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Some new criteria for identifying H-matrices

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Abstract. Given a matrix, whether it is a nonsingular H-matrix or not is very important in many applications. In this paper, some new simple criteria, as well as a necessary condition of nonsingular H-matrices, are proposed. Moreover, they only depend on the elements of a given matrix. Finally, some numerical examples are given to illustrate the validity of our results.

1. Introduction

The class of H-matrices, proposed by Ostrowski to study the convergence of matrix iterative schemes, plays an important role in many subjects, such as computational mathematics, mathematical physics, economics and dynamical system theory[1]. For a given matrix, an interesting problem is to check whether it is a H-matrix or not, as it can be used in numerical linear algebra, control theory, economic model and etc.(see [2, 3]).

In 2008, R. Bru et al. proposed a partition in the family of H-matrices, that is, "invertible class", "singular class" and "mixed class". The "invertible class" contains all H-matrices such that its comparison matrix is nonsingular, and these H-matrices are invertible, therefore, they also be called "nonsingular H-matrices". The second class contains H-matrices with a singular comparison matrix. Note that the H-matrices in the third class, the mixed class, all have diagonal elements different from zero, but their comparison matrices are singular. These matrices may be singular or not, as well as reducible or not[1]. For nonsingular H-matrices, many criteria for determining nonsingular H-matrices are obtained (see [4–16]). However, most of these criteria are sufficient conditions.

In this paper, we still focus on the judgement of nonsingular H-matrix. In section 2, several simple criteria which only depend on the elements of a given matrix are obtained, and a necessary condition for nonsingular H-matrix is also presented. In section 3, some numerical examples are given to illustrate our results.

For the convenience of discussion, some notations, definitions and lemmas are listed firstly in the following.

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Let n be an integer number, $N = \{1, 2, \ldots, n\}$ and $C^{n\times n}$ be the set of all complex matrices of order n. Denote
$r_i(A) = \sum_{j\neq i} |a_{ij}|$,
$N_1 = \{i \in N : 0 < |a_{ii}| \leq \alpha r_i(A)\}$,
$N_2 = \{i \in N : \alpha r_i(A) < |a_{ii}| < r_i(A)\}$,
$N_3 = \{i \in N : |a_{ii}| = r_i(A)\}$,
$N_4 = \{i \in N : |a_{ii}| > r_i(A)\}$,
where $0 < \alpha < 1$.

Definition 1.1. *[2] A matrix $A = (a_{ij}) \in C^{n\times n}$ is called a diagonally dominant matrix if for each $i \in N$,*

$$|a_{ii}| \geq r_i(A). \tag{1}$$

Definition 1.2. *[2] A matrix $A = (a_{ij}) \in C^{n\times n}$ is called a strictly diagonally dominant (shortly as SDD) matrix if for each $i \in N$,*

$$|a_{ii}| > r_i(A). \tag{2}$$

Definition 1.3. *[17] A matrix $A = (a_{ij}) \in R^{n\times n}$ is a nonsingular M-matrix if its inverse is nonnegative and all its off-diagonal entries are nonpositive.*

Definition 1.4. *[18] A matrix $A = (a_{ij}) \in C^{n\times n}$ is called a nonsingular H-matrix if its comparison matrix $\mu(A) = (m_{ij}) \in R^{n\times n}$,*

$$m_{ij} = \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & i \neq j, \end{cases}$$

is a nonsingular M-matrix.

From Definition 1.2, it is easy to obtain that A is a nonsingular H-matrix if $\bigcup_{i=1}^{3} N_i = \emptyset$. In addition, for a nonsingular H-matrix, there exists at least one strict diagonally dominant row, i.e., $N_4 \neq \emptyset$ [4].

Definition 1.5. *[19] A matrix $A = (a_{ij}) \in C^{n\times n}$ is called reducible if there exists a $\emptyset \neq S \subset N$ such that $a_{ij} = 0$, for any $i \in S$ and $j \notin S$. Otherwise, A is called irreducible.*

Definition 1.6. *([2, 3]) A matrix $A = (a_{ij}) \in C^{n\times n}$ is called irreducibly diagonally dominant if A is a irreducible and diagonally dominant matrix with strict inequality (2) holds for at least one $i \in N$.*

Definition 1.7. *[5] A matrix $A = (a_{ij}) \in C^{n\times n}$ is called a diagonally dominant matrix with nonzero elements chain if A is a diagonally dominant matrix with at least one strict inequality (2) holds, and for every $i \in N_3$, there exists a nonzero elements chain $a_{ij_1}a_{j_1j_2}\ldots a_{j_{k-1}j_k} \neq 0$ such that $|a_{j_kj_k}| > r_{j_k}(A)$.*

Lemma 1.8. *[20] A matrix $A = (a_{ij}) \in C^{n\times n}$ is a nonsingular H-matrix if and only if there exists a positive diagonal matrix X such that AX is an SDD matrix.*

Lemma 1.9. *[7] Let $A = (a_{ij}) \in C^{n\times n}$, if A is an irreducible diagonally dominant matrix, then it is a nonsingular H-matrix.*

Lemma 1.10. *[7] Let $A = (a_{ij}) \in C^{n\times n}$, if A is a nonzero element chain diagonally dominant matrix, then it is a nonsingular H-matrix.*

Lemma 1.11. *[6] Let $A = (a_{ij}) \in C^{n\times n}$, if A has nonzero diagonal elements, then A is a nonsingular H-matrix if and only if $A(\theta)$ is a nonsingular H-matrix, where $A(\theta)$ be the submatrix of A whose rows and columns are indexed by θ and $\theta = \{i | i \in N, r_i(A) > 0\}$.*

Lemma 1.12. *[6] Let $A = (a_{ij}) \in C^{n\times n}$ and P be a permutation matrix. Then A is a nonsingular H-matrix if and only if P^TAP is a nonsingular H-matrix.*

2. Main results

To begin with, a new criteria for nonsingular H-matrix is introduced in Theorem 2.1.

