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Endomorphism rings and formal matrix rings of pseudo-projective modules

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Abstract. A module M is called pseudo-projective if every epimorphism from M to each quotient module of M can be lifted to an endomorphism of M. In this paper, we study some properties of pseudo-projective modules and their endomorphism rings. It shows that if M is a self-cogenerator pseudo-projective module with finite hollow dimension, End(M) is a semilocal ring and every maximal right ideal of End(M) has of the form $\{s \in \mathrm{End}(M) | \mathrm{Im}(s) + \mathrm{Ker}(h) \neq M\}$ for some endomorphism h of M with $h(M)$ hollow. Moreover, it shows that a pseudo-projective R-module M is an SSP-module if and only if the product of any two regular elements of End(M) is a regular element. Finally, we investigate the pseudo-projectivity of modules over a formal triangular matrix ring.

1. Introduction

Throughout this article all rings are associative rings with unity and all modules are right unital modules over a ring. We denote by $|X|$ the cardinality of a set X. For a submodule N of M, we write $N \leq M$ ($N < M, N \ll M$) iff N is a submodule of M (respectively, a proper submodule, a small submodule). We denote by $J(R)$ the Jacobson radical of the ring R. For any term not defined here the reader is referred to [3] and [12].

A module M is called *pseudo-injective* if every monomorphism from each submodule of M to M is extended to an endomorphism of M. It is well-known that M is pseudo-injective if M is invariant under all automorphisms of its injective envelope ([17]). These modules are called *automorphism-invariant* ([11]). Some properties of pseudo-injective modules and structure of rings via automorphism-invariant modules are studied ([1, 9, 13, 17, 18]). Dualizing the notion of a pseudo-injective module, a module M is called *pseudo-projective* if every epimorphism from M to each quotient module of M can be lifted to an endomorphism of M ([19]). A right R-module M is called *quasi-principally injective* if for every endomorphism α of M, any homomorphism from $\alpha(M)$ to M can extended to an endomorphism of M. In [16, Theorem 4], the authors Sanh and Shum proved that if M is a quasi-principally injective module which is a self-generator

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with finite Goldie dimension, then $\mathrm{End}(M)/J(\mathrm{End}(M))$ is semisimple. This result is extended for general quasi-principally injective modules which are studied by Quynh and Sanh (see [14]). From this result, endomorphism rings of automorphism-invariant modules are studied in [20]. It shows that if M is an automorphism-invariant self-generator module with finite Goldie dimension, then every maximal left ideal of $\mathrm{End}(M)$ has the form of $\{s \in \mathrm{End}(M) | \mathrm{Ker}(s) \cap \mathrm{Im}(u) \neq 0\}$ for some $u \in \mathrm{End}(M)$ with $u(M)$ uniform. Motivated by these results, in this paper, we show, in Theorem 2.7, that if M is a pseudo-projective self-cogenerator module with finite hollow dimension and $S = \mathrm{End}(M)$ then

1. Every maximal right ideal of S has of the form

$$\{s \in S | \mathrm{Im}(s) + \mathrm{Ker}(h) \neq M\}$$

for some endomorphism h of M with $h(M)$ hollow.

2. S is semilocal (i.e., $S/J(S)$ is semisimple artinian).

In [14], the authors proved that if M is a general quasi-principally injective self-generator module with $S = \mathrm{End}(M)$, S is right perfect if and only if for any infinite sequence $s_1, s_2, \cdots \in S$, the chain $\mathrm{Ker}(s_1) \leq \mathrm{Ker}(s_2 s_1) \leq \cdots$ is stationary. By the dual method for pseudo-projective modules, it shows that for a pseudo-projective self-cogenerator right R-module M, $\mathrm{End}(M)$ is left perfect if and only if any infinite sequence $s_1, s_2, \cdots \in \mathrm{End}(M)$, the chain $\mathrm{Im}(s_1) \geq \mathrm{Im}(s_1 s_2) \geq \cdots$ is stationary (see Theorem 2.13). Consider the summand intersection property and the summand sum property of modules, we show that if M is a pseudo-projective (resp, pseudo-injective) module, M has the summand sum property (resp., the summand intersection property) if and only if the product of any two regular elements of $\mathrm{End}(M)$ is a regular element (see Theorem 3.4, 3.6). In section 4, we investigate the pseudo-projectivity of modules over a formal triangular matrix ring $K = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}$. It is shown that if $V = (X; Y)_f$ is a right K−module such that X is a pseudo-projective right A-module and the reduced map $\widetilde{f} : Y \to Hom_A(M, X)$ is an isomorphism, then V is a pseudo-projective right K-module (see Theorem 4.1).

2. On maximal ideals

Recall that a module M is called *quasi-projective* if every homomorphism from M to each quotient module of M can be lifted to an endomorphism of M. A module M is called *quasi-injective* if every homomorphism from each submodule of M to M is extended to an endomorphism of M. It is well-known that a module M is quasi-injective if and only if M is invariant under all endomorphisms of its injective envelope. One can check that every quasi-projective module is pseudo-projective. The following example shows that the converse is not true in general.

Example 2.1 ([9, Example 5.1]). *Let* $R = \begin{bmatrix} \mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2 \\ 0 & \mathbb{F}_2 & 0 \\ 0 & 0 & \mathbb{F}_2 \end{bmatrix}$ *where* $\mathbb{F}_2$ *is the field of two elements and* $M = e_{11}R$. *As* R *is a finite-dimensional algebra over* $\mathbb{F}_2$, *the functors*

$$\mathrm{Hom}_{\mathbb{F}_2}(-, \mathbb{F}_2) : \text{Mod-R} \to \text{R-Mod}$$

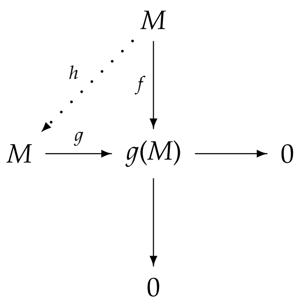
and

$$\mathrm{Hom}_{\mathbb{F}_2}(-, \mathbb{F}_2) : \text{R-Mod} \to \text{Mod-R}$$

establish a contravariant equivalence between the subcategories of left and right finitely generated modules over R. Then, $\mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_2)$ *is a pseudo-projective left R-module and it is not quasi-projective.*

Lemma 2.2. *Let* M *be a pseudo-projective module with* $S = \mathrm{End}(M)$. *If* f *and* g *are endomorphisms of* M *with* $\mathrm{Im}(f) = \mathrm{Im}(g)$, *then* $fS = gS$.

