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Ideal epi-convergence of sequences of functions

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Abstract. There are many definitions of epi-convergence in the literature. These definitions can be categorized as sequential, topological and epigraphical. In this paper, we will first give the epigraphical definition of ideal epi-convergence, and from this definition we will derive other definitions. In addition, we will see that the definition of epi-convergence can be given over the level sets. We will also answer the question of what is the relationship between ideal epi-convergence and ordinary epi-convergence.

1. Introduction and Preliminaries

The notion of statistical convergence is first studied by Zygmund [26] in 1935 and introduced by Steinhaus [19], Fast [7] and Schoenberg [17] independently. Then it is generalized by Kostyrko et al. [10] with the help of ideal of subsets of the set of natural numbers $\mathbb{N}$. Kostyrko et al. [9] and Aytar et al. [4] proved some of basic properties of $\mathcal{I}$-convergence. Also, Demirci [6] presented the notions of $\mathcal{I}$-limit superior and inferior of a real sequence and gave some properties.

Wijsman [24, 25] studied epi-convergence in the late of 1960's. At that time, it was called infimal convergence. Later, it is studied by Mosco [13] on variational inequalities, by Joly [8] on topological structures compatible with epi-convergence, by Salinetti and Wets [16] on equisemicontinuous families of convex functions, by Attouch [3] on the relationship between the epi-convergence of convex functions and the graphical convergence of their subgradient mappings, and by McLinden and Bergstrom [12] on the preservation of epi-convergence under various operations performed on convex functions. In the time, it is also called Γ-convergence by Dal Maso [11]. For the first time, Wets [23] called it epi-convergence in 1980. A characterization of epi-convergence in terms of level sets is given by Beer et al. [5]. Sever et al. [18] have focused on statistical epi-convergence as a generalization of epi-convergence. The connection between epi-Cesaro convergence of sequences of functions and Kuratowski Cesaro convergence of their epigraphs matched by Nuray et al. [14]. Moreover, Tortop et al. [22] studied the sequential characterization of statistical epi-convergence. In recent years, Nuray [15] has also derived some results of epi-convergence on double sequence of functions. Solutions of some mathematical problems including stochastic optimization, variational problems and partial differential equations need epi-convergence.

In this part, fundamental definitions and theorems will be given. First of all, let (X, d) be a metric space and f, (f_n) are functions defined on X with $n \in \mathbb{N}$. If it is not mentioned explicitly the symbol d stands for the metric on X.

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Statistical convergence of a sequence of scalars was introduced by Fast [7]. Let (x_n) be a sequence of real or complex numbers. If for all $\varepsilon > 0$, there exists L such that,

$$\lim_{k\to\infty} \frac{1}{k}|\{n \le k : |x_n - L| \ge \varepsilon\}| = 0,$$

then the sequence (x_n) is statistically convergent to L.

Now, let us recall the definitions of basic concepts (see [1, 2, 6, 9, 10, 20, 21]).
Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:
i) $\emptyset \in \mathcal{I}$, ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.
$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.
Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:
i) $\emptyset \notin \mathcal{F}$, ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.
If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

is a filter on X, called the filter associated with $\mathcal{I}$.
A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.
Denote by $\mathcal{I}_d$ the class of all $A \subset \mathbb{N}$ with $d(A) = 0$. Then $\mathcal{I}_d$ is non-trivial admissible ideal and $\mathcal{I}_d$-convergence coincides with the statistical convergence.
Throughout the paper we take $\mathcal{I}$ as a nontrivial admissible ideal in $\mathbb{N}$.
Let (X, ρ) be a linear metric space. A sequence (x_n) of elements of X is said to be $\mathcal{I}$-convergent to $\xi \in X$ ($\mathcal{I} - \lim_{n\to\infty} x_n = \xi$) if and only if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \rho(x_n, \xi) \ge \varepsilon\}$ belongs to $\mathcal{I}$. The element ξ is called the $\mathcal{I}$-limit of the sequence (x_n).
Note that if $\mathcal{I}$ is an admissible ideal, then usual convergence in X implies $\mathcal{I}$-convergence in X.
For a sequence $x = (x_n)$ of real numbers, the notions of ideal limit superior and ideal limit inferior are defined as follows:

