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Some homological properties on generalized amalgamated Banach algebras

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Abstract. In the present paper, first we characterize the multiplier algebra, (maximal) ideals, minimal idempotents and spectrum of the generalized amalgamated Banach algebra $A \boxtimes_\Theta X$ in terms of A and X and the bilinear mapping Θ from $X \times X$ into A. Then, we show that there are a strong relationship between some of homological properties of $A \boxtimes_\Theta X$, such as Connes-amenability, flatness and projectivity, ϕ-biprojectivity, and the Banach algebras A and X and the mapping Θ. The results of this paper extend several results in the literature.

1. Introductions and Preliminaries

Let A and X be Banach algebras and $\Theta : X \times X \to A$ be a bounded bilinear mapping. If also X is an algebraic Banach A-module with respect to Θ, which is a Banach A-module with compatible operations, that is for each $a, a' \in A$ and $x, x', x'' \in X$

$$a\Theta(x, x') = \Theta(ax, x'), \Theta(x, x')a = \Theta(x, x'a), \Theta(xa, x') = \Theta(x, ax'), \Theta(xx', x'') = \Theta(x, x'x''),$$

in A and

$$(xx')a = x(x'a), a(xx') = (ax)x', (xa)x' = x(ax'), \Theta(x, x')x'' = x\Theta(x', x''),$$

in X, then a direct verification shows that the ℓ^1-direct product $A \times X$ as a linear space with the product

$$(a, x)(a', x') = (aa' + \Theta(x, x'), ax' + xa' + xx') \qquad (a, a' \in A, x, x' \in X),$$

is a Banach algebra. We call this Banach algebra the generalized amalgamated Banach algebra with respect to Θ and we denote it by $A \boxtimes_\Theta X$ in this paper.
If X be an algebraic Banach A-module (that is an algebraic A-module with respect to the zero bilinear mapping). Then the generalized module extension Banach algebra $A \bowtie X$ is a generalized amalgamated Banach algebra with respect to $\Theta = 0$, See [24]. The module extension Banach algebras, unitization of Banach algebras, Lau product of Banach algebras and direct product of Banach algebras are the main examples of generalized amalgamated Banach algebras; see for more details about this Banach algebras [4],

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[5], [17] and [32].

Many properties of these Banach algebras such as n-weak amenability, Connes amenability, topological centers, Biflatness, and Biprojectivity and other properties have been studied by many authors, see [1], [24], [18], [6], [7], [8], [12], [13], [14], [20], [22], [23], [26], [30], [31], [32], [9] and references therein. Another important class of these Banach algebras is the generalized matrix Banach algebra $G = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$, see [16], where A and B are Banach algebras, M is a (A, B)-module and N is a (B, A)-module can be identify with the generalized amalgamated Banach algebra $(A \otimes B) \boxtimes_\Theta (M \otimes N)$, see [15] for details. Also see [16] for some homological properties of generalized matrix Banach algebras.

Consider also $G = A \boxtimes_{-\pi} A^0$ where A^0 is A with zero product and the actions of A on A^0 are the product π of the Banach algebra A and $\Theta = -\pi$ (Where $A = \mathbb{R}$ this product is the usual product on $\mathbb{C}$). Then G is a generalized amalgamated Banach algebra.

For a generalized amalgamated Banach algebra $A \boxtimes_\Theta X$ one can directly checked that the dual $(A \boxtimes_\Theta X)^*$ as a Banach $(A \boxtimes_\Theta X)$-module enjoys the following module operations:

$$(f, g)(a, x) = (fa + gx, gx + ga + f.x),$$
$$(a, x)(f, g) = (af + xg, xg + ag + x.f),$$

for all $a \in A, f \in A^*$ and $x \in X, g \in X^*$, where "." is denoted for the corresponding bilinear mapping induced by Θ. In the sequel for simply of our notations we omit ".".