Theorem 2.1. *Let $A = (a_{ij}) \in C^{n\times n}$, if there exists a $\alpha \in (0,1)$ such that A satisfies the following conditions,*

$$\begin{aligned} |a_{ii}| >& \frac{\alpha r_i(A)}{|a_{ii}|}\Big[\sum_{t\in N_1\backslash\{i\}} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t\in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t\in N_3}|a_{it}| \\ &+ \sum_{t\in N_4}\frac{r_t(A)}{|a_{tt}|}|a_{it}|\Big], \qquad (for \ \ all \ \ i\in N_1), \end{aligned} \tag{3}$$

$$\begin{aligned} |a_{ii}| >& \frac{r_i(A)}{|a_{ii}| - \alpha r_i(A)}\Big[\sum_{t\in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t\in N_2\backslash\{i\}} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| \\ &+ \sum_{t\in N_3}|a_{it}| + \sum_{t\in N_4}\frac{r_t(A)}{|a_{tt}|}|a_{it}|\Big], \qquad (for \ \ all \ \ i\in N_2), \end{aligned} \tag{4}$$

$$\begin{aligned} |a_{ii}| >& \sum_{t\in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t\in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t\in N_3\backslash\{i\}}|a_{it}| \\ &+ \sum_{t\in N_4}\frac{r_t(A)}{|a_{tt}|}|a_{it}|, \quad (for \ \ all \ \ i\in N_3), \end{aligned} \tag{5}$$

then A is a nonsingular H-matrix.

Proof. From (3)-(5), we obtain that

$$\begin{aligned} R_i =& \frac{1}{\sum\limits_{t\in N_4}|a_{it}|}\Big[\frac{|a_{ii}|}{\alpha r_i(A)}|a_{ii}| - \sum_{t\in N_1\backslash\{i\}} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| - \sum_{t\in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| \\ &- \sum_{t\in N_3}|a_{it}| - \sum_{t\in N_4}\frac{r_t(A)}{|a_{tt}|}|a_{it}|\Big] > 0, \qquad (for \ \ all \ \ i\in N_1), \end{aligned} \tag{6}$$

$$\begin{aligned} S_i =& \frac{1}{\sum\limits_{t\in N_4}|a_{it}|}\Big[\frac{|a_{ii}| - \alpha r_i(A)}{r_i(A)}|a_{ii}| - \sum_{t\in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| - \sum_{t\in N_2\backslash\{i\}} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| \\ &- \sum_{t\in N_3}|a_{it}| - \sum_{t\in N_4}\frac{r_t(A)}{|a_{tt}|}|a_{it}|\Big] > 0, \qquad (for \ \ all \ \ i\in N_2), \end{aligned} \tag{7}$$

$$\begin{aligned} P_i =& \frac{1}{\sum\limits_{t\in N_4}|a_{it}|}\Big[|a_{ii}| - \sum_{t\in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| - \sum_{t\in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| - \sum_{t\in N_3\backslash\{i\}}|a_{it}| \\ &- \sum_{t\in N_4}\frac{r_t(A)}{|a_{tt}|}|a_{it}|\Big] > 0, \qquad (for \ \ all \ \ i\in N_3). \end{aligned} \tag{8}$$

If $\sum\limits_{t\in N_4}|a_{it}|$=0, we showed $R_i = +\infty$, $S_i = +\infty$, $P_i = +\infty$. Obviously, $R_i > 0$, $S_i > 0$, $P_i > 0$, for all $i \in \bigcup\limits_{i=1}^{3} N_i$. Then there exists a positive number $\varepsilon > 0$ such that

$$0 < \varepsilon < min\{\min_{i\in N_1} R_i, \min_{i\in N_2} S_i, \min_{i\in N_3} P_i\}. \tag{9}$$

Construct a matrix $X = diag(x_1, \dots, x_n)$, where

$$x_t = \begin{cases} \frac{|a_{tt}|}{\alpha r_t(A)}, & t \in N_1, \\ \frac{|a_{tt}|-\alpha r_t(A)}{r_t(A)}, & t \in N_2, \\ 1, & t \in N_3, \\ \varepsilon + \frac{r_t(A)}{|a_{tt}|}, & t \in N_4. \end{cases}$$

As $\varepsilon \neq +\infty$, we obtain $x_i \neq +\infty$, then X is a positive diagonal matrix.

Let $B = (b_{ij}) = AX$, then $b_{ij} = a_{ij}x_j$, for all $i, j \in N$. Next, we prove that B is an SDD matrix. From the division of N, the proof can be divided four cases as follows:

Case 1: for all $i \in N_1$.

(i) If $\sum\limits_{t\in N_4} |a_{it}| = 0$, then $a_{it} = 0$, for all $i \in N_1$, $t \in N_4$.