$$\mathcal{I} - \limsup x = \begin{cases} \sup B_x & , \; if \;\; B_x \neq \emptyset \\ -\infty & , \; if \;\; B_x = \emptyset \end{cases}$$

and

$$\mathcal{I} - \liminf x = \begin{cases} \inf A_x & , \; if \;\; A_x \neq \emptyset \\ +\infty & , \; if \;\; A_x = \emptyset \end{cases},$$

where $A_x = \{a \in \mathbb{R} : \{n \in \mathbb{N} : x_n < a\} \notin \mathcal{I}\}$ and $B_x = \{b \in \mathbb{R} : \{n \in \mathbb{N} : x_n > b\} \notin \mathcal{I}\}$.
A point $\lambda \in X$ is called an ideal limit point of a sequence (x_n) if there is a set $K = \{n_1 < n_2 < n_3 < ...\} \subset \mathbb{N}$ with $K \notin \mathcal{I}$ such that $x_{n_k} \to \lambda$ as $k \to \infty$. The set of all ideal limit points of a sequence (x_n) will be denoted by $\mathcal{I}(\Lambda_{(x_n)})$.
A point $\gamma \in X$ is called an ideal cluster point of (x_n) if for any $\varepsilon > 0$,

$$\{n \in \mathbb{N} : d(x_n, \gamma) < \varepsilon\} \notin \mathcal{I}.$$

The set of all statistical cluster points of (x_n) will be denoted by $\mathcal{I}(\Gamma_{(x_n)})$.
Obviously we have $\mathcal{I}(\Lambda_{(x_n)}) \subseteq \mathcal{I}(\Gamma_{(x_n)})$.
$\alpha = \mathcal{I}\text{-}\liminf x_n \in \mathbb{R}$ if and only if

$$\{n : x_n < \alpha + \varepsilon\} \notin \mathcal{I} \text{ and } \{n : x_n < \alpha - \varepsilon\} \in \mathcal{I}. \tag{1}$$

$\beta = \mathcal{I}\text{-}\limsup x_n \in \mathbb{R}$ if and only if

$$\{n : x_n > \beta - \varepsilon\} \notin \mathcal{I} \text{ and } \{n : x_n > \beta + \varepsilon\} \in \mathcal{I}. \tag{2}$$

Following collections of subsets of $\mathbb{N}$ need to be defined before we mention about ideal convergence of sequence of sets.

$\mathcal{N}_{\mathcal{I}} = \{N \subset \mathbb{N} : \mathbb{N} \setminus N \in \mathcal{I}\} = \mathcal{F}(\mathcal{I})$ and $\mathcal{N}_{\mathcal{I}}^{\#} = \{N \subset \mathbb{N} : N \notin \mathcal{I}\}$.

Following definitions and propositions of ideal inner and outer limit of a sequence (A_n) of closed subsets of X are referred to [21].

Let (X, d) be a metric space. The ideal inner and outer limit of a sequence (A_n) of closed subsets of X are defined as follows:

$$\mathcal{I}\text{-}\liminf_n A_n : = \{x \mid \forall V \in \mathcal{N}(x), \exists N \in \mathcal{N}_{\mathcal{I}}, \forall n \in N : A_n \cap V \neq \emptyset\},$$
$$\mathcal{I}\text{-}\limsup_n A_n : = \{x \mid \forall V \in \mathcal{N}(x), \exists N \in \mathcal{N}_{\mathcal{I}}^{\#}, \forall n \in N : A_n \cap V \neq \emptyset\}.$$

Proposition 1.1. *Let (X, d) be a metric space and (A_n) be a sequence of closed subsets of X. Then*

$$\mathcal{I}\text{-}\liminf_n A_n = \{x \mid \exists N \in \mathcal{N}_{\mathcal{I}}, \forall n \in N, \exists y_n \in A_n : \lim_n y_n = x\}.$$