The generalized amalgamated Banach algebra $G = A \boxtimes_\Theta X$ was introduced by authors in [15] and many important properties such as n-weak amenability, topological centers, bounded approximate identity, and the ideal structure have been studied.

The present paper divides into five sections. In sections 2 and 3, we characterize the multiplier algebra, (maximal) ideals, minimal idempotents and spectrum of $A \boxtimes_\Theta X$ in terms of A and X and the mapping Θ. Then, in sections 4 and 5, we show that there are a strong relationship between some of homological properties of generalized amalgamated Banach algebra $A \boxtimes_\Theta X$, such as Connes-amenability, flatness, projectivity and ϕ-biprojectivity, and the corresponding properties of Banach algebras A and X and the mapping Θ. These results extend some previous results in this field.

2. Some Primary results on $A \boxtimes_\Theta X$

In this section we obtain some primary results on the generalized amalgamated Banach algebra $G = A \boxtimes_\Theta X$.

Proposition 2.1. *Let $G = A \boxtimes_\Theta X$ be a generalized amalgamated Banach algebra. Then the following statements hold.*

(i) G is commutative if and only if both A and X are commutative, X is a symmetric A-bimodule and $\Theta = \Theta^t$.

(ii) Suppose that A, A', X and X' are Banach algebras such that X and X' are algebric Banach A and A'-modles with respect to Θ and Θ', respectively and there exist isomomorphisms $\varphi : A \to A'$ and $\psi : X \to X'$ such that $\psi(ax) = \varphi(a)\psi(x)$ and $\psi(xa) = \psi(x)\varphi(a)$ and $\Theta'(\psi(x), \psi(y)) = \varphi \circ \Theta(x, y)$. Then the generalized amalgamated Banach algebras $A \boxtimes_\Theta X$ and $A' \boxtimes_{\Theta'} X'$ are isomorphic.

(iii) If $A \boxtimes_\Theta X$ has an identity (a_0, x_0), then A has the identity a_0, $x_0 A = A x_0 = 0$, $a_0 x + x_0 x = x a_0 + x x_0 = x$ and $\Theta(x_0, x) = \Theta(x, x_0) = 0$, for each $x \in X$.

(iv) If A is unital and X is a unital Banach A-module, then $A \boxtimes_\Theta X$ is unital.

Similar results of the parts (iii) and (iv), can be given for the (left or right) approximate identities.

Proof. (i) It is obvious.

(ii) It is sufficient to verify that the map F from $A \boxtimes_\Theta X$ into $A' \boxtimes_{\Theta'} X'$ defined by $F(a, x) = (\varphi(a), \psi(x))$ is an isomorphism.

(iii) For each $a \in A$ and $x \in X$, we have $(a, 0) = (a_0, x_0)(a, 0) = (a_0 a, x_0 a)$ and $(0, x) = (a_0, x_0)(0, x) = (\Theta(x_0, x), a_0 x + x_0 x)$. Again repeat this process for $(a, 0)(a_0, x_0)$ and $(0, x)(a_0, x_0)$.

(iv) If A has an identity e then $(e, 0)$ is the identity of $A \boxtimes_\Theta X$. □

In the following proposition we will characterize the minimal idempotent of generalized amalgamated Banach algebras.

Proposition 2.2. *Suppose that $G = A \boxtimes_\Theta X$ is a generalized amalgamated Banach algebra and $x_0 \in X$ is arbitrary and fixed. Let $\Theta(X, x_0) = \Theta(x_0, X) = 0$ and $x_0 A = A x_0 \in \langle x_0 \rangle$. Then G has a minimal idempotent (a_0, x_0) if and only if one of the following items hold.*

(i) $a_0 = 0$ and x_0 is a minimal idempotent of X.