Proof. Assume that f and g are endomorphisms of M with $\mathrm{Im}(f) = \mathrm{Im}(g)$. We consider the following diagram



As M is pseudo-projective, there is an endomorphism h of M such that $f = gh \in gS$. Similarly, we also have $g \in fS$. Thus, $fS = gS$. □

Let M be a right R-module with $S = \mathrm{End}(M)$. A nonzero module M is said to be *hollow* if every proper submodule is small in M. An element h in S is called a *right hollow* element of S if h is nonzero and $\mathrm{Im}(h)$ is a hollow submodule of M.

Let h be a right hollow element of S. We call

$$\mathcal{M}_h = \{s \in S \mid \mathrm{Im}(s) + \mathrm{Ker}(h) \neq M\}$$

One can check that $\mathcal{M}_h$ is a proper right ideal of S.

Let α be an endomorphism of M with $S = \mathrm{End}(M)$. We denote by

$$r_S(\alpha) = \{s \in S \mid \alpha s = 0\}$$

the annihilator of α in S. If α is a right hollow element of S, then $r_S(\alpha)$ is a right ideal of S contained in $\mathcal{M}_\alpha$.

Lemma 2.3. *Assume that M is a pseudo-projective module. If h is a right hollow element of S, $\mathcal{M}_h$ is the unique maximal right ideal of S containing $r_S(h)$.*

Proof. Take s an element of S and $s \notin \mathcal{M}_h$. From the definition of $\mathcal{M}_h$, it infers that $\mathrm{Im}(s) + \mathrm{Ker}(h) = M$. Then, $hs(M) = h(M)$. By Lemma 2.2, we have that $hsS = hS$ and obtain that $h = hsk$ for some k in S. It follows that $S = r_S(h) + sS \leq \mathcal{M}_h + sS$, and so $S = \mathcal{M}_h + sS$. It is shown that $\mathcal{M}_h$ is a maximal of S. It remains to show that $\mathcal{M}_h$ is the unique right ideal of S containing $r_S(h)$. Indeed, let I be an another maximal ideal of S containing $r_S(h)$ and $I \neq \mathcal{M}_h$. Then, there exists an element $\alpha \in I \setminus \mathcal{M}_h$. It follows that $\mathrm{Im}(\alpha) + \mathrm{Ker}(h) = M$. By the similar process proof as above, we have $S = \alpha S + r_S(h) \leq I$ and so $S = I$, a contradiction. □

A family $\{M_\lambda\}_\Lambda$ of proper submodules of M is called *coindependent* if, for any $\lambda \in \Lambda$ and any finite subset $I \subseteq \Lambda \setminus \{\lambda\}, M_\lambda + \bigcap_{i \in F} M_i = M$.

Lemma 2.4 ([15, Lemma 3.5]). *Assume that M has coindependent submodules $M_1, M_2, \ldots, M_k$ such that $\bigcap_{i=1}^{k} M_i \ll M$ and M/M_i is hollow for every $1 \leq i \leq k$. If M has a submodule L such that $L + M_i \neq M$ for every $1 \leq i \leq k$, then L is small in M.*

Lemma 2.5. *Let M be a pseudo-projective right R-module with $S = \mathrm{End}(M)$ and $\{\varphi_1, \varphi_2, \ldots, \varphi_k\}$ be a family of nonzero elements of S with $\{\mathrm{Ker}(\varphi_1), \mathrm{Ker}(\varphi_2), \ldots, \mathrm{Ker}(\varphi_k)\}$ a finite coindependent family in M and $\{\mathrm{Im}(\varphi_1), \mathrm{Im}(\varphi_2), \ldots, \mathrm{Im}(\varphi_k)\}$ hollow modules. If I is a maximal right ideal of S which is not of the form $\mathcal{M}_h$ for some right hollow element h of S, then there is an endomorphism $\psi \in I$ such that*

$$[\mathrm{Im}(1-\psi) + \bigcap_{i=1}^{k} \mathrm{Ker}(\varphi_i)] / \bigcap_{i=1}^{k} \mathrm{Ker}(\varphi_i) \ll M / \bigcap_{i=1}^{k} \mathrm{Ker}(\varphi_i)$$

Proof. Take $W = \bigcap_{i=1}^{k} \mathrm{Ker}(\varphi_i)$. Let $\alpha \in I \setminus \mathcal{M}_{\varphi_1}$ and so $M = \mathrm{Im}(\alpha) + \mathrm{Ker}(\varphi_1)$. Then $\varphi_1(M) = (\varphi_1\alpha)(M)$. From Lemma 2.2, it immediately infers that $\varphi_1 S = (\varphi_1\alpha)S$. Thus, $\varphi_1 = (\varphi_1\alpha)s_1 = \varphi_1(\alpha s_1)$ for some $s_1 \in S$. Call $\psi_1 = \alpha s_1 \in I$, and so $\varphi_1(1-\psi_1) = 0$. This implies that $\mathrm{Im}(1-\psi_1) + \mathrm{Ker}(\varphi_1) = \mathrm{Ker}(\varphi_1) \neq M$. Suppose that $\mathrm{Im}(1-\psi_1) + \mathrm{Ker}(\varphi_j) \neq M$ for all $2 \leq j \leq k$. We have $\{\mathrm{Ker}(\varphi_1), \mathrm{Ker}(\varphi_2), \ldots, \mathrm{Ker}(\varphi_k)\}$ is a finite coindependent family in M and obtain that there is an isomorphism $\phi : M/W \to \bigoplus_{i=1}^{k} M/\mathrm{Ker}(\varphi_i)$ defined by

$$\phi(m + W) = (m + \mathrm{Ker}(\varphi_1), m + \mathrm{Ker}(\varphi_2), \ldots, m + \mathrm{Ker}(\varphi_k))$$

One can check that $\phi^{-1}[\bigoplus_{i=1}^{k} \frac{\mathrm{Im}(1-\psi_1) + \mathrm{Ker}(\varphi_i)}{\mathrm{Ker}(\varphi_i)}] = \frac{\mathrm{Im}(1-\psi_1) + W}{W}$. Since every $M/\mathrm{Ker}(\varphi_j) \cong \mathrm{Im}(\varphi_j)$ is hollow, $(\mathrm{Im}(1-\psi_1)+W)/W \ll M/W$. Without loss of generality, we now assume that $\mathrm{Im}(1-\psi_1)+\mathrm{Ker}(\varphi_2) = M$. Then $\varphi_2(1-\psi_1)(M) = \varphi_2(M)$. Since $\varphi_2(M)$ is hollow, $\varphi_2(1-\psi_1)(M)$ is hollow. Thus $\varphi_2(1-\psi_1)$ is a right hollow element of S. Since $I \neq \mathcal{M}_{\varphi_2(1-\psi_1)}$ and $\mathcal{M}_{\varphi_2(1-\psi_1)}$ is a maximal right ideal of S, we take $h \in I \setminus \mathcal{M}_{\varphi_2(1-\psi_1)}$. By using the above argument, we can find $s_2 \in S$ such that $\varphi_2(1-\psi_1) = \varphi_2(1-\psi_1)hs_2$, and so $\varphi_2(1-(\psi_1+(1-\psi_1)hs_2) = 0$. Put $\psi_2 = \psi_1 + (1-\psi_1)hs_2$. Then, we have $\varphi_i(1-\psi_2) = 0$ for all $i = 1, 2$. Continuing this process, we eventually get a $\psi \in I$ such that $\varphi_i(1-\psi) = 0$ for all $i = 1, 2, \ldots, k$. Thus, $\mathrm{Im}(1-\psi) \leq W$. We deduce that $(\mathrm{Im}(1-\psi) + W)/W \ll M/W$. □

From the proof of [22, 22.2], we have the following result of the Jacobson radical of a pseudo-projective module.