By (3), we have

$$\begin{aligned} r_i(B) &= \sum_{t\in N_1\backslash\{i\}} x_t|a_{it}| + \sum_{t\in N_2} x_t|a_{it}| + \sum_{t\in N_3} x_t|a_{it}| + \sum_{t\in N_4} x_t|a_{it}| \\ &= \sum_{t\in N_1\backslash\{i\}} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t\in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t\in N_3} |a_{it}| \\ &< \frac{|a_{ii}|}{\alpha r_i(A)}|a_{ii}| = |b_{ii}|. \end{aligned}$$

(ii) If $\sum\limits_{t\in N_4} |a_{it}| \neq 0$, by (6) and (9), we have

$$\begin{aligned} r_i(B) &= \sum_{t\in N_1\backslash\{i\}} x_t|a_{it}| + \sum_{t\in N_2} x_t|a_{it}| + \sum_{t\in N_3} x_t|a_{it}| + \sum_{t\in N_4} x_t|a_{it}| \\ &= \sum_{t\in N_1\backslash\{i\}} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t\in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t\in N_3} |a_{it}| + \sum_{t\in N_4} (\varepsilon + \frac{r_t(A)}{|a_{tt}|})|a_{it}| \\ &< R_i \sum_{t\in N_4} |a_{it}| + \sum_{t\in N_1\backslash\{i\}} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t\in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t\in N_3} |a_{it}| + \sum_{t\in N_4} \frac{r_t(A)}{|a_{tt}|}|a_{it}| \\ &= \frac{|a_{ii}|}{\alpha r_i(A)}|a_{ii}| = |b_{ii}|. \end{aligned}$$

Case 2: for all $i \in N_2$.

(i) If $\sum\limits_{t\in N_4} |a_{it}| = 0$, then $a_{it} = 0$, for all $i \in N_2$, $t \in N_4$, similar to the proof of (i) of Case 1, and by (4), we obtain

$$r_i(B) < \frac{|a_{ii}| - \alpha r_i(A)}{r_i(A)}|a_{ii}| = |b_{ii}|.$$

(ii) If $\sum\limits_{t\in N_4} |a_{it}| \neq 0$, by (7) and (9), we have

$$\begin{aligned} r_i(B) &= \sum_{t\in N_1} x_t|a_{it}| + \sum_{t\in N_2\backslash\{i\}} x_t|a_{it}| + \sum_{t\in N_3} x_t|a_{it}| + \sum_{t\in N_4} x_t|a_{it}| \\ &= \sum_{t\in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t\in N_2\backslash\{i\}} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t\in N_3} |a_{it}| + \sum_{t\in N_4} (\varepsilon + \frac{r_t(A)}{|a_{tt}|})|a_{it}| \\ &< S_i \sum_{t\in N_4} |a_{it}| + \sum_{t\in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t\in N_2\backslash\{i\}} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t\in N_3} |a_{it}| + \sum_{t\in N_4} \frac{r_t(A)}{|a_{tt}|}|a_{it}| \\ &= \frac{|a_{ii}| - \alpha r_i(A)}{r_i(A)}|a_{ii}| = |b_{ii}|. \end{aligned}$$

Case 3: for all $i \in N_3$.

(i) If $\sum\limits_{i \in N_4} |a_{it}| = 0$, then $a_{it} = 0$, for all $i \in N_3$, $t \in N_4$, similar to the proof of (i) of Case 1, and by (5), we have

$$r_i(B) < |a_{ii}| = |b_{ii}|.$$

(ii) If $\sum\limits_{i \in N_4} |a_{it}| \neq 0$, by (8) and (9), we have

$$\begin{aligned} r_i(B) &= \sum_{t \in N_1} x_t|a_{it}| + \sum_{t \in N_2} x_t|a_{it}| + \sum_{t \in N_3\backslash\{i\}} x_t|a_{it}| + \sum_{t \in N_4} x_t|a_{it}| \\ &= \sum_{t \in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t \in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t \in N_3\backslash\{i\}} |a_{it}| + \sum_{t \in N_4}(\varepsilon + \frac{r_t(A)}{|a_{tt}|})|a_{it}| \\ &< P_i \sum_{t \in N_4} |a_{it}| + \sum_{t \in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t \in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t \in N_3\backslash\{i\}} |a_{it}| + \sum_{t \in N_4} \frac{r_t(A)}{|a_{tt}|}|a_{it}| \\ &= |a_{ii}| = |b_{ii}|. \end{aligned}$$

Case 4: for all $i \in N_4$, we have $|a_{ii}| > r_i(A)$, further, it is easy to obtain that

$$|a_{ii}| - \sum_{t \in N_4\backslash\{i\}} |a_{it}| > 0, for\ all\ i \in N_4.$$

From the definition of N_1, N_2 and N_3, we obtain that

$$0 < \frac{|a_{tt}|}{\alpha r_t(A)} < 1 \quad (for\ all\ t \in N_1),$$
$$0 < \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)} < 1 \quad (for\ all\ t \in N_2),$$
$$0 < \frac{r_t(A)}{|a_{tt}|} < 1 \quad (for\ all\ t \in N_4).$$

Therefore, we get that

$$\begin{aligned} &\sum_{t \in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t \in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t \in N_3} |a_{it}| + \sum_{t \in N_4\backslash\{i\}} \frac{r_t(A)}{|a_{tt}|}|a_{it}| - r_i(A) \\ &\quad \le \sum_{t \in N_1} |a_{it}| + \sum_{t \in N_2} |a_{it}| + \sum_{t \in N_3} |a_{it}| + \sum_{t \in N_4\backslash\{i\}} |a_{it}| - r_i(A) = 0, \end{aligned}$$

further, we obtain that

$$\frac{1}{|a_{ii}| - \sum\limits_{t \in N_4\backslash\{i\}} |a_{it}|}\Big[\sum_{t \in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t \in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t \in N_3} |a_{it}| + \sum_{t \in N_4\backslash\{i\}} \frac{r_t(A)}{|a_{tt}|}|a_{it}| - r_i(A)\Big] \le 0.$$

As $\varepsilon > 0$, then

$$\varepsilon > \frac{1}{|a_{ii}| - \sum\limits_{t \in N_4\backslash\{i\}} |a_{it}|}\Big[\sum_{t \in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t \in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t \in N_3} |a_{it}| + \sum_{t \in N_4\backslash\{i\}} \frac{r_t(A)}{|a_{tt}|}|a_{it}| - r_i(A)\Big]. \tag{10}$$

From (10), for all $i \in N_4$,

$$\begin{aligned} |b_{ii}| - r_i(B) &= x_i|a_{ii}| - \sum_{t \in N_1} x_t|a_{it}| - \sum_{t \in N_2} x_t|a_{it}| - \sum_{t \in N_3} x_t|a_{it}| - \sum_{t \in N_4\backslash\{i\}} x_t|a_{it}| \\ &= \varepsilon(|a_{ii}| - \sum_{t \in N_4\backslash\{i\}} |a_{it}|) + r_i(A) - \sum_{t \in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| \\ &- \sum_{t \in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| - \sum_{t \in N_3} |a_{it}| - \sum_{t \in N_4\backslash\{i\}} \frac{r_t(A)}{|a_{tt}|}|a_{it}| > 0. \end{aligned}$$

That is, $|b_{ii}| > r_i(B)$, for all $i \in N_4$.