Proposition 1.2. *Let (X, d) be a metric space and (A_n) be a sequence of closed subsets of X. Then*

$$\mathcal{I}\text{-}\limsup_n A_n = \{x \mid \exists N \in \mathcal{N}_{\mathcal{I}}^{\#}, \forall n \in N, \exists y_n \in A_n : x \in \mathcal{I}(\Gamma_{(y_n)})\}.$$

Let f be a function defined on X, the epigraph of f is the set $epif := \{(x, \alpha) \in X \times \mathbb{R} \mid \alpha \geq f(x)\}$ and its level set is defined by $lev_{\leq\alpha} f := \{x \in X \mid f(x) \leq \alpha\}$. Hence for functions f and g from X to $\mathbb{R}$, if $f \leq g$ for all $x \in X$ it is obvious that

$$epif \supseteq epig. \tag{3}$$

For any sequence (f_n) of functions on X, statistical epi-limit inferior, $e_{st}\text{-}\liminf_n f_n$:

$$epi(e_{st}\text{-}\liminf_n f_n) = st\text{-}\limsup_n (epif_n).$$

Statistical epi-limit superior, $e_{st}\text{-}\limsup_n f_n$:

$$epi(e_{st}\text{-}\limsup_n f_n) := st\text{-}\liminf_n (epif_n).$$

When these two functions equal to each other, we have $e_{st}\text{-}\lim_n f_n = e_{st}\text{-}\liminf_n f_n = e_{st}\text{-}\limsup_n f_n$. Hence the functions f_n are said to statistical epi convergent to the function f (see [18]). It is symbolized by $f_n \xrightarrow{e_{st}} f$. Moreover, the relation between set convergence and convergence of sequence of functions appears in the following equality.

$$f_n \xrightarrow{e_{st}} f \Leftrightarrow epif_n \xrightarrow{st} epif.$$

For every function $f : X \to \overline{\mathbb{R}}$ the lower semicontinuous envelope $sc^- f$ of f is defined for every $x \in X$ by $(sc^- f)(x) = \sup_{g \in \mathcal{G}(f)} g(x)$, where $\mathcal{G}(f)$ is the set of all lower semicontinuous functions g on X such that $g(y) \leq f(y)$ for every $y \in X$.

Proposition 1.3. *[11] Let $f : X \to \overline{\mathbb{R}}$ be a function. Then,*

$$(sc^- f)(x) = \sup_{V \in \mathcal{N}(x)} \inf_{y \in V} f(y)$$

for every $x \in X$ where $\mathcal{N}(x)$ is the neighbourhood of x.

2. Main Result

In this part, we define ideal epi-convergence by using epigraph of the functions. The functions are chosen to be lower semicontinuous since epigraphs of lower semicontinuous functions are closed. After that, we derive the topological definition of ideal epi-convergence. Level sets which are important instruments in set theory are also included in our calculations for lower and upper ideal epi-limits. At the end, we answer the question of what is the relationship between ideal epi-convergence and ordinary epi-convergence.

Definition 2.1. *Let (X,d) be a metric space and $f_n : X \to \overline{\mathbb{R}}$ a sequence of lower semicontinuous functions. The ideal epi-limit inferior, $e_{\mathcal{I}}\text{-}\liminf_n f_n$ is defined by:*

$$epi(e_{\mathcal{I}}\text{-}\liminf_n f_n) := \mathcal{I}\text{-}\limsup_n (epi f_n).$$

Similarly, the ideal epi-limit superior $e_{\mathcal{I}}\text{-}\limsup_n f_n$ is defined by:

$$epi(e_{\mathcal{I}}\text{-}\limsup_n f_n) := \mathcal{I}\text{-}\liminf_n (epi f_n).$$

When these two functions are equal, we get ideal epi-limit function:

$$f = e_{\mathcal{I}}\text{-}\lim_n f_n := e_{\mathcal{I}}\text{-}\limsup_n f_n = e_{\mathcal{I}}\text{-}\liminf_n f_n.$$

As defined in above and by (3)*, it is obvious that $e_{\mathcal{I}}\text{-}\liminf_n f_n \le e_{\mathcal{I}}\text{-}\limsup_n f_n$.*