Lemma 2.6. *Let M be a right R-module. If M is a pseudo-projective module with $S = \mathrm{End}(M)$, then $J(S) = \{f \in S \mid \mathrm{Im}(f) \ll M\}$.*

A right R-module is called a *self-cogenerator* if it cogenerates all its factor modules ([22]). If M has coindependent submodules $\{M_1, M_2, ..., M_k\}$ such that $\bigcap_{i=1}^{k} M_i \ll M$ and M/M_i is hollow for every $1 \leq i \leq k$, M is said to have *hollow dimension k*, denoting this by $hdim(M) = k$.

Theorem 2.7. *Let M be a self-cogenerator pseudo-projective module with finite hollow dimension with $S = \mathrm{End}(M)$.*

1. *If I is a maximal right ideal, then $I = \mathcal{M}_h$ for some right hollow element $h \in S$.*
2. *S is semilocal (i.e., $S/J(S)$ is semisimple artinian).*

Proof. Assume that M has finite hollow dimension, there exists a coindependent family $\{N_1, N_2, \ldots, N_n\}$ of submodules of M such that $M/N_1, M/N_2, \ldots, M/N_n$ are hollow, $\bigcap_{i=1}^{n} N_i \ll M$ and an isomorphism $M/(\bigcap_{i=1}^{n} N_i) \cong \bigoplus_{i=1}^{n}(M/N_i)$. Take $\pi_j : M \to M/M_j$ the natural projections for all $j = 1, 2, \ldots, n$. We have that M is self-cogenerator, there is a nonzero homomorphism $f_j : M/N_j \to M$. Then, we have the homomorphisms $h_j = f_j\pi_j \in S$ for all $j = 1, 2, \ldots, n$. One can check that $N_j \leq \mathrm{Ker}(h_j)$ for all $j = 1, 2, \ldots, n$. We deduce that $M/\mathrm{Ker}(h_j)$ is hollow and the family $\{\mathrm{Ker}(h_1), \mathrm{Ker}(h_2), \ldots, \mathrm{Ker}(h_n)\}$ is coindependent. Take $W = \bigcap_{i=1}^{n} \mathrm{Ker}(h_i)$, and so $\bigcap_{i=1}^{n} N_i \leq W$. We have that $M/(\bigcap_{i=1}^{n} \mathrm{Ker}(h_i)) \cong \bigoplus_{i=1}^{n} M/\mathrm{Ker}(h_i)$ and obtain that $hdim(M/(\bigcap_{i=1}^{n} Kerh_i)) = n = hdim(M)$. Thus, $W \ll M$ by [6, 5.4(2)].

(1) Suppose that I is a maximal right ideal of S with $I \neq \mathcal{M}_h$ for every right hollow element h of S. Then by Lemma 2.5, there is an endomorphism φ in I such that $(\mathrm{Im}(1-\varphi) + W)/W \ll M/W$. We have that $W \ll M$ and obtain that $\mathrm{Im}(1-\varphi) \ll M$. From Lemma 2.6, it immediately infers that $1 - \varphi \in J(S) \leq I$, and so $1 \in I$, a contradiction.

(2) We have $J(S) \leq \bigcap_{i=1}^{n} \mathcal{M}_{h_i}$. If $f \in \bigcap_{i=1}^{n} \mathcal{M}_{h_i}$, then $\mathrm{Im}(f) + \mathrm{Ker}(h_j) \neq M$ for each $j = 1, 2, \ldots, n$. It follows that $\mathrm{Im}(f) \ll M$ by Lemma 2.4, and so $f \in J(S)$ by Lemma 2.6. Thus, $J(S) = \bigcap_{i=1}^{n} \mathcal{M}_{h_i}$. We deduce that S is semilocal. □

Corollary 2.8. *Let R be a self-cogenerator ring with finite hollow dimension. If I is a maximal right ideal of R,* $I = \mathcal{M}_h$ *for some right hollow element* $h \in R$.

Remark 2.9. *Theorem 2.7 holds if we replace the condition "self-cogenerator" by the condition "*$\operatorname{Hom}(M/K, M)$ *nonzero for all proper submodules K of M".*

Example 2.10. *(1) Let R be the ring of integers* $\mathbb{Z}$*. Take* $M = \mathbb{Z}$*. Then M is pseudo-projective with infinite hollow dimension. Note that* $\operatorname{End}(M)$ *contains no hollow elements. Thus the statements (1) and (2) of Theorem 2.7 are not satisfied. This shows that the hypothesis "M has finite hollow dimension" in Theorem 2.7 is not superfluous.*

(2) Let R be a nonlocal commutative domain with finitely many maximal ideals. Then, every nonzero element h in R is not hollow. So $\operatorname{End}(R)$ *contains no hollow elements. Thus the statements (1) and (2) of Theorem 2.7 are not satisfied. Note that R is pseudo-projective with finite hollow dimension. But R is not self-cogenerator because* $\operatorname{Hom}(R/J(R), R) = 0$*. This example shows that Theorem 2.7 is not true if M is not self-cogenerator.*

We denote by $\nabla(M) = \{f \in S \mid \operatorname{Im}(f) \ll M\}$ the set of all endomorphisms of M with small image.

Recall that an element $a \in R$ is said to be *regular* (in the sense of von Neumann) if there exists $x \in R$ such that $axa = a$. A ring R is called *regular* if every element of R is regular.

Lemma 2.11 (McCoy's Lemma). *Let R be a ring and* $a, c \in R$*. If* $b = a - aca$ *is a regular element of R, then so is a.*

Proof. This is by definition. □

Lemma 2.12. *Let M be a pseudo-projective module which is a self-cogenerator,* $S = \operatorname{End}(M)$*. If* $a \notin \nabla(M)$*, then* $\operatorname{Im}(a - asa) < \operatorname{Im}(a)$ *for some* $s \in S$.