From Case 1-4, we get that $|b_{ii}| > r_i(B)$, for all $i \in N$, i.e., B is an SDD matrix, from Lemma 1.8, A is a nonsingular H-matrix. □

Next, based on the conclusion of Lemma 1.9, we have that an irreducible diagonally dominant matrix is a nonsingular H-matrix, therefore, we obtain the following Theorem 2.2.

Theorem 2.2. *Let $A = (a_{ij}) \in C^{n\times n}$ be irreducible matrix, if there exists a $\alpha \in (0, 1)$ such that*

$$|a_{ii}| \geq \frac{\alpha r_i(A)}{|a_{ii}|}\Big[\sum_{t\in N_1\setminus\{i\}} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t\in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t\in N_3} |a_{it}| + \sum_{t\in N_4} \frac{r_t(A)}{|a_{tt}|}|a_{it}|\Big], \quad (for\ \ all\ \ i \in N_1), \tag{11}$$

$$|a_{ii}| \geq \frac{r_i(A)}{|a_{ii}| - \alpha r_i(A)}\Big[\sum_{t\in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t\in N_2\setminus\{i\}} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t\in N_3} |a_{it}| + \sum_{t\in N_4} \frac{r_t(A)}{|a_{tt}|}|a_{it}|\Big], \quad (for\ \ all\ \ i \in N_2), \tag{12}$$

$$|a_{ii}| \geq \sum_{t\in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t\in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t\in N_3\setminus\{i\}} |a_{it}| + \sum_{t\in N_4} \frac{r_t(A)}{|a_{tt}|}|a_{it}|, \quad (for\ \ all\ \ i \in N_3), \tag{13}$$

and there exists at least one strict inequality holds for $i \in \bigcup_{i\in 1}^{3} N_i$, then A is a nonsingular H-matrix.

Proof. By the irreducibility of A, we can get $r_i(A) > 0$, for all $i \in N$. Construct $X = diag(x_1, x_2, \ldots, x_n)$, where

$$x_t = \begin{cases} \frac{|a_{tt}|}{\alpha r_t(A)}, & t \in N_1, \\ \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}, & t \in N_2, \\ 1, & t \in N_3, \\ \frac{r_t(A)}{|a_{tt}|}, & t \in N_4. \end{cases}$$

Obviously $x_i \neq +\infty$, then X is a positive diagonal matrix.

Let $B = (b_{ij}) = AX$, then $b_{ij} = a_{ij}x_j$, for all $i, j \in N$. Similar to the proof of Theorem 2.1, the proof can be divided four cases as follows:

Case 1: By (11), we have for all $i \in N_1$

$$\begin{aligned} r_i(B) &= \sum_{t\in N_1\setminus\{i\}} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t\in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t\in N_3} |a_{it}| + \sum_{t\in N_4} \frac{r_t(A)}{|a_{tt}|}|a_{it}| \\ &\leq \frac{|a_{ii}|}{\alpha r_i(A)}|a_{ii}| = |b_{ii}|. \end{aligned}$$

Case 2: By (12), we have for all $i \in N_2$

$$r_i(B) \leq \frac{|a_{ii}| - \alpha r_i(A)}{r_i(A)}|a_{ii}| = |b_{ii}|.$$

Case 3: By (13), we have for all $i \in N_3$

$$r_i(B) \leq |a_{ii}| = |b_{ii}|,$$

and there exists at least one $i \in \bigcup_{i=1}^{3} N_i$, such that $|b_{ii}| > r_i(B)$.

Case 4: Since $0 < \frac{|a_{tt}|}{\alpha r_t(A)} < 1$ (for all $t \in N_1$), $0 < \frac{|a_{tt}|-\alpha r_t(A)}{r_t(A)} < 1$ (for all $t \in N_2$), $0 < \frac{r_t(A)}{|a_{tt}|} < 1$ (for all $t \in N_4$), from the irreducibility of A, we get that for all $i \in N_4$

$$r_i(A) - \sum_{t \in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| - \sum_{t \in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| - \sum_{t \in N_3}|a_{it}| - \sum_{t \in N_4 \setminus \{i\}} \frac{r_t(A)}{|a_{tt}|}|a_{it}| \geq 0.$$

Therefore

$$\begin{aligned}
|b_{ii}| - r_i(B) &= x_i|a_{ii}| - \sum_{t \in N_1} x_t|a_{it}| - \sum_{t \in N_2} x_t|a_{it}| - \sum_{t \in N_3} x_t|a_{it}| - \sum_{t \in N_4 \setminus \{i\}} x_t|a_{it}| \\
&= r_i(A) - \sum_{t \in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| - \sum_{t \in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| \\
&- \sum_{t \in N_3}|a_{it}| - \sum_{t \in N_4 \setminus \{i\}} \frac{r_t(A)}{|a_{tt}|}|a_{it}| \geq 0,
\end{aligned}$$

and we obtain $|b_{ii}| \geq r_i(B)$, for all $i \in N_4$.