Here we use statistical Painlevé-Kuratowski convergence. Whenever (f_n) is ideal epi-convergent to f we can use the inclusion

$$\mathcal{I}\text{-}\limsup_n (epi f_n) \subset epi f \subset \mathcal{I}\text{-}\liminf_n (epi f_n).$$

Moreover, following comparisons with epi-limit are valid for any $f : X \to \overline{\mathbb{R}}$.

$$e\text{-}\liminf_n f_n \le e_{\mathcal{I}}\text{-}\liminf_n f_n \text{ and } e\text{-}\limsup_n f_n \le e_{\mathcal{I}}\text{-}\limsup_n f_n.$$

Lemma 2.2. *Let (X,d) be a metric space and $f_n : X \to \mathbb{R}$ a sequence of lower semicontinuous functions, for every $x \in X$, define $g : X \to \overline{\mathbb{R}}$ by*

$$g(x) = \sup_{V \in \mathcal{N}(x)} \mathcal{I}\text{-}\liminf_n \inf_{y \in V} f_n(y).$$

Then, $\mathcal{I}\text{-}\limsup_n (epi f_n) = epi g$.

Proof. We need to prove the inclusions

$$\mathcal{I}\text{-}\limsup_n (epi f_n) \subset epi g \text{ and } epi g \subset \mathcal{I}\text{-}\limsup_n (epi f_n).$$

For the first inclusion, let us choose arbitrary $(x,\alpha) \in \mathcal{I}\text{-}\limsup_n (epi f_n)$, $V_0 \in \mathcal{N}(x)$ and $\varepsilon > 0$. According to the definition of $\mathcal{I}$-outer limit, $\exists N \in \mathcal{N}_{\mathcal{I}}^{\#}$ such that $\forall n \in N$ we have

$$V_0 \times (-\infty, \alpha + \varepsilon) \bigcap epi f_n \neq \emptyset.$$

As a result,

$$\{n \in \mathbb{N} : \inf_{y \in V_0} f_n(y) < \alpha + \varepsilon\} \notin \mathcal{I}.$$

By (1) we have,

$$\mathcal{I}\text{-}\liminf_n \inf_{y \in V_0} f_n(y) \le \alpha + \varepsilon.$$

Since V_0 and ε are arbitrary, we have $g(x) \leq \alpha$ and hence $(x, \alpha) \in epig$.

For the second inclusion let $(x, \alpha) \in epig$, for all $V_0 \in \mathcal{N}(x)$ and for all $\varepsilon > 0$ we have,

$$\alpha + \varepsilon > g(x) \geq \mathcal{I}\text{-}\liminf_n \inf_{y \in V_0} f_n(y).$$

Again by (1) we get

$$\{n \in \mathbb{N} : \inf_{y \in V_0} f_n(y) < \alpha + \varepsilon\} \notin \mathcal{I}.$$

It means, $\exists N \in \mathcal{N}_{\mathcal{I}}^{\#}$ such that $\forall n \in N$

$$V_0 \times (-\infty, \alpha + \varepsilon) \bigcap epif_n \neq \emptyset.$$

and as epigraphs lie in the vertical direction, we have

$$V_0 \times (\alpha - \varepsilon, \alpha + \varepsilon) \bigcap epif_n \neq \emptyset,$$

and so

$$(x, \alpha) \in \mathcal{I}\text{-}\limsup_n (epif_n).$$

□

Lemma 2.3. *Let (X, d) be a metric space and $f_n : X \to \mathbb{R}$ a sequence of lower semicontinuous functions, for every $x \in X$, define $h : X \to \overline{\mathbb{R}}$ by*

$$h(x) = \sup_{V \in \mathcal{N}(x)} \mathcal{I}\text{-}\limsup_n \inf_{y \in V} f_n(y).$$

Then, $\mathcal{I}\text{-}\liminf_n(epif_n) = epih$.

