1+z^2}{nz}\sum_{j=1}^{[\sqrt{n}]}\left(\bigvee_{z-\frac{z}{j}}^{z}\Lambda'_z\right) + \frac{z}{\sqrt{n}}\left(\bigvee_{z-\frac{z}{\sqrt{n}}}^{z}\Lambda'_z\right). \tag{15}$$

Also, using integration by parts in $F_{n,\theta}^{(\tau)}(\Lambda'_z, z)$ and using Lemma 5.1, we have

$$\begin{aligned}
|F_{n,\theta}^{(\tau)}(\Lambda'_z, z)| &\leq \left|\int_z^{2z}\left(\int_z^w \Lambda'_z(v)dv\right)\frac{\partial}{\partial w}\left(1-\lambda_{n,\theta}^{(\tau)}(z,w)\right)dw\right| + \left|\int_{2z}^{\infty}\left(\int_z^w \Lambda'_z(v)dv\right)\mathcal{J}_{n,\tau}^{\theta}(z,w)dw\right| \\
&\leq \left|\int_z^{2z}\Lambda'_z(v)dv\right|\left|1-\lambda_{n,\theta}^{(\tau)}(z,2z)\right| + \int_z^{2z}\left|\Lambda'_z(w)\right|\left(1-\lambda_{n,\theta}^{(\tau)}(z,w)\right)dw \\
&+ \left|\int_{2z}^{\infty}(\Lambda(w)-\Lambda(z))\mathcal{J}_{n,\tau}^{\theta}(z,w)dw\right| + |\Lambda'(z+)|\left|\int_{2z}^{\infty}(w-z)\mathcal{J}_{n,\tau}^{\theta}(z,w)dw\right|.
\end{aligned}$$

We have

$$\begin{aligned}
&\int_z^{2z}\left|\Lambda'_z(w)\right|\left(1-\lambda_{n,\theta}^{(\tau)}(z,w)\right)dw = \int_z^{z+\frac{z}{\sqrt{n}}}\left|\Lambda'_z(w)\right|\left(1-\lambda_{n,\theta}^{(\tau)}(z,w)\right)dw \\
&+ \int_{z+\frac{z}{\sqrt{n}}}^{2z}\left|\Lambda'_z(w)\right|\left(1-\lambda_{n,\theta}^{(\tau)}(z,w)\right)dw = J_1 + J_2 \text{ (say)}.
\end{aligned} \tag{16}$$

Since $\Lambda'_z(z) = 0$ and $1-\lambda_{n,\theta}^{(\tau)} \leq 1$, we get

$$J_1 = \int_z^{z+\frac{z}{\sqrt{n}}}\left|\Lambda'_z(w)-\Lambda'_z(z)\right|\left(1-\lambda_{n,\theta}^{(\tau)}(z,w)\right)dw \leq \int_z^{z+\frac{z}{\sqrt{n}}}\left(\bigvee_z^{z+\frac{z}{\sqrt{n}}}\Lambda'_z\right)dw = \frac{z}{\sqrt{n}}\bigvee_z^{z+\frac{z}{\sqrt{n}}}\Lambda'_z.$$

From Lemma 5.1 and considering $w = z+\dfrac{z}{v}$, we find that

$$\begin{aligned}
J_2 &\leq M(\theta,d)\frac{1+z^2}{n}\int_{z+\frac{z}{\sqrt{n}}}^{2z}\frac{1}{(w-z)^2}|\Lambda'_z(w)-\Lambda'_z(z)|dw \\
&\leq M(\theta,d)\frac{1+z^2}{n}\int_{z+\frac{z}{\sqrt{n}}}^{2z}\frac{1}{(w-z)^2}\left(\bigvee_z^t f'_z\right)dw = M(\theta,d)\frac{1+z^2}{nz}\int_1^{\sqrt{n}}\bigvee_z^{z+\frac{z}{v}}\Lambda'_z dv \\
&\leq M(\theta,d)\frac{1+z^2}{nz}\sum_{j=1}^{[\sqrt{n}]}\int_j^{j+1}\left(\bigvee_z^{z+\frac{z}{v}}\Lambda'_z\right)dv \leq M(\theta,d)\frac{1+z^2}{nz}\sum_{j=1}^{[\sqrt{n}]}\left(\bigvee_z^{z+\frac{z}{j}}\Lambda'_z\right).
\end{aligned}$$