Proof. If $a \notin \nabla(M)$, then $\operatorname{Im}(a)$ is not a small submodule of M. Hence there exists a proper submodule A of M such that $A + \operatorname{Im}(a) = M$. We have the natural isomorphism

$$M/(A \cap \operatorname{Im}(a)) \cong M/\operatorname{Im}(a) \oplus M/A$$

Since M is a self-cogenerator, there exists a nonzero homomorphism $M/A \to M$. It follows that there is a nonzero endomorphism λ of M such that A is contained in $\operatorname{Ker}(\lambda)$. Then, we have $\operatorname{Im}(a) + \operatorname{Ker}(\lambda) = M$, and so $(\lambda a)(M) = \lambda(M)$. Since M is pseudo-projective, $(\lambda a)S = \lambda S$ and so $\lambda = \lambda as$ for some $s \in S$. On the other hand, as λ is nonzero, there is $m \in M$ such that $\lambda(m)$ is nonzero. Call $y = as(m) \in \operatorname{Im}(a)$. One can check that y and $\lambda(y)$ are nonzero. Next, we show that y is not in $\operatorname{Im}(a - asa)$. Indeed, suppose that $y = (a - asa)(x) \in \operatorname{Im}(a - asa)$ for some $x \in M$. Then, we have

$$\lambda(y) = \lambda(a - asa)(x) = (\lambda a - \lambda asa)(x) = (\lambda a - \lambda a)(x) = 0$$

This is a contradiction, and so $y \in \operatorname{Im}(a) \setminus \operatorname{Im}(a - asa)$. □

Theorem 2.13. *Let M be a pseudo-projective right R-module which is a self-cogenerator and* $S = \operatorname{End}(M)$*. Then the following conditions are equivalent:*

(1) *S is left perfect.*
(2) *For any infinite sequence* $s_1, s_2, \cdots \in S$*, the chain*

$$\operatorname{Im}(s_1) \geq \operatorname{Im}(s_1 s_2) \geq \cdots$$

is stationary.

Proof. (1) ⇒ (2). Let $s_i \in S$, $i = 1, 2....$ Since S is left perfect, S satisfies DCC on finitely generated right ideals. So the chain $s_1 S \geq s_1 s_2 S \geq \cdots$ terminates. Thus, there exists $n > 0$ such that $s_1 s_2 .... s_n S = s_1 s_2 ... s_k S$ for all $k > n$. It follows that $s_1 s_2 .... s_n = s_1 s_2 ... s_k f$ and $s_1 s_2 ... s_k = s_1 s_2 .... s_n g$ for some $f, g \in S$. Thus, $s_1 s_2 .... s_n(M) = s_1 s_2 ... s_k(M)$ for all $k > n$.

(2) ⇒ (1). We first prove that $S/\nabla(M)$ is a von Neumann regular ring. Let $a_1 \notin \nabla(M)$. Then by Lemma 2.12, there is $c_1 \in S$ such that $\mathrm{Im}(a_1 - a_1c_1a_1) < \mathrm{Im}(a_1)$. Put $a_2 = a_1 - a_1c_1a_1$, and so $\mathrm{Im}(a_2) < \mathrm{Im}(a_1)$. If $a_2 \in \nabla(M)$, then we have $\bar{a}_1 = \bar{a}_1\bar{c}_1\bar{a}_1$, i.e., $\bar{a}_1$ is a regular element of $S/\nabla(M)$ (where $\bar{s} = s + \nabla(M)$ for all $s \in S$). If $a_2 \notin \nabla(M)$, there exists $a_3 \in S$ such that $\mathrm{Im}(a_3) < \mathrm{Im}(a_2)$ with $a_3 = a_2 - a_2c_2a_2$ for some $c_2 \in S$ by the preceding proof. Repeating the above-mentioned process, we get a strictly ascending chain

$$\mathrm{Im}(a_1) > \mathrm{Im}(a_2) > \cdots,$$

where $a_{i+1} = a_i - a_ic_ia_i$ for some $c_i \in S$, $i = 1, 2....$ Let

$$b_1 = a_1, b_2 = 1 - c_1a_1, ..., b_{i+1} = 1 - c_ia_i, ...,$$

then

$$a_1 = b_1, a_2 = b_1b_2, ..., a_{i+1} = b_1b_2...b_{i+1}, ...$$

and we have the following strictly ascending chain

$$\mathrm{Im}(b_1) > \mathrm{Im}(b_1b_2) > \cdots,$$

which contradicts the hypothesis. Hence there exists a positive integer m such that $a_{m+1} \in \nabla(M)$, i.e., $a_m - a_mc_ma_m \in J(S)$. This shows that $\bar{a}_m$ is a regular element of $S/\nabla(M)$, and hence $\bar{a}_{m-1}, \bar{a}_{m-2}, ..., \bar{a}_1$ are regular elements of $S/\nabla(M)$ by Lemma 2.11, i.e., $S/\nabla(M)$ is von Neumann regular. We have $J(S) = \nabla(M)$ by Lemma 2.6, proving that $S/J(S)$ is von Neumann regular.

We show that $J(S)$ is left T-nilpotent. In fact, if for any sequence $a_1, a_2, \ldots$ from $J(S)$, the chain

$$\mathrm{Im}(a_1) \geq \mathrm{Im}(a_1a_2) \geq \cdots$$

is stationary. Thus, there exists n such that $a_1a_2...a_n(M) = a_1a_2...a_k(M)$ for all $k > n$. We have that M is pseudo-projective and obtain that $a_1a_2...a_nS = a_1a_2...a_kS$ for all $k > n$. Then, $a_1a_2...a_n(1 - a_{n+1}s) = 0$ for some $s \in S$, and so $a_1a_2...a_n = 0$ (since $1 - a_{n+1}s$ is a unit of S). It means that $J(S)$ is left T-nilpotent.