From Case 1-4, we get that $|b_{ii}| \geq r_i(B)$, for all $i \in N$, i.e. B is a diagonally dominant matrix. In addition, since A is irreducible, then we get B is also irreducible. Hence, B is an irreducible diagonally dominant matrix. By Lemma 1.9, we conclude that B is a nonsingular H-matrix, and then there exists a positive diagonal matrix X_1, such that BX_1 is an SDD matrix, that is, $BX_1 = A(XX_1)$ and (XX_1) is a positive diagonal matrix, then we get that A is a nonsingular H-matrix. □

It is shown that a diagonally dominant matrix with nonzero elements chain is a nonsingular H-matrix by Lemma 1.10, therefore, the following Theorem 2.3 is obtained.

Let

$$\begin{aligned}
J_1 = \Big\{i \in N_1 : |a_{ii}| &> \frac{\alpha r_i(A)}{|a_{ii}|}\Big[\sum_{t \in N_1 \setminus \{i\}} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t \in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t \in N_3}|a_{it}| + \sum_{t \in N_4} \frac{r_t(A)}{|a_{tt}|}|a_{it}|\Big]\Big\}, \\
J_2 = \Big\{i \in N_2 : |a_{ii}| &> \frac{r_i(A)}{|a_{ii}| - \alpha r_i(A)}\Big[\sum_{t \in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t \in N_2 \setminus \{i\}} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t \in N_3}|a_{it}| \\
&+ \sum_{t \in N_4} \frac{r_t(A)}{|a_{tt}|}|a_{it}|\Big]\Big\}, \\
J_3 = \Big\{i \in N_3 : |a_{ii}| &> \sum_{t \in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t \in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t \in N_3 \setminus \{i\}}|a_{it}| \\
&+ \sum_{t \in N_4} \frac{r_t(A)}{|a_{tt}|}|a_{it}|\Big\}, \\
J_4 = \Big\{i \in N_4 : |a_{ii}| &> \frac{|a_{ii}|}{\alpha r_i(A)}\Big[\sum_{t \in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t \in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t \in N_3}|a_{it}| + \sum_{t \in N_4 \setminus \{i\}} \frac{r_t(A)}{|a_{tt}|}|a_{it}|\Big]\Big\}.
\end{aligned}$$

Theorem 2.3. *Let $A = (a_{ij}) \in C^{n\times n}$, if there exists a $\alpha \in (0,1)$such that*

$$\begin{aligned} |a_{ii}| \geq & \frac{\alpha r_i(A)}{|a_{ii}|}\Big[\sum_{t\in N_1\setminus\{i\}} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t\in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t\in N_3} |a_{it}| \\ & + \sum_{t\in N_4} \frac{r_t(A)}{|a_{tt}|}|a_{it}|\Big], \qquad (for\ \ all\ \ i \in N_1), \end{aligned} \tag{14}$$

$$\begin{aligned} |a_{ii}| \geq & \frac{r_i(A)}{|a_{ii}| - \alpha r_i(A)}\Big[\sum_{t\in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t\in N_2\setminus\{i\}} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| \\ & + \sum_{t\in N_3} |a_{it}| + \sum_{t\in N_4} \frac{r_t(A)}{|a_{tt}|}|a_{it}|\Big], \qquad (for\ \ all\ \ i \in N_2), \end{aligned} \tag{15}$$

$$\begin{aligned} |a_{ii}| \geq & \sum_{t\in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t\in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t\in N_3\setminus\{i\}} |a_{it}| \\ & + \sum_{t\in N_4} \frac{r_t(A)}{|a_{tt}|}|a_{it}|, \qquad (for\ \ all\ \ i \in N_3), \end{aligned} \tag{16}$$

and if for all $i \in \bigcup_{i=1}^{4}[N_i - Ji]$, there exists a nonzero elements chain $a_{ij_1}a_{j_1j_2}\ldots a_{j_{k-1}k} \neq 0$ such that $k \in \bigcup_{i=1}^{4} J_i$, then A is a nonsingular H-matrix.

Proof. From Lemma 1.11, we assume that $r_i(A) > 0$ (for all $i \in N$). Construct $X = diag(x_1, x_2, \ldots, x_n)$, where

$$x_t = \begin{cases} \frac{|a_{tt}|}{\alpha r_t(A)}, & t \in N_1, \\ \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}, & t \in N_2, \\ 1, & t \in N_3, \\ \frac{r_t(A)}{|a_{tt}|}, & t \in N_4. \end{cases}$$

Obviously $x_i \neq +\infty$, thus X is a diagonal matrix that has positive diagonal entries. Let $B = (b_{ij}) = AX$, we can write $b_{ij} = a_{ij}x_j$, for all $i, j \in N$. This proof is similar to the proof of Theorem 2.2, we can obtain that $|b_{ii}| \geq r_i(B)$, for all $i \in N$, and there exists at least an $i \in \bigcup_{i=1}^{3} N_i$ such that $|b_{ii}| > r_i(B)$.