Putting the values of J_1 and J_2 in (16), we get

$$\int_z^{2z}\left|\Lambda'_z(w)\right|\left(1-\lambda_{n,\theta}^{(\tau)}(z,w)\right)dw \leq \frac{z}{\sqrt{n}}\bigvee_z^{z+\frac{z}{\sqrt{n}}}\Lambda'_z + M(\theta,d)\frac{1+z^2}{nz}\sum_{j=1}^{[\sqrt{n}]}\left(\bigvee_z^{z+\frac{z}{j}}\Lambda'_z\right).$$

Therefore,

$$
\begin{aligned}
|F^{(\tau)}_{n,\theta}(\Lambda'_z, z)| \leq\ & N_\Lambda \int_{2z}^{\infty} (w^2+1)\mathcal{J}^{\theta}_{n,\tau}(z,w)dw + |\Lambda(z)| \int_{2z}^{\infty} \mathcal{J}^{\theta}_{n,\tau}(z,w)dw \\
& + |\Lambda'(z+)| \sqrt{M(\theta,d)\frac{1+z^2}{n}} + M(\theta,d)\frac{1+z^2}{nz^2}|\Lambda(2z)-\Lambda(z)-xf'(z+)| \\
& + \frac{z}{\sqrt{n}} \bigvee_{z}^{z+\frac{z}{\sqrt{n}}} \Lambda'_z + M(\theta,d)\frac{1+z^2}{nz}\sum_{j=1}^{[\sqrt{n}]}\left(\bigvee_{z}^{z+\frac{z}{j}}\Lambda'_z\right). \qquad (17)
\end{aligned}
$$

Since $w \leq 2(w-z)$ and $z \leq w-z$ when $w \geq 2z$, we obtain

$$
\begin{aligned}
& N_\Lambda \int_{2z}^{\infty} (w^2+1)\mathcal{J}^{\theta}_{n,\tau}(z,w)dw + |\Lambda(z)| \int_{2z}^{\infty} \mathcal{J}^{\theta}_{n,\tau}(z,w)dw \\
& \leq (N_\Lambda + |\Lambda(z)|)\int_{2z}^{\infty} \mathcal{J}^{\theta}_{n,\tau}(z,w)dw + 4N_\Lambda \int_{2z}^{\infty}(w-z)^2\mathcal{J}^{\theta}_{n,\tau}(z,w)dw \\
& \leq \frac{N_\Lambda + |\Lambda(z)|}{z^2}\int_{0}^{\infty}(w-z)^2\mathcal{J}^{\theta}_{n,\tau}(z,w)dw + 4N_\Lambda \int_{0}^{\infty}(w-z)^2\mathcal{J}^{\theta}_{n,\tau}(z,w)dw \\
& \leq \left(4N_\Lambda + \frac{N_\Lambda + |\Lambda(z)|}{z^2}\right)M(\theta,d)\frac{1+z^2}{n}. \qquad (18)
\end{aligned}
$$

Using the inequality (18), it follows

$$
\begin{aligned}
|F^{(\tau)}_{n,\theta}(\Lambda'_z, z)| \leq\ & \left(4N_\Lambda + \frac{N_\Lambda + |\Lambda(z)|}{z^2}\right)M(\theta,d)\frac{1+z^2}{n} + |\Lambda'(z+)|\sqrt{M(\theta,d)\frac{1+z^2}{n}} \\
& + M(\theta,d)\frac{1+z^2}{nz^2}|\Lambda(2z)-\Lambda(z)-xf'(z+)| + \frac{z}{\sqrt{n}}\bigvee_{z}^{z+\frac{z}{\sqrt{n}}}\Lambda'_z + M(\theta,d)\frac{1+z^2}{nz}\sum_{j=1}^{[\sqrt{n}]}\left(\bigvee_{z}^{z+\frac{z}{j}}\Lambda'_z\right). \qquad (19)
\end{aligned}
$$

From (14), (15) and (19), we get the required result. □

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