(ii) a_0 is a minimal idempotent of A and for each $x \in X$,

$$(a_0 x + x_0 x)a_0 = -(a_0 x + x_0 x)x_0 \quad \text{and} \quad x_0^2 + a_0 x_0 + x_0 a_0 = x_0.$$

Proof. (a_0, x_0) is a minimal idempotent if and only if for each $(a, x) \in G$, we have $(a_0, x_0)^2 = (a_0, x_0)$ and $(a_0, x_0)(a, x)(a_0, x_0) = \lambda_{a,x}(a_0, x_0)$, for some $\lambda_{a,x}$. This is equivalent to

$$a_0^2 + \Theta(x_0, x_0) = a_0, \tag{2.1}$$

$$x_0^2 + a_0 x_0 + x_0 a_0 = x_0 \tag{2.2}$$

and

$$a_0 a a_0 + \Theta(x_0, x)a_0 + \Theta(a_0 x + x_0 a + x_0 x, x_0) = \lambda_{a,x} a_0, \tag{2.3}$$

$$a_0 x a_0 + x_0 a a_0 + x_0 x a_0 + a_0 a x_0 + \Theta(x_0, x)x_0 + a_0 x x_0 + x_0 a x_0 + x_0 x x_0 = \lambda_{a,x} x_0 \tag{2.4}$$

for each $a \in A$ and $x \in X$.

Obviously if one of (i) is true, then (a_0, x_0) is a minimal idempotent. Also if (ii) is true, then since $x_0 A, A x_0 \in \langle x_0 \rangle$, (a_0, x_0) is a minimal idempotent.

For the converse, if (a_0, x_0) is a minimal idempotent, then 2.1 and 2.3 imply that $a_0^2 = a_0$ and $a_0 a a_0 = \lambda_{a,x} a_0$, for each $x \in X$; and for $x = 0$, we have $a_0 a a_0 = \lambda_{a,0} a_0$. Therefore we have two cases:

• $a_0 = 0$, and from 2.2 and 2.4 with $a = 0$ we obtain x_0 is a minimal idempotent, which is (i).

•• a_0 is a minimal idempotent. Putting $x = 0$ in 2.3, we conclude that $\lambda_{a,x} = \lambda_{a,0}$, for each $a \in A$ and $x \in X$. Thus by taking $a = 0$ in 2.4, we have

$$a_0 x a_0 + x_0 x a_0 + a_0 x x_0 + x_0 x x_0 = \lambda_{0,0} x_0, \tag{2.5}$$

and by putting $x = 0$ in (2.5), we have $\lambda_{0,0} = 0$. Therefore (ii) is valid by 2.2 and (2.5). □

3. Characters and Spectrum of $A \boxtimes_\Theta X$

In this section, we will obtain a characterization of the left multipliers of $A \boxtimes_\Theta X$ and then we conclude its spectrum. At the end of this section, we will compute its spectrum by computing of both spectrums A and X.

The following result characterized the left multipliers of $A \boxtimes_\Theta X$ which is noted by $LM(A \boxtimes_\Theta X)$.

Proposition 3.1. *The operator T is in $LM(A \boxtimes_\Theta X)$ if and only if there exists some $U_A \in Hom_A(A, A)$, $U_X \in Hom_X(X, A)$, $V_X \in LM(X)$ and $V_A \in Hom_X(A, X)$ such that for each $a, b \in A$ and $x \in X$, we have* *(i)* $T((a,x)) = \Big(U_A(a) + U_X(x), V_A(a) + V_X(x)\Big)$.

(ii) $U_A\Big(aa'\Big) = aU_A(a')$, *and* $V_A(aa') = aV_A(a')$.

(iii) $U_X(xx') + U_A(\Theta(x,x')) = \Theta(x, V_X(x'))$ *and* $V_X(xx') + V_A(\Theta(x,x')) = xU_X(x') + xV_X(x')$.

(iv) $U_A(xa') = xU_X(a') + \Theta(x, V_A(a'))$ *and* $V_A(xa') = xU_A(a') + xV_A(a')$.

(v) $U_X(ax') = aU_X(x')$ *and* $V_X(ax') = aV_X(x')$.