The other hand, if $|b_{ii}| = r_i(B)$, then for all $i \in \bigcup_{i=1}^{4}[N_i - J_i]$, by the hypotheses, we get that there exists a non-zero entries chain of A, which is $a_{ij_1}a_{j_1j_2}\ldots a_{j_{k-1}k} \neq 0$ such that $k \in \bigcup_{i=1}^{4} J_i$. Then there exists a non-zero entries chain of B, which is $b_{ij_1}b_{j_1j_2}\ldots b_{j_{k-1}k} \neq 0$, such that $k \in \bigcup_{i=1}^{4} J_i$ satisfying $|b_{kk}| > r_k(B)$. This means that B is a nonsingular H-matrix. By a similar proof of Theorem 2.2, we obtain that A is a nonsingular H-matrix. □

Theorem 2.4. *Let $A = (a_{ij}) \in C^{n\times n}$, if there exists a $\alpha \in (0,1)$ such that for all $j \in N_4$, $t \in \bigcup_{i=1}^{3} N_i$, $a_{jt} = 0$ and*

$$\begin{aligned} |a_{ii}| \quad > \quad & \frac{\alpha r_i(A)}{|a_{ii}|}\Big[\sum_{t\in N_1\setminus\{i\}} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t\in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| \\ + \quad & \sum_{t\in N_3} |a_{it}|\Big], \qquad (for\ \ all\ \ i \in N_1), \end{aligned}$$

$$|a_{ii}| > \frac{r_i(A)}{|a_{ii}| - \alpha r_i(A)}\Big[\sum_{t\in N_1}\frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t\in N_2\setminus\{i\}}\frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t\in N_3}|a_{it}|\Big], \quad (for\ \ all\ \ i \in N_2),$$

$$|a_{ii}| > \sum_{t\in N_1}\frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t\in N_2}\frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t\in N_3\setminus\{i\}}|a_{it}|, \quad (for\ \ all\ \ i \in N_3),$$

then A is a nonsingular H-matrix.

Proof. By assuming that there exists a positive number $k > 0$ such that

$$R_i = \frac{1}{\sum\limits_{t\in N_4}|a_{it}|}\Big[\frac{|a_{ii}|}{\alpha r_i(A)}|a_{ii}| - \sum_{t\in N_1\setminus\{i\}}\frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| - \sum_{t\in N_2}\frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| - \sum_{t\in N_3}|a_{it}| - \frac{1}{k}\sum_{t\in N_4}\frac{r_t(A)}{|a_{tt}|}|a_{it}|\Big] > 0, \quad (for\ \ all\ \ i \in N_1),$$

$$S_i = \frac{1}{\sum\limits_{t\in N_4}|a_{it}|}\Big[\frac{|a_{ii}| - \alpha r_i(A)}{r_i(A)}|a_{ii}| - \sum_{t\in N_1}\frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| - \sum_{t\in N_2\setminus\{i\}}\frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| - \sum_{t\in N_3}|a_{it}| - \frac{1}{k}\sum_{t\in N_4}\frac{r_t(A)}{|a_{tt}|}|a_{it}|\Big] > 0, \quad (for\ \ all\ \ i \in N_2),$$

$$P_i = \frac{1}{\sum\limits_{t\in N_4}|a_{it}|}\Big[|a_{ii}| - \sum_{t\in N_1}\frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| - \sum_{t\in N_2}\frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| - \sum_{t\in N_3\setminus\{i\}}|a_{it}| - \frac{1}{k}\sum_{t\in N_4}\frac{r_t(A)}{|a_{tt}|}|a_{it}|\Big] > 0, \quad (for\ \ all\ \ i \in N_3).$$

Then there exists a positive number $\varepsilon > 0$ such that

$$0 < \varepsilon < min\{\min_{i\in N_1} R_i, \min_{i\in N_2} S_i, \min_{i\in N_3} P_i\}.$$

Construct a matrix $X = diag(x_1, \ldots, x_n)$, where

$$x_t = \begin{cases} \frac{|a_{tt}|}{\alpha r_t(A)}, & t \in N_1, \\ \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}, & t \in N_2, \\ 1, & t \in N_3, \\ \varepsilon + \frac{r_t(A)}{k|a_{tt}|}, & t \in N_4. \end{cases}$$

As $\varepsilon \neq +\infty$, so we get $x_i \neq +\infty$, then X has positive diagonal elements and it is a diagonal matrix. Let $B = (b_{ij}) = AX$, where $b_{ij} = a_{ij}x_j$, for all $i, j \in N$, this proof is similar to the proof of Theorem 2.1, we obtain that A is a nonsingular H-matrix. □

Remark: For a matrix A of the form

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^{\bigcup_{i=1}^{3} N_i}_{N_4},$$

if the inequalities in Theorem 2.4 hold in A_{11}, then A is a nonsingular H-matrix, no matter how large the elements of A_{12} are.

In theorem 2.1-2.4, we obtain several simple criteria for nonsingular H-matrix. Finally, a necessary condition for nonsingular H-matrix is given in theorem 2.5.

Theorem 2.5. *If $A = (a_{ij}) \in C^{n\times n}$ is a nonsingular H-matrix, then there exists at least one $i \in N$ such that for $i \in N_1$,*

$$|a_{ii}| > \frac{\alpha r_i(A)}{|a_{ii}|}\Big[\sum_{t\in N_1\setminus\{i\}} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t\in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t\in N_3} |a_{it}|\Big], \tag{17}$$

or for $i \in N_2$,

$$|a_{ii}| > \frac{r_i(A)}{|a_{ii}| - \alpha r_i(A)}\Big[\sum_{t\in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t\in N_2\setminus\{i\}} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t\in N_3} |a_{it}|\Big], \tag{18}$$

or for $i \in N_3$,

$$|a_{ii}| > \sum_{t\in N_1} \frac{|a_{tt}|}{\alpha r_t(A)}|a_{it}| + \sum_{t\in N_2} \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}|a_{it}| + \sum_{t\in N_3\setminus\{i\}} |a_{it}|, \tag{19}$$

holds, where $\alpha \in (0, 1)$.