Next, we prove that $S/J(S)$ contains no infinite sets of non-zero orthogonal idempotents. Indeed, let $\varepsilon_1, \varepsilon_2, ..., \varepsilon_k...$ be a countably infinite set of non-zero orthogonal idempotents in $S/J(S)$. Then, there exist non-zero orthogonal idempotents $e_1, e_2, ..., e_k...$ in S such that $\varepsilon_i = e_i + J(S)$, $i = 1, 2, ...$ by [3, Proposition 27.1]. Put $a_i = 1 - (e_1 + e_2 + \cdots + e_i)$, $i = 1, 2, ...$ Then $a_{i+1} = a_i - a_ie_{i+1}a_i$. One can check that $e_{i+1}a_{i+1} = 0$ and $e_{i+1}a_i = e_{i+1} \neq 0$. Take $m \in M$ with $e_{i+1}(m) \neq 0$. Call $y = a_i(m)$, and so y is nonzero in $\mathrm{Im}(a_i)$. Suppose that $y \in \mathrm{Im}(a_{i+1})$, $y = a_{i+1}(t)$ for some $t \in M$. Then, we have

$$e_{i+1}a_i(m) = e_{i+1}(y) = e_{i+1}a_{i+1}(t) = 0$$

Thus, $e_{i+1}(m) = e_{i+1}a_i(m) = 0$, a contradiction. It means that we have the strict sequence $\mathrm{Im}(a_i) > \mathrm{Im}(a_{i+1})$, $i = 1, 2, ...$ Let $b_i = 1 - e_i$, $i = 1, 2, ...$ Then $a_i = b_1b_2...b_i$ and $\mathrm{Im}(b_1b_2...b_i) > \mathrm{Im}(b_1b_2...b_{i+1})$, $i = 1, 2, ...$ We obtain the following strictly ascending chain $\mathrm{Im}(b_1) > \mathrm{Im}(b_1b_2) > ...$, a contradiction. Hence $S/J(S)$ contains no infinite sets of non-zero orthogonal idempotents. We deduce that $S/J(S)$ is semisimple. Thus S is left perfect. □

Corollary 2.14. *Let R be a self-cogenerator. If for any infinite sequence $r_1, r_2, \cdots$ in R, the chain $r_1R \geq r_1r_2R \geq \cdots$ is stationary then R is left perfect.*

Note that if M has DCC on the submodules of the form IM, where I is a right ideal of $\mathrm{End}(M)$, $\nabla(M)$ is nilpotent. Thus, we have the following corollary

Corollary 2.15. *Let M be a self-cogenerator pseudo-projective module with $S = \mathrm{End}(M)$. If M has DCC on the submodules of the form IM, where I is a right ideal of S then S is semiprimary.*

Lemma 2.16. *Let N be a submodule of a pseudo-projective module M. Then N is a direct summand of M if and only if M/N is isomorphic to a direct summand of M.*

Proof. The necessary condition is obvious. Now, assume that M/N is isomorphic to a direct summand of M. Take $\phi : K \to M/N$ an isomorphism with $M = K \oplus K'$. Let $\pi : M \to K$ be the canonical projection, $\iota : K \to M$ be the inclusion map and $p : M \to M/N$ the natural projection. Since M is pseudo-projective, $pg = \phi\pi$ for some an endomorphism g of M. Then, we have $pg\iota\phi^{-1} = 1_{M/N}$. It means that p splits, and so N is a direct summand of M. □

A module M is called a *D2-module* if A is an arbitrary submodule of M such that M/A is isomorphic to a summand of M, A is a direct summand of M.

Corollary 2.17. *Every pseudo-projective module is a D2-module.*

Corollary 2.18. *Let $M = A \oplus B$ be a pseudo-projective module. Then, every epimorphism $A \to B$ splits.*

Proof. Let $f : A \to B$ be an epimorphism. Then, $A/\operatorname{Ker}(f) \cong B$ is a direct summand of M. From Lemma 2.16, $\operatorname{Ker}(f)$ is a direct summand of M, and so it is a direct summand of A. We deduce that f splits. □

Let N and L be submodules of a right R-module M. N is called a *supplement* of L, if $N + L = M$ and $N \cap L \ll N$. Recall that a submodule U of the R-module M has *ample supplement* in M if, for every $V \leq M$ with $U + V = M$, there is a supplement V_0 of U with $V_0 \leq V$. M is called *supplemented (resp., ample supplemented)* if each of its submodules has a supplement (resp., ample supplement) in M (see [22]).

From Corollary 2.18, we have the following results:

Proposition 2.19. *For a ring R, the following statements are equivalent:*

1. *R is right perfect.*
2. *Every pseudo-projective right R-module is amply supplemented.*
3. *Every pseudo-projective right R-module is supplemented.*

Proposition 2.20. *For a ring R, the following statements are equivalent:*

1. *Every pseudo-projective right R-module is projective.*
2. *The direct sum of any family of pseudo-projective right R-modules is projective.*
3. *The direct sum of any two pseudo-projective right R-modules is projective.*
4. *Every right R-module is pseudo-projective;*
5. *Every finitely generated R-module is pseudo-projective.*
6. *R is semisimple artinian.*

3. On SSP-modules anf SIP-modules

In this section, we study direct sums and intersections of two direct summands of a pseudo-projective module. A right module M is said to have *summand intersection property* (in short, an SIP-module) if the intersection of every pair of direct summands of M is again a direct summand of M. A right R-module M is said to have *summand sum property* (in short, an SSP-module) if the sum of every pair of direct summands of M is again a direct summand of M ([7, 21]).

Lemma 3.1. *Let M be a right R-module and let e and f be idempotents of* $\operatorname{End}(M)$*. Then*

1. $e(M) + f(M)$ *is a direct summand of M if and only if* $(1 - e)f(M)$ *is a direct summand of M.*
2. $e(M) \cap f(M)$ *is a direct summand of M if and only if* $\operatorname{Ker}[(1 - f)e]$ *is a direct summand of M.*

Proof. (1) On can check that $e(M) + f(M) = e(M) \oplus (1 - e)f(M)$. Assume that $e(M) + f(M)$ is a direct summand of M. It follows that $(1 - e)f(M)$ is a direct summand of M. Conversely, let $M = (1 - e)f(M) \oplus K$ with K a submodule of M. Then, we have $(1 - e)(M) = (1 - e)f(M) \oplus [K \cap (1 - e)(M)]$. It follows that $M = e(M) \oplus (1 - e)f(M) \oplus [K \cap (1 - e)(M)] = [e(M) + f(M)] \oplus [K \cap (1 - e)(M)$. Thus, $e(M) + f(M)$ is a direct summand of M.

(2) We can check that $\mathrm{Ker}[(1 - f)e] = [e(M) \cap f(M)] \oplus (1 - e)(M)$. Thus, if $\mathrm{Ker}[(1 - f)e]$ is a direct summand of M, then $e(M) \cap f(M)$ is a direct summand of M. Conversely, let $M = [e(M) \cap f(M)] \oplus H$ with H a submodule of M. It follows that $e(M) = [e(M) \cap f(M)] \oplus [H \cap e(M)]$, and so

$$M = [e(M) \cap f(M)] \oplus [H \cap e(M)] \oplus (1 - e)(M) = \mathrm{Ker}[(1 - f)e] \oplus [H \cap e(M)]$$

We deduce that $\mathrm{Ker}[(1 - f)e]$ is a direct summand of M. □

It is well known that an endomorphism $f \in \mathrm{End}(M)$ is regular if and only if $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are direct summands of M.