Proof. We need to show that

$$\mathcal{I}\text{-}\liminf_n(epif_n) \subset epih \text{ and } epih \subset \mathcal{I}\text{-}\liminf_n(epif_n).$$

For the first inclusion, let us choose arbitrary $(x, \alpha) \in \mathcal{I}\text{-}\liminf_n(epif_n)$, $V_0 \in \mathcal{N}(x)$ and $\varepsilon > 0$. According to the definition of $\mathcal{I}$-inner limit, $\exists N \in \mathcal{N}_{\mathcal{I}}$ such that $\forall n \in N$ we have

$$V_0 \times (-\infty, \alpha + \varepsilon) \bigcap epif_n \neq \emptyset.$$

Hence, we get

$$\{n \in \mathbb{N} : \inf_{y \in V_0} f_n(y) > \alpha + \varepsilon\} \in \mathcal{I}.$$

By (2) we obtain

$$\mathcal{I}\text{-}\limsup_n \inf_{y \in V_0} f_n(y) \leq \alpha + \varepsilon.$$

Since V_0 and ε are arbitrary, we have $h(x) \leq \alpha$ and hence $(x, \alpha) \in epih$.

For the second inclusion, fix $(x, \alpha) \in epih$. Given $V_0 \in \mathcal{N}(x)$ and $\varepsilon > 0$, $\exists N \in \mathcal{N}_{\mathcal{I}}$ such that $\forall n \in N$ we have

$$\mathcal{I}\text{-}\limsup_n \inf_{y \in V_0} f_n(y) \leq h(x) < \alpha + \varepsilon$$

and it equals to the following equality

$$\{n \in \mathbb{N} : \inf_{y \in V_0} f_n(y) < \alpha + \varepsilon\} \in \mathcal{N}_{\mathcal{I}}.$$

Hence,

$$\{n \in \mathbb{N} : V_0 \times (-\infty, \alpha + \varepsilon) \bigcap epif_n \neq \emptyset\} \in \mathcal{N}_{\mathcal{I}}.$$

It can be written as

$$\{n \in \mathbb{N} : V_0 \times (\alpha - \varepsilon, \alpha + \varepsilon) \bigcap epif_n \neq \emptyset\} \in \mathcal{N}_{\mathcal{I}}.$$

It gives $(x, \alpha) \in \mathcal{I}\text{-}\liminf_n(epif_n)$ and concludes the proof. □

The following definition is a direct result of Lemma 2.2 and Lemma 2.3.

Definition 2.4. *Let (X, d) be a metric space and $f_n : X \to \mathbb{R}$ a sequence of lower semicontinuous functions, for every $x \in X$, ideal epi-limit inferior and superior functions are defined by*

$$\left(e_{\mathcal{I}}\text{-}\liminf_n f_n\right)(x) := \sup_{V \in \mathcal{N}(x)} \mathcal{I}\text{-}\liminf_n \inf_{y \in V} f_n(y),$$

$$\left(e_{\mathcal{I}}\text{-}\limsup_n f_n\right)(x) := \sup_{V \in \mathcal{N}(x)} \mathcal{I}\text{-}\limsup_n \inf_{y \in V} f_n(y)$$

If there exists a function $f : X \to \overline{\mathbb{R}}$ such that

$$e_{\mathcal{I}}\text{-}\liminf_n f_n = e_{\mathcal{I}}\text{-}\limsup_n f_n = f$$

then, we write $f = e_{\mathcal{I}}\text{-}\lim_n f_n$ and we say that (f_n) is $e_{\mathcal{I}}$-convergent to f on X.

Lemma 2.5. *Let $x = (x_n)$ be a real sequence. Then,*

$$\mathcal{I}\text{-}\liminf_n x_n = \inf_{N \in \mathcal{N}_{\mathcal{I}}^{\#}} \sup_{n \in N} x_n = \sup_{N \in \mathcal{N}_{\mathcal{I}}} \inf_{n \in N} x_n,$$

$$\mathcal{I}\text{-}\limsup_n x_n = \sup_{N \in \mathcal{N}_{\mathcal{I}}^{\#}} \inf_{n \in N} x_n = \inf_{N \in \mathcal{N}_{\mathcal{I}}} \sup_{n \in N} x_n$$