Proof. Assume that $T \in LM(A \boxtimes_\Theta X)$. Then, there exists bounded linear maps $U : A \boxtimes_\Theta X \to A$ and $V : A \boxtimes_\Theta X \to A$ such that $T = (U, V)$. Taking $U_A(a) = U((a,0))$, $V_A(a) = V((a,0))$ for each $a \in A$, and $U_X(x) = U((0,x))$, $V_X(x) = V((0,x))$, for each $x \in A$. Then clearly these mappings are linear and satisfy in (*i*). For another parts, we have

$$\begin{aligned} T((a,x)(a',x')) &= T((aa' + \Theta(x,x'), ax' + xa' + xx')) \\ &= \Big(U_A(aa' + \Theta(x,x')) + U_X(ax' + xa' + xx') \\ &\quad , V_A(aa' + \Theta(x,x')) + V_X(ax' + xa' + xx')\Big) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} (a,x)T((a',x')) &= (a,x)\Big(U_A(a') + U_X(x'), V_A(a') + V_X(x')\Big) \\ &= \Big(aU_A(a') + aU_X(x') + \Theta(x, V_A(a') + V_X(x')) \\ &, aV_A(a') + aV_X(x') + xU_A(a') + xU_X(x') + xV_A(a') + xV_X(x')\Big), \end{aligned} \tag{3.2}$$

for each $a, a' \in A$ and $x, x' \in X$. From (3.1) and (3.2), we get

$$U_A(aa' + \Theta(x,x')) + U_X(ax' + xa' + xx') = aU_A(a') + aU_X(x') + \Theta(x, V_A(a') + V_X(x')) \tag{3.3}$$

and

$$\begin{aligned} V_A(aa' + \Theta(x,x')) + V_X(ax' + xa' + xx') \quad &= \quad aV_A(a') + aV_X(x') + xU_A(a') \\ +xU_X(x') + xV_A(a') + xV_X(x'), \end{aligned} \tag{3.4}$$

for each $a, a' \in A$ and $x, x' \in X$. Putting $x = x' = 0$ in (3.3) and (3.4), we have the statement (*ii*), again taking $a = a' = 0$ in (3.4) and (3.4), we get (*iii*). For the parts of (*iv*) and (*v*), put $a = 0$, $x' = 0$ and $a' = 0$, $x = 0$, in (3.3) and (3.4) respectively.

The converse can be prove easily. □

A similar argument as the proof of the last proposition implies the following proposition, which we omit its proof.

Proposition 3.2. $(\alpha, \beta) \in \sigma(A \boxtimes_\Theta X)$ *if and only if for each* $a, b \in A$ *and* $x, y \in X$*, the following conditions hold.*

(i) $\alpha \circ \Theta(x,y) + \beta(xy) = \beta(x)\beta(y)$.

(ii) $\alpha(ab) = \alpha(a)\alpha(b)$.

(iii) $\alpha(a)\beta(x) = \beta(ax)$.

(iv) $\beta(x)\alpha(a) = \beta(xa)$.

In the next theorem, we characterize the character space of $A \boxtimes_\Theta M$, where M is a closed ideal of X. Note that $A \boxtimes_\Theta M$ is a Banach algebra, if $AM \cup MA \subseteq M$.

Theorem 3.3. *Suppose that* $\sigma(A) \neq \emptyset$ *and* $\overline{span(AM \cup MA)} = M$*, where* M *is a closed ideal of* X*. If for each* $\alpha \in \sigma(A)$*,* $\alpha \circ \Theta|_{M\times M} = 0$*, then* $\sigma(A \boxtimes_\Theta M) = U \cup V$*, where*

$$U = \{(m.\beta,\ \beta) : \ \beta \in \sigma(M),\ m \in M,\ \beta(m) = 1,\ m.\beta \circ \Theta|_{M\times M} = 0\},$$