From Lemma 3.1, we have the following results in [2].

Corollary 3.2 ([2, Theorem 2.3]). *For a right R-module M, the following conditions are equivalent.*

1. *M is an SSP-module.*
2. *For any two regular homomorphisms $f, g \in \mathrm{End}(M)$, the module $\mathrm{Im}(fg)$ is a direct summand of the module M.*

Corollary 3.3 ([2, Theorem 2.4]). *The following conditions are equivalent for a right R-module M.*

1. *M is an SIP-module.*
2. *For any two regular homomorphisms $f, g \in \mathrm{End}(M)$, the module $\mathrm{Ker}(fg)$ is a direct summand of the module M.*

Next, we give characterizations the product of any two regular elements of endomorphism rings of pseudo-projective modules.

Theorem 3.4. *The following conditions are equivalent for a pseudo-projective right R-module M.*

1. *M is an SSP-module.*
2. *The product of any two regular elements of $\mathrm{End}(M)$ is a regular element.*

Proof. (1) ⇒ (2). Assume that M is an SSP-module. Let $f, g \in \mathrm{End}(M)$ be regular endomorphisms. By Lemma 3.1 or Corollary 3.2, $fg(M)$ is a direct summand of the module M. Moreover, we have $M/\mathrm{Ker}(fg) \cong fg(M)$. It follows that $\mathrm{Ker}(fg)$ is a direct summand of the module M by Lemma 2.16. We deduce that fg is regular.

(2) ⇒ (1) by Corollary 3.2. □

Corollary 3.5. *Every pseudo-projective SSP-module is an SIP-module*

The dual of Theorem 3.4, we have the following result for pseudo-injective modules.

Theorem 3.6. *The following conditions are equivalent for a pseudo-injective right R-module M.*

1. *M is an SIP-module.*
2. *The product of any two regular elements of $\mathrm{End}(M)$ is a regular element.*

Proof. We only prove (1) ⇒ (2). Assume that M is an SIP-module. Let $f, g \in \mathrm{End}(M)$ be regular endomorphisms. Then, $\mathrm{Ker}(fg)$ is a direct summand of M. It follows that $\mathrm{Im}(fg)$ is isomorphic to a direct summand of M, and so $\mathrm{Im}(fg)$ is a direct summand of M. We deduce that fg is regular.

(2) ⇒ (1) by Corollary 3.2. □

Corollary 3.7. *Every pseudo-injective SIP-module is an SSP-module*

From above results, we have the following proposition:

Proposition 3.8. *The following statements are equivalent for a ring R:*

1. *R is semisimple artinian.*
2. *Every pseudo-projective right R-module is an SSP-module.*
3. *Every pseudo-projective right R-module is semisimple.*

4. Some study of modules over formal triangular matrix rings

Let A and B be rings and ${}_BM_A$ be a bimodule. Take $K = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}$ a formal triangular matrix ring. It is well known that ([8]) the category of right K-modules and the category $\mathcal{W}$ of triples $(X;Y)_f$ are equivalent, where X is a right A-module and Y is a right B-module and $f : Y \otimes_B M \to X$ is a right A-homomorphism. If $(X;Y)_f$ and $(U;V)_g$ are two objects in $\mathcal{W}$, then a morphism from $(X;Y)_f$ to $(U;V)_g$ in W are pairs $(\varphi_1;\varphi_2)$ where $\varphi_1 : X \to U$ is a right $A-$homomorphism, $\varphi_2 : Y \to V$ is a right $B-$homomorphism satisfying the condition $\varphi_1 \circ f = g \circ (\varphi_2 \otimes 1_M)$. The right $K-$module corresponding to the triple $(X;Y)_f$ is the additive group $X \oplus Y$ with the right action given by

$$(x,y)\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} = (xa + f(y \otimes m), yb).$$

We write $(X \oplus Y)_K$ is the right $K-$module. On the other hand, if $(\varphi_1;\varphi_2) : (X;Y)_f \to (U;V)_g$ is a map in $\mathcal{W}$, the associated right $K-$homomorphism $\varphi : (X \oplus Y)T \to (U \oplus V)_K$ is given by $\varphi(x;y) = (\varphi_1(x);\varphi_2(y))$ for any $x \in X$ and $y \in Y$. One can check that φ is injective (resp., surjective) if and only if $\varphi_1 : X \to U, \varphi_2 : Y \to V$ are injective (resp., surjective). It is convenient to view such triples as $K-$modules and the morphisms between them as $K-$homomorphisms. Here we should note that the $K-$module K_K corresponds to $(A \oplus M;B)_f$, where f is the right $A-$homomorphism $B \otimes_B M \to A \oplus M$ given by $f(b \otimes m) = (0;bm)$.

Let $(X;Y)_f \in Ob(\mathcal{W})$ and $(X \oplus Y)_K$ be the associated right $K-$module. Under the right $K-$action on $X \oplus Y$ we have

$$(0 \oplus Y)\begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} = (f(Y \otimes M), 0).$$

In general the submodule $f(Y \otimes M)$ of X_A is denoted by YM. Now consider $Y' \leq Y_B$ and let $j_2 : Y' \to Y$ denote the inclusion map. Then

$$(0 \oplus Y')\begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} = (f(j_2 \otimes 1_M)(Y' \otimes M), 0).$$

In general, the submodule $f(j_2 \otimes 1_M)(Y' \otimes M)$ of X_A is denoted by $Y'M$. Let $X' \leq X_A$ satisfy $Y'M \subseteq X'$. Writing f' for $f(j_2 \otimes 1_M)$ and denoting the inclusion $X' \to X$ by j_1 we see that $(X';Y')_{f'} \in Ob(\mathcal{W})$ and $(j_1;j_2) : (X';Y')_{f'} \to (X;Y)_f$ is a map in $\mathcal{W}$ realizing $(X' \oplus Y')_K$ as a $K-$submodule of $(X \oplus Y)_K$. Therefore when we take a submodule $(X' \oplus Y')_K$ of $(X \oplus Y)_K$ we have $X' \leq X_A, Y' \leq Y_B, f(j_2 \otimes 1_M)(Y' \otimes M) \leq X'$. The map $f' : Y' \otimes M \to X'$ is completely determined; it has to be $f(j_2 \otimes 1_M)$. Let X'' (resp. Y'') be a quotient of X_A (resp. Y_B) with $\eta_1 : X \to X''$ (resp. $\eta_2 : Y \to Y''$) the canonical maps. Let $\ker \eta_1 = X'$ and $\ker \eta_2 = Y'$. Assume that $Y'M \subseteq X'$. Let $j_1 : X' \to X, j_2 : Y' \to Y$ be the inclusion maps. Clearly, we have the $A-$homomorphism $f'' : Y'' \otimes M \to X''$ rendering the following diagram commutative

$$\begin{array}{ccccccc}
Y' \otimes M & \xrightarrow{j_2 \otimes 1_M} & Y \otimes M & \xrightarrow{\eta_2 \otimes 1_M} & Y'' \otimes M & \longrightarrow & 0 \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
X' & \xrightarrow{j_1} & X & \xrightarrow{\eta_1} & X'' & \longrightarrow & 0
\end{array}$$