By lemma 2.5, the ideal epi-limit inferior can be expressed as follows:

$$(e_{\mathcal{I}}\text{-}\liminf_n f_n)(x) = \sup_{V \in \mathcal{N}(x)} \inf_{N \in \mathcal{N}_{\mathcal{I}}^{\#}} \sup_{n \in N} \inf_{y \in V} f_n(y) = \sup_{V \in \mathcal{N}(x)} \sup_{N \in \mathcal{N}_{\mathcal{I}}} \inf_{n \in N} \inf_{y \in V} f_n(y).$$

Similarly, the ideal epi-limit superior can be expressed as follows:

$$(e_{\mathcal{I}}\text{-}\limsup_n f_n)(x) = \sup_{V \in \mathcal{N}(x)} \sup_{N \in \mathcal{N}_{\mathcal{I}}^{\#}} \inf_{n \in N} \inf_{y \in V} f_n(y) = \sup_{V \in \mathcal{N}(x)} \inf_{N \in \mathcal{N}_{\mathcal{I}}} \sup_{n \in N} \inf_{y \in V} f_n(y).$$

Remark 2.6. *If the functions $f_n(x)$ are independent of x, for every $n \in \mathbb{N}$ there exists a constant $\alpha_n \in \overline{\mathbb{R}}$ such that $f_n(x) = \alpha_n$ for every $x \in X$,*

$$e_{\mathcal{I}}\text{-}\liminf_n f_n(x) = \mathcal{I}\text{-}\liminf_n \alpha_n \;\; \textit{and} \;\; e_{\mathcal{I}}\text{-}\limsup_n f_n(x) = \mathcal{I}\text{-}\limsup_n \alpha_n.$$

If the sequence of functions are independent of n, there exists $f : X \to \overline{\mathbb{R}}$ such that $f_n(x) = f(x)$ for every $x \in X$,

$$e_{\mathcal{I}}\text{-}\liminf_n f_n = e_{\mathcal{I}}\text{-}\limsup_n f_n = sc^- f$$

for every $n \in \mathbb{N}$.

Proposition 2.7. *In a metric space (X, d) the following inequalities hold:*

$$(e_{\mathcal{I}}\text{-}\liminf_n f_n)(x) \leq \mathcal{I}\text{-}\liminf_n f_n(x) \;\; \textit{and} \;\; (e_{\mathcal{I}}\text{-}\limsup_n f_n)(x) \leq \mathcal{I}\text{-}\limsup_n f_n(x).$$

for every $x \in X$.

Proof. $\forall x \in X$ and $\forall V \in \mathcal{N}(x)$, $\exists N \in \mathcal{N}_{\mathcal{I}}$ such that $\forall n \in N$ we have

$$\inf_{y\in V} f_n(y) \leq f_n(x) \text{ and } inf_{y\in V} f_n(y) \leq f_n(x).$$

Since by the choice of our index set ($n \in N$), we get the following inequalities,

$$\mathcal{I}\text{-}\liminf_n \inf_{y\in V} f_n(y) \leq \mathcal{I}\text{-}\liminf_n f_n(x) \text{ and } \mathcal{I}\text{-}\limsup_n \inf_{y\in V} f_n(y) \leq \mathcal{I}\text{-}\limsup_n f_n(x).$$

After taking the supremum over all $V \in \mathcal{N}(x)$ we get the desired conclusion. □

Theorem 2.8. *Let (X, d) be a metric space and let (f_n) be a sequence of lower semicontinuous functions. Suppose that for each $\alpha \in \mathbb{R}$, $\exists(\alpha_n)$ of reals ideal convergent to α with $lev_{\leq\alpha} f = \mathcal{I}\text{-}\lim_n(lev_{\leq\alpha_n} f_n)$, then $f = e_{\mathcal{I}}\text{-}\lim_n f_n$.*

Proof. Since $lev_{\leq\alpha} f = \mathcal{I}\text{-}\lim_n(lev_{\leq\alpha_n} f_n)$ then, we can use the inclusion

$$lev_{\leq\alpha} f \subset \mathcal{I}\text{-}\liminf_n(lev_{\leq\alpha_n} f_n)$$

for each $\alpha \in \mathbb{R}$ and for some sequence $\alpha_n \xrightarrow{\mathcal{I}} \alpha$. Let $(x, \alpha) \in epif$, then there exists a sequence α_n ideal convergent to α such that

$$lev_{\leq\alpha} f \subset \mathcal{I}\text{-}\liminf_n(lev_{\leq\alpha_n} f_n).$$