$$V = \{(\alpha, 0) : \quad \alpha \in \sigma(A),\ \alpha \circ \Theta = 0\}.$$

Proof. Obviously $V \subseteq \sigma(A \boxtimes_\Theta M)$ and sincce for each $m \in M$ and $\beta \in \sigma(M)$ with $\beta(m) = 1$ we have $\beta(mm') = \beta(m'm)$, therefore $U \subseteq \sigma(A \boxtimes_\Theta M)$ and thus $U \cup V \subseteq \sigma(A \boxtimes_\Theta M)$.
Now let $(\alpha, \beta) \in \sigma(A \boxtimes_\Theta M)$ and $(a, m), (a', m') \in A \boxtimes_\Theta M$. Then

$$\begin{aligned} \langle(\alpha,\beta),(a,m).(a',m')\rangle &= \langle(\alpha,\beta),(aa' + \Theta(m,m'), am' + ma' + mm')\rangle \\ &= \alpha(aa') + \alpha(\Theta(m,m')) + \beta(am') + \beta(ma') + \beta(mm'). \end{aligned} \tag{3.5}$$

On the other hand,

$$\begin{aligned} \langle(\alpha,\beta),(a,m).(a',m')\rangle &= (\alpha,\beta)(a,m) \times (\alpha,\beta)(a',m') \\ &= (\alpha(a) + \beta(m))(\alpha(a') + \beta(m')) \\ &= \alpha(a)\alpha(a') + \alpha(a)\beta(m') + \beta(m)\alpha(a') + \beta(m)\beta(m'). \end{aligned} \tag{3.6}$$

From equality of (3.5) and (3.6), for $a = a' = 0$, we have $\alpha(\Theta(m, m')) + \beta(mm') = \beta(m)\beta(m')$. Since for each $\alpha \in \sigma(A)$, $\alpha \circ \Theta|_{M\times M} = 0$, we get $\beta(mm') = \beta(m)\beta(m')$. So $\beta \in \sigma(A) \cup \{0\}$.
Again, by equality of (3.5) and (3.6), for $m = m' = 0$, we have $\alpha(a, a') = \alpha(a)\alpha(a')$. So $\alpha \in \sigma(M) \cup \{0\}$. Using (3.5) and (3.6), $\alpha = 0$ implies that $\beta(am') + \beta(ma') = 0$ for each $a, a' \in A$, $m, m' \in M$, and so $\beta(am' + ma') = 0$. Which implies $\beta = 0$ on $\overline{span(AM \cup MA)} = M$. This is a contradiction by $(\alpha, \beta) \in \sigma(A \boxtimes_\Theta M)$. Thus we have two cases:

- If $\beta = 0$ then form equality of (3.5) and (3.6), we have $\alpha \circ \Theta = 0$ on $M \times M$ and so $(\alpha, \beta) \in V$, thus $\alpha \in \sigma(A)$.

•• If $\beta \neq 0$ then by equality of (3.5) and (3.6), we have

$$\alpha(\Theta(m, m')) + \beta(am') + \beta(ma') + \beta(mm') = \alpha(a)\beta(m') + \beta(m)\alpha(a') + \beta(m)\beta(m'),$$

and by $a' = 0$, $m = 0$, we have

$$\beta(am') = \alpha(a)\beta(m'),$$

for each $a \in A$, $m' \in M$. Choose $m' \in M$ such that $\beta(m') = 1$, then for each $a \in A$, we have $\alpha(a) = \beta(am') = (m'.\beta)(a)$. Therefore $(\alpha, \beta) \in U$.
□

Taking $\Theta = 0$ in Theorem 3.3, we obtain the following result.

Corollary 3.4. *Suppose that* $\sigma(A) \neq \emptyset$ *and* $\overline{span(AM \cup MA)} = M$*, where* M *is a closed ideal of* X*. Then* $\sigma(A \bowtie M) = U \cup V$*, where*

$$U = \{(m.\beta,\ \beta) : \quad \beta \in \sigma(M),\ m \in M,\ \beta(m) = 1\}, \qquad V = \{(\alpha, 0) : \quad \alpha \in \sigma(A)\}.$$

For each θ-Lau product of Banach algebra $A \times_\theta B$, we have the following characterization.