In this diagram $f' = f(j_2 \otimes 1_M)$ and the rows are exact. Also it is clear that $(\eta_1; \eta_2) : (X;Y)_f \to (X'';Y'')_{f''}$ is a map in $\mathcal{W}$ realizing $(X'' \oplus Y'')_K$ as a quotient of $(X \oplus Y)_K$. The kernel of the associated K−homomorphism $\eta : (X \oplus Y)_K \to (X'' \oplus Y'')_K$ is precisely $(X' \oplus Y')_K$. Now when we deal with a quotient $(X'' \oplus Y'')_K$ of $(X \oplus Y)_K$ the A−homomorphism $f'' : Y'' \otimes M \to X''$ is completely determined.

Let $V = (X;Y)_f$ be a right K−module. Take $\widetilde{f} : Y \to \mathrm{Hom}_A(M, X)$ defined by $\widetilde{f}(y)(m) = f(y \otimes m)$ for all $y \in Y$ and $m \in M$. Then, $\widetilde{f}$ is the B-homomorphism.

Theorem 4.1. *Let $V = (X;Y)_f$ be a right K−module. If X is a pseudo-projective right A-module and $\widetilde{f}_{|Y'}$ is an isomorphism for every submodule $(X', Y')_{f'}$ of V_K, then V is a pseudo-projective right K-module.*

Proof. Let $V'' = (X'';Y'')_{f''}$ be a quotient of V_K. Then $X'' = X/X'; Y'' = Y/Y', \eta_1 : X \to X''$ and $\eta_2 : Y \to Y''$ are the natural epimorphisms, $(X';Y')_{f'}$ is a submodule of V with the homomorphism $f' = f(j_2 \otimes 1_M)$ (with $j_2 : Y' \to Y$ the inclusion map) and $f'' : Y'' \otimes M \to X''$ is the A−homomorphism which makes the following diagram commutative:

$$\begin{array}{ccccccc} Y' \otimes M & \xrightarrow{j_2 \otimes 1_M} & Y \otimes M & \xrightarrow{\eta_2 \otimes 1_M} & Y'' \otimes M & \longrightarrow & 0 \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ X' & \xrightarrow{j_1} & X & \xrightarrow{\eta_1} & X'' & \longrightarrow & 0 \end{array}$$

where $j_1 : X' \to X$ is the inclusion map. Then, $\eta = (\eta_1; \eta_2) : V \to V''$ is the corresponding natural K−homomorphism. Let $\sigma : V \to V''$ be an arbitrary K−epimorphism. Then σ corresponds to the pair $(\sigma_1; \sigma_2)$ such that $\sigma_1 : X \to X''$ is an A−epimorphism, $\sigma_2 : Y \to Y''$ is a B−epimorphism and $\sigma_1 f = f''(\sigma_2 \otimes 1_M)$ and $\sigma(x;y) = (\sigma_1(x); \sigma_2(y))$. We have that X is pseudo-projective and obtain that there exists a right A−homomorphism $\bar{\sigma_1} : X \to X$ such that $\eta_1\bar{\sigma_1} = \sigma_1$. Now we want to define a right B−homomorphism $\bar{\sigma_2} : Y \to Y$ such that the pair $(\bar{\sigma_1}; \bar{\sigma_2})$ lifts σ with the corresponding K−homomorphism $\bar{\sigma}$. For any element $y \in Y$, we can define a right B-homomorphism $\theta : M \to X$ with $\theta(m) = \bar{\sigma_1} f(y \otimes m)$ for all $m \in M$. By the hypothesis, $\widetilde{f}$ is an isomorphism, and so there exists a unique $y_1 \in Y$ such that $\widetilde{f}(y_1) = \theta$. Now let $\bar{\sigma_2} : y \to y_1$. One can check that $\bar{\sigma_2}$ is an B−endomorphism of Y. For every $y \in Y$ and $m \in M$, we have $f(\bar{\sigma_2} \otimes 1_M)(y \otimes m) = f(\bar{\sigma_2}(y) \otimes m) = f(y_1 \otimes m) = \widetilde{f}(y_1)(m) = \theta(m) = \bar{\sigma_1} f(y \otimes m)$, where $\bar{\sigma_2}(y) = y_1$ and $\widetilde{f}(y_1) = \theta$. Therefore $f(\bar{\sigma_2} \otimes 1_M) = \bar{\sigma_1} f$. Thus $\bar{\sigma} = (\bar{\sigma_1}; \bar{\sigma_2}) : (X;Y)_f \to (X;Y)_f$ is a right K-homomorphism. Now we should see that $\eta\bar{\sigma} = \sigma$. It is enough to show that $\eta_2\bar{\sigma_2} = \sigma_2$. Take $y \in Y$ an arbitrary element. We have that $\sigma_1 f = f''(\sigma_2 \otimes 1_M)$ for all $m \in M$, $(\sigma_1 f)(y \otimes m) = \sigma_1(f(y \otimes m)) = f''(\sigma_2(y) \otimes m)$ and obtain $\eta_1\bar{\sigma_1}(f(y \otimes m)) = f''(\sigma_2(y) \otimes m)$. Let $\sigma_2(y) = z + Y'$ for some $z \in Y$. On the other hand, $f''(\eta_2 \otimes 1_M) = \eta_1 f$ and so $f''((\eta_2 \otimes 1_M)(z \otimes m)) = \eta_1 f(z \otimes m) = \eta_1 \widetilde{f}(z)(m) = \eta_1\bar{\sigma_1} f(y \otimes m)$ for all $m \in M$. Since $f(\bar{\sigma_2} \otimes 1_M) = \bar{\sigma_1} f$, $\eta_1\bar{\sigma_1} f(y \otimes m) = \eta_1 f(\bar{\sigma_2} \otimes 1_M)(y \otimes m) = \eta_1 f(\bar{\sigma_2}(y) \otimes m) = \eta_1 \widetilde{f}(\bar{\sigma_2}(y))(m)$ for all $m \in M$. Now $\eta_1 \widetilde{f}(z)(m) = \eta_1 \widetilde{f}(\bar{\sigma_2}(y))(m)$ for all $m \in M$. This means that $\widetilde{f}(z - \bar{\sigma_2}(y))$ is a right A−homomorphism from M to X'. Since $\widetilde{f}_{|Y'}$ is an isomorphism, there exists an element $y' \in Y'$ such that $\widetilde{f}_{|Y'}(y') = \widetilde{f}(z - \bar{\sigma_2}(y))$ and so $y' = z - \bar{\sigma_2}(y)$. Thus, $\sigma_2(y) = \eta_2\bar{\sigma_2}(y)$ or $\sigma_2 = \eta_2\bar{\sigma_2}$. □