Hence, $x \in \mathcal{I}\text{-}\liminf_n(lev_{\leq\alpha_n} f_n)$. It means there exists a sequence (x_n) ideal convergent to x such that $x_n \in (lev_{\leq\alpha_n} f_n)$. Finally we get

$$(x_n, \alpha_n) \xrightarrow{\mathcal{I}} (x, \alpha) \text{ and } (x, \alpha) \in \mathcal{I}\text{-}\liminf_n epif_n.$$

In order to get $\mathcal{I}\text{-}\limsup epif_n \subset epif$, suppose to the contrary that $(x, \beta) \in \mathcal{I}\text{-}\limsup epif_n$ but that $(x, \beta) \notin epif$. Then, $\beta < f(x)$. We can find $N \in \mathcal{N}_{\mathcal{I}}^{\#}$ such that $\forall n \in N$, $(x_n, \beta_n) \in epif_n$ such that $(x, \beta) \in \mathcal{I}(\Gamma_{(x_n,\beta_n)})$. Choose a scalar α between β and $f(x)$ and let (α_n) be a sequence ideal convergent to α for which

$$lev_{\leq\alpha} f \supset \mathcal{I}\text{-}\limsup_n(lev_{\leq\alpha_n} f_n).$$

We have

$$\{n : \beta_n < \alpha_n\} \notin \mathcal{I} \text{ and } (x_n, \beta_n) \in epif_n.$$

$\exists N \in \mathcal{N}_I^{\#}$, $\forall n \in N$, $x_n \in lev_{\leq\alpha_n} f_n$ which means

$$x \in \mathcal{I}\text{-}\limsup_n lev_{\alpha_n} f_n.$$

By the inclusion $\mathcal{I}\text{-}\limsup_n(lev_{\leq\alpha_n} f_n) \subset lev_{\leq\alpha} f$ we get

$$x \in lev_{\leq\alpha} f \text{ and } f(x) \leq \alpha$$

which is a contradiction. □

Theorem 2.9. *The following properties hold for any sequence of lower semicontinuous functions (f_n) defined on X.*
(i) The functions $e_{\mathcal{I}}\text{-}\liminf_n f_n$ and $e_{\mathcal{I}}\text{-}\limsup_n f_n$ are lower semicontinuous and so too is $e_{\mathcal{I}}\text{-}\lim_n f_n$ when it exists.
(ii) If the sequence (f_n) is monotone ideal decreasing, then $e_{\mathcal{I}}\text{-}\lim_n f_n$ exists and equals $sc^-[\inf_n f_n]$.
(iii) If the sequence (f_n) is monotone ideal increasing, then $e_{\mathcal{I}}\text{-}\lim_n f_n$ exists and equals $\sup_n[sc^- f_n]$.

Proof. (i) Let U be a family of open subsets of X, $\alpha : U \to \mathbb{R}$ be an arbitrary function and $f : X \to \mathbb{R}$ be defined by $f(x) = \sup_{U\in N(x)} \alpha(U)$. $\forall U \subseteq X$, $\forall y \in U$ and $\forall U \in N(y)$ it is clear that $f(y) \geq \alpha(U)$. Since the inequality is satisfied by for all $U \in N(x)$ we have

$$\inf_{y\in U} f(y) \geq \alpha(U).$$

Taking supremum of both sides

$$f(x) = \sup_{U\in N(x)} \alpha(U) \leq \sup_{U\in N(x)} \inf_{y\in U} f(y)$$

for every $x \in X$. Since the opposite inequality trivial we get

$$\sup_{U\in N(x)} \alpha(U) = \sup_{U\in N(x)} \inf_{y\in U} f(y).$$

If we write

$$\alpha(U) = \mathcal{I}\text{-}\liminf_{n} \inf_{y\in U} f_n(y),$$

we get the desired conclusion. The proof is similar for functions $e_{\mathcal{I}}\text{-}\limsup_n f_n$ and $e_{\mathcal{I}}\text{-}\lim_n f_n$.