Proof. We prove the opposite. Let us assume inequations (17)-(19) are all false, we construct $X = diag(x_1, x_2, \ldots, x_n)$, where

$$x_t = \begin{cases} \frac{|a_{tt}|}{\alpha r_t(A)}, & t \in N_1, \\ \frac{|a_{tt}| - \alpha r_t(A)}{r_t(A)}, & t \in N_2, \\ 1, & t \in N_3, \\ 1, & t \in N_4. \end{cases}$$

Obviously, $x_i \neq +\infty$ and X is a positive diagonal entries. Let $B = (b_{ij}) = AX$, where $b_{ij} = a_{ij}x_j$, for all $i, j \in N$. Because inequations (17)-(19) are all invalid, we known that

$$|b_{jj}| \leq \sum_{t\in\bigcup_{i=1}^{3} N_i, t\neq j} |b_{jt}|, \qquad (for\ \ all\ \ j \in \bigcup_{i=1}^{3} N_i). \tag{20}$$

The other hand, because A is a nonsingular H-matrix, and so is B. Let P be a permutation matrix, such that

$$P^T BP = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}^{\bigcup_{i=1}^{3} N_i}_{N_4}.$$

By Lemma1.12, we know that

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}^{\bigcup_{i=1}^{3} N_i}_{N_4}$$

is a nonsingular H-matrix. Then

$$\begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix}^{\bigcup_{i=1}^{3} N_i}_{N_4}$$

is a nonsingular H-matrix. Therefore, B_{11} is a nonsingular H-matrix, which contradicts inequation (20). Then the conclusion follows. □

3. Example

In this section, some numerical examples are given to illustrate our results.

Example 3.1. *Let*

$$A_1 = \begin{bmatrix} 3 & 0 & 0 & 2 & 4 \\ 0 & 6 & 0 & 5 & 3 \\ 1 & 2 & 3 & 0 & 0 \\ 1 & 1 & 1 & 100 & 0 \\ 1 & 1 & 1 & 0 & 100 \end{bmatrix}.$$

We choose $\alpha = \frac{1}{2}$ and by simple calculation, it is easy to obtain that $N_1 = \{1\}, N_2 = \{2\}, N_3 = \{3\}, N_4 = \{4,5\}$, and $r_1(A_1) = 6, r_2(A_1) = 8, r_3(A_1) = 3, r_4(A_1) = 3, r_5(A_1) = 3$.
For all $i \in N_1$,

$$|a_{11}| = 3 > \frac{9}{50} = \frac{\frac{1}{2}r_1(A_1)}{|a_{11}|}[\frac{|a_{22}| - \frac{1}{2}r_2(A_1)}{r_2(A_1)}|a_{12}| + |a_{13}| + \frac{r_4(A_1)}{|a_{44}|}|a_{14}| + \frac{r_5(A_1)}{|a_{55}|}|a_{15}|],$$

for all $i \in N_2$,

$$|a_{22}| = 6 > \frac{24}{25} = \frac{r_2(A_1)}{|a_{22}| - \frac{1}{2}r_2(A_1)}[\frac{|a_{11}|}{\frac{1}{2}r_1(A_1)}|a_{21}| + |a_{23}| + \frac{r_4(A_1)}{|a_{44}|}|a_{24}| + \frac{r_5(A_1)}{|a_{55}|}|a_{25}|],$$

and for all $i \in N_3$,

$$|a_{33}| = 3 > \frac{3}{2} = \frac{|a_{11}|}{\frac{1}{2}r_1(A_1)}|a_{31}| + \frac{|a_{22}| - \frac{1}{2}r_2(A_1)}{r_2(A_1)}|a_{32}| + \frac{r_4(A_1)}{|a_{44}|}|a_{34}| + \frac{r_5(A_1)}{|a_{55}|}|a_{35}|,$$

therefore, from Theorem 2.1 it is easy to obtain that A is a nonsingular H-matrix. In fact, let $X = diag\{1, 0.25, 1, 0.18, 0.18\}$, one can also obtain that A is a nonsingular H-matrix from Lemma 1.8.

Next, we compare the Theorem 1.1 in [21] with our Theorem 2.1.

Example 3.2. *Let*

$$A_2 = \begin{bmatrix} 3 & 5 & 1 & 0 & 0 \\ 0 & 6 & 0 & 5 & 3 \\ 1 & 2 & 3 & 0 & 0 \\ 1 & 1 & 1 & 100 & 0 \\ 1 & 1 & 1 & 0 & 100 \end{bmatrix}.$$

We choose $\alpha = \frac{1}{2}$ and by calculation, we have $N_1 = \{1\}, N_2 = \{2\}, N_3 = \{3\}, N_4 = \{4,5\}$ and $r_1(A_2) = 6, r_2(A_2) = 8, r_3(A_2) = 3, r_4(A_2) = 3, r_5(A_2) = 3$.
For all $i \in N_1$,

$$|a_{11}| = 3 > \frac{9}{4} = \frac{\frac{1}{2}r_1(A_2)}{|a_{11}|}[\frac{|a_{22}| - \frac{1}{2}r_2(A_2)}{r_2(A_2)}|a_{12}| + |a_{13}| + \frac{r_4(A_2)}{|a_{44}|}|a_{14}| + \frac{r_5(A_2)}{|a_{55}|}|a_{15}|],$$

for all $i \in N_2$,

$$|a_{22}| = 6 > \frac{24}{25} = \frac{r_2(A_2)}{|a_{22}| - \frac{1}{2}r_2(A_2)}[\frac{|a_{11}|}{\frac{1}{2}r_1(A_2)}|a_{21}| + |a_{23}| + \frac{r_4(A_2)}{|a_{44}|}|a_{24}| + \frac{r_5(A_2)}{|a_{55}|}|a_{25}|],$$

and for all $i \in N_3$,

$$|a_{33}| = 3 > \frac{3}{2} = \frac{|a_{11}|}{\frac{1}{2}r_1(A_2)}|a_{31}| + \frac{|a_{22}| - \frac{1}{2}r_2(A_2)}{r_2(A_2)}|a_{32}| + \frac{r_4(A_2)}{|a_{44}|}|a_{34}| + \frac{r_5(A_2)}{|a_{55}|}|a_{35}|.$$

Therefore, from Theorem 2.1, we get A_2 is a nonsingular H-matrix.