Corollary 4.2. *Let $V = (X;Y)_f$ be a right K−module. If X is a pseudo-projective right A-module and $\widetilde{f}$ is an isomorphism, then V is a pseudo-projective right K-module.*

Example 4.3. *Let A be a ring and M be a right A−module such that ${}_{\mathbb{Z}}M$ is torsion-free which is not pseudo-projective. Let $K = \begin{bmatrix} A & 0 \\ M & \mathbb{Z} \end{bmatrix}$ and consider the right K−module $V_K = (M;Z)_f$ where $f : Z \otimes M \to M$ defined by $n \otimes m \mapsto nm$ for all $n \in \mathbb{Z}$ and $m \in M$. Clearly, f is an R−isomorphism. Therefore, V_K is pseudo-projective by [5, 4.1.1]. On the other hand, M is not pseudo-projective.*

Theorem 4.4. *Let $V = (X;Y)_f$ be a right K−module. If V is a pseudo-projective right K-module, then Y is a pseudo-projective right B-module and $X/f(Y \otimes M)$ is a pseudo-projective right A-module.*

Proof. Let $\eta : Y \to Y/K$ be the natural epimorphism and $\alpha : Y \to Y/K$ be any B−epimorphism, where $K \leq Y$. Then we can construct the quotient $(0, Y/K)_0$ of $(X, Y)_f$ with the following commutative diagram:

$$\begin{array}{ccccccc}
K \otimes M & \xrightarrow{j\otimes 1_M} & Y \otimes M & \xrightarrow{\eta\otimes 1_M} & Y/K \otimes M & \longrightarrow & 0 \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 0} & & \\
X & \xrightarrow{1_X} & X & \xrightarrow{0} & 0 & \longrightarrow & 0
\end{array}$$

with $j : K \to Y$ the inclusion map and $f' = f(j \otimes 1_M)$.

Now we have the natural K-epimorphism

$$\bar{\eta} = (0; \eta) : (X; Y)_f \to (0; Y/K)_0$$

and a right K-epimorphism

$$\bar{\alpha} = (0; \alpha) : (X; Y)_f \to (0; Y/K)_0$$

Since V is pseudo-projective, there is a right K−homomorphism $\beta : V \to V$ such that $\bar{\eta}\beta = \bar{\alpha}$. Take $\beta = (\beta_1, \beta_2)$ with $\beta_2 : Y \to Y$ a right B−homomorphism and $\beta_1 : X_2 \to X_1$ a right A−homomorphism such that $\beta_1 f = f(\beta_2 \otimes 1_M)$ and $\beta(x; y) = (\beta_1(x); \beta_2(y))$ for all $x \in X$ and $y \in Y$. Thus $\eta\beta_2 = \alpha$. We deduce that Y is pseudo-projective.

Let $X'/f(Y\otimes M)$ be a submodule of $X/f(Y\otimes M)$. Now consider the natural epimorphism $\nu : X/f(Y\otimes M) \to \frac{X/f(Y\otimes M)}{X'/f(Y\otimes M)}$ and a right A−epimorphism $\mu : X/f(Y\otimes M) \to \frac{X/f(Y\otimes M)}{X'/f(Y\otimes M)}$. Let $\gamma : [X/f(Y\otimes M)]/[X'/f(Y\otimes M)] \to X/X'$ be the isomorphism and $\pi : X \to X/f(Y \otimes M)$ be the natural epimorphism. One can check that $(X'; Y)_{f'}$ is a submodule of V with $f' = f$ and $(X/X', 0)_0$ is a factor module of V.

Now $(\gamma\mu\pi, 0) : (X; Y)_f \to (X/X'; 0)_0$ is a right K−epimorphism and $(\gamma\nu\pi; 0) : (X; Y)_f \to (X/X'; 0)_0$ is the natural epimorphism. We have that V is pseudo-projective and obtain that a right K−homomorphism with the pair $(\mu_1; \mu_2) : (X; Y)_f \to (X; Y)_f$ such that $(\gamma\mu\pi, 0) = (\gamma\nu\pi, 0)(\mu_1, \mu_2)$

Note that we have the compositions $\mu_1 f = f(\mu_2\otimes 1_M)$ and $\nu\pi\mu_1 = \mu\pi$. Let us define the A−homomorphism $\bar{\mu} : X/f(Y \otimes M) \to X/f(Y \otimes M)$ by $x + f(Y \otimes M) \mapsto \mu_1(x) + f(Y \otimes M)$. Since $\mu_1 f = f(\mu_2 \otimes 1_M)$, $\bar{\mu}$ is well-defined and since $\nu\pi\mu_1 = \mu\pi$, $\nu\bar{\mu} = \mu$. Therefore we have $\nu\bar{\mu} = \mu$.

We deduce that $X/f(Y \otimes M)$ is pseudo-projective. □

We say that a module P is a *pseudo-projective cover* of any module U if, there exists an epimorphism $\varphi : P \to U$ such that P is pseudo-projective and $\mathrm{Ker}(\varphi)$ is small in P.

Corollary 4.5. *If $(X, Y)_f$ has a pseudo-projective cover as a right K-module, then $(X/f(Y \otimes M)_A$ and Y_B have pseudo-projective covers.*

Proof. Let $\varphi : (U, V)_g \to (X, Y)_f$ be a pseudo-projective cover of $(X, Y)_f$. Then there exist homomorphisms $\varphi_1 : U_A \to X_A, \varphi_2 : V_B \to Y_B$ such that $\varphi = (\varphi_1, \varphi_2) : (U; V)_g \to (X; Y)_f$ is a right K-epimorphism with $\varphi_1 g = f(\varphi_2 \otimes 1_M)$ and $(\varphi_1(u); \varphi_2(v)) = \varphi(u; v)$. By [4, Theorem 2.4], the epimorphism $\varphi_2 : V_B \to Y_B$ has small kernel and we have the epimorphism $\bar{\varphi}_1 : U/g(V \otimes M) \to X/f(Y \otimes M)$ with small kernel. Thus $(X/f(Y \otimes M)_A$ and Y_B have pseudo-projective covers with the epimorphisms $\bar{\varphi}_1$ and φ_2 respectively by Theorem 4.4. □

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