Now we will prove (*ii*), the proof of (*iii*) is similar. Since the sequence (f_n) is ideal decreasing, then there exists a subset $K = \{k_1 < k_2 < k_3 < \cdots\} \subseteq \mathbb{N}$ such that $K \in \mathcal{F}(\mathcal{I})$ and $f_{k_n} \geq f_{k_{n+1}}$ for all $n \in \mathbb{N}$ and its epigraph $epi f_n$ will increase that is $epi f_{k_n} \subseteq epi f_{k_{n+1}}$. Then, we have

$$epi(sc^{-}[\inf_n f_n]) = cl \bigcup_{n\in\mathbb{N}} epi f_{k_n}. \quad (4)$$

Moreover, Theorem 3.15 in [21] makes clear the following equality for increasing sequences

$$\mathcal{I}\text{-}\lim_n (epi f_n) = cl \bigcup_{n\in\mathbb{N}} epi f_{k_n}. \quad (5)$$

By using (4) and (5) combining with Definition 2.1,

$$\mathcal{I}\text{-}\lim_n (epi f_n) = epi(sc^{-}[\inf_n f_n]) = epi(e_{\mathcal{I}}\text{-}\lim_n f_n).$$

Finally, we get the desired equation

$$sc^{-}[\inf_n f_n] = e_{\mathcal{I}}\text{-}\lim_n f_n.$$

□

Definition 2.10. *The sequence (f_n) is called ideal equi-lower semicontinuous at a point x if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ and $N \subset \mathcal{N}_{\mathcal{I}}$ such that for all $y \in B(x, \delta)$ we have,*

$$f_n(x) - f_n(y) < \varepsilon$$

for each $n \in N$.

In general, ideal epi-convergence is neither stronger nor weaker than ideal convergence. The obvious difference between these convergence types is obtaining minimums, but we can say that these two convergence types coincide under some conditions. The next theorem states the conditions for this overlap.

Theorem 2.11. *(f_n) and f are functions from X to $\mathbb{R}$ and (f_n) be ideal equi-lower semicontinuous at x. (f_n) is ideal epi-convergent to f at x if and only if (f_n) is ideal convergent to f at x.*

Proof. Assuming (f_n) is ideal equi-lower semicontinuous at x, we have that for all $\varepsilon > 0$, there exists $V \in \mathcal{N}(x)$ and $N \in \mathcal{N}_{\mathcal{I}}$ such that

$$f_n(x) - \varepsilon < \inf_{y \in V} f_n(y)$$

for all $n \in N$. This implies

$$\mathcal{I}\text{-}\liminf_n f_n(x) - \varepsilon \leq \sup_{V \in \mathcal{N}(x)} \mathcal{I}\text{-}\liminf_n \inf_{y \in V} f_n(y)$$

for every $\varepsilon > 0$. Combining with Proposition 2.7 we get

$$\mathcal{I}\text{-}\liminf_n f_n(x) = \sup_{V \in \mathcal{N}(x)} \mathcal{I}\text{-}\liminf_n \inf_{y \in V} f_n(y)$$

which means,

$$\mathcal{I}\text{-}\liminf_n f_n(x) = e_{\mathcal{I}}\text{-}\liminf_n f_n(x).$$

In similar way, we get

$$\mathcal{I}\text{-}\limsup_n f_n(x) = e_{\mathcal{I}}\text{-}\limsup_n f_n(x)$$

and finally we reach the desired equality as follows

$$\mathcal{I}\text{-}\lim_n f_n(x) = e_{\mathcal{I}}\text{-}\lim_n f_n(x).$$

□

3. Conclusion, future work

So far, topological and epigraphical definitions of ideal epi-convergence have been focused on. These definitions form the framework for understanding ideal epi-convergence and prepare the ground for the sequential characterization. The difference of sequential characterization of ideal epi-convergence from the definitions given in this paper is that it is not biconditional contrary to the ordinary characterization of epi-convergence. In our future work, we will try to understand how this difference will affect the optimization theorems.

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