However, it is difficult to prove whether A_2 is a nonsingular H-matrix using [21]. Since it is hard to find parameters α that satisfy the condition, we will illustrate using the following two figures.

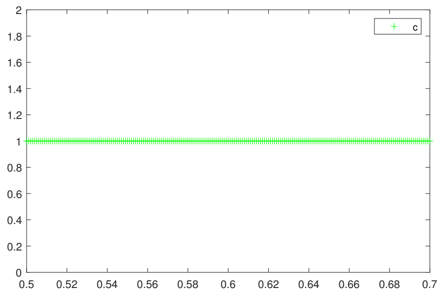

Figure 1: **The parameter α that satisfies all conditions of Theorem 2.1.**

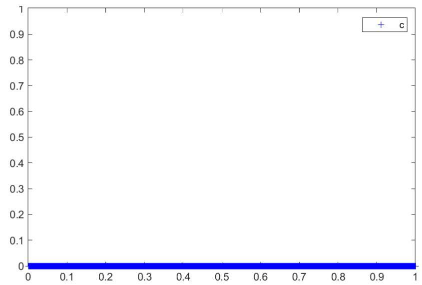

Figure 2: **Parameter α that does not satisfy the condition of Theorem 1.1 in [21].**

In figures 1 and 2, 1 signifies the parameter α that matches the criterion of Theorem 2.1, while 0 indicates parameter α that does not conform to the stipulations of Theorem 1.1 as mentioned in [21]. It is easy to observe when implementing Theorem 2.1 to assess whether A_2 is a nonsingular H-matrix, a substantial number of α values fulfill the prerequisites when the $0.5 < \alpha < 0.7$ and the step size is 0.01.

While applying Theorem 1.1 from [21] to check whether A_2 is a nonsingular H-matrix, it is extremely challenging to locate a value of $0 < \alpha < 1$ that satisfies the condition when the step size is set to 0.01. Consequently, our criterion yield better results in determining parameter α for A_2.

At the same time, using Theorem 1 in [6], it is not possible to determine whether matrix A_2 is a nonsingular H-matrix since $|a_{11}| = 3 < 9.5 = \frac{r_1(A_2)}{|a_{11}|}[\frac{|a_{22}|}{r_2(A_2)}|a_{12}| + \frac{|a_{33}|}{r_3(A_2)}|a_{13}| + \frac{r_4(A_2)}{|a_{44}|}|a_{14}| + \frac{r_5(A_2)}{|a_{55}|}|a_{15}|]$.

And the use of Theorem 2 in [7] does not provide a definitive judgment on whether matrix A_2 is a nonsingular H-matrix since $|a_{11}|\frac{|a_{11}|}{r_1(A_2)-|a_{11}|} = 3 < 5.375 = |a_{12}|\frac{r_2(A_2)+|a_{22}|}{2r_2(A_2)} + |a_{13}|$.

Finally, we have verified Theorem 1 in [20], since $|a_{11}| = 3 < 4.5 = \frac{r_1(A_2)}{|a_{11}|}[\frac{r_2(A_2)-|a_{22}|}{r_2(A_2)}|a_{12}| + |a_{13}| + \frac{r_4(A_2)}{|a_{44}|}|a_{14}| + \frac{r_5(A_2)}{|a_{55}|}|a_{15}|]$, then it is impossible to determine if matrix A_2 is a nonsingular H-matrix.

Example 3.3. *Let*

$$A_3 = \begin{bmatrix} 2 & 5 & 0 & 0 & 0 \\ 2 & 6 & 5 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}.$$

We choose $\alpha = \frac{1}{2}$, and by calculation, it is easy to obtain that $N_1 = \{1\}, N_2 = \{2\}, N_3 = \{3,4\}, N_4 = \{5\}$, and $r_1(A_3) = 5, r_2(A_3) = 7, r_3(A_3) = 3, r_4(A_3) = 4, r_5(A_3) = 1$. We get that for all $i \in N_1$,

$$|a_{11}| = 2 \leq \frac{125}{56} = \frac{\frac{1}{2}r_1(A_3)}{|a_{11}|}[\frac{|a_{22}| - \frac{1}{2}r_2(A_3)}{r_2(A_3)}|a_{12}| + |a_{13}|],$$

for all $i \in N_2$,

$$|a_{22}| = 6 \leq \frac{462}{25} = \frac{r_2(A_3)}{|a_{22}| - \frac{1}{2}r_2(A_3)}[\frac{|a_{11}|}{\frac{1}{2}r_1(A_3)}|a_{21}| + |a_{23}|],$$

and for all $i \in N_3$,

$$|a_{33}| = 3 = 3 = \frac{|a_{11}|}{\frac{1}{2}r_1(A_3)}|a_{31}| + \frac{|a_{22}| - \frac{1}{2}r_2(A_3)}{r_2(A_3)}|a_{32}| + |a_{34}|,$$

$$|a_{44}| = 4 = 4 = \frac{|a_{11}|}{\frac{1}{2}r_1(A_3)}|a_{41}| + \frac{|a_{22}| - \frac{1}{2}r_2(A_3)}{r_2(A_3)}|a_{42}| + |a_{43}|.$$

From Theorem 2.5, we obtain that A_3 is not nonsingular H-matrix. In fact, it is easy to verify that A_3 is not nonsingular H-matrix, since its comparison matrix $\mu(A_3) = (m_{ij}) \in R^{n\times n}$ is a singular matrix, that is $det(\mu(A_3)) = 0$.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

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Author's contributions

All authors jointly worked on the results and they read and approved the final manuscript.

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