Corollary 3.5. [31, Proposition 2.4] *Suppose that* $\sigma(A) \neq \emptyset$ *and* $\theta \in \sigma(A)$*. Then* $\sigma(A \times_\theta X) = U \cup V$*, where*

$$U = \{(\theta, \beta) : \quad \beta \in \sigma(M)\}, \qquad V = \{(\alpha, 0) : \quad \alpha \in \sigma(A)\}.$$

Also we have the following characterization.

Corollary 3.6. [22, Theorem 5.1] *Let* $\theta : A \to X$ *be a homomorphism,* $\sigma(A) \neq \emptyset$ *and* $\overline{\theta(A)M \cup M\theta(A)} = M$*, where* M *is a closed ideal of* X*. Then* $\sigma(A \times_\theta B) = U \cup V$*, where*

$$U = \{((m.\beta) \circ \theta,\ \beta) : \quad \beta \in \sigma(M),\ m \in M,\ \beta(m) = 1\}, \qquad V = \{(\alpha, 0) : \quad \alpha \in \sigma(A)\}.$$

4. Connes-amenability of $A \boxtimes_{\Theta} X$

The concept of amenability for W^*-algebras was defined by Johnson et al. in [19]. Then Connes in [2] and [3], introduced another notion of amenability which is called Connes-amenability by Helemskii [10]. Next Runde in [27] extended this notion of Connes-amenability from W^*-algebras to dual Banach algebras. Any Connes-amenable dual Banach algebra A, is unital. In Theorem 4.4.8 of [27] it is proved that for any Arens regular Banach algebra A which is an ideal in A^{**}, A is amenable if and only if A^{**} is Connes-amenable.
In this section we will characterize Connes-amenability of $A \boxtimes_{\Theta} X$ in terms of Connes-amenability of A and X and the mapping Θ.

Definition 4.1. *Let $f : X \times Y \to X$ (or Y) be a bilinear map and $V \subseteq X$ (or $W \subseteq Y$) as a subspace. We say that f is stable on V (or W) if $f(V, Y) \subseteq V$ (or $f(X, W) \subseteq W$).*

Definition 4.2. *[27]*

(i) A Banach algebra A is called a dual Banach algebra if there exists a closed submodule A_* of A^* such that $A \cong (A_*)^*$.

(ii) suppose that A is a dual Banach algebra and E is a dual Banach A-bimodule. An element $x \in E$ is normal if the following maps from A into E are w^*-w^*-continuous:

$$a \mapsto a.x \quad \text{and} \quad a \mapsto x.a.$$

E is normal, if any element of E is normal.

(iii) A dual Banach algebra A is Connes-amenable if, for every normal, dual Banach A-bimodule E, every w^*-w^*-continuous derivation $D \in Z^1(A, E)$ is inner.

Lemma 4.3. *Let A and X be dual Banach algebrs and $G = A \boxtimes_{\Theta} X$. If π_ℓ^*, π_r^{t*}, Θ^* and Θ^{t*} are stable on X_* and π_ℓ^{t*} and π_r^* are stable on A_*, then G is a dual Banach algebra.*

Proof. It is easy to see that $(A_* \times X_*)$ is a closed submodule of $A^* \times X^* \cong (A \boxtimes_{\Theta} X)^* = G^*$ such that $(A \times X_*)^* \cong G$. □

Lemma 4.4. *Let G be a dual Banach algebra. Then A is a dual Banach algebra. If π_ℓ and π_r are zero, then X is a dual Banach algebra.*

Proof. Let V be a closed submodule of G^* such that $V^* = G$. Put $V_A = \{a^* \in A^* : \quad (a^*, 0) \in V\}$. Suppose that $\{a^*_\alpha\}$ is a net in V_A, such that $a^*_\alpha \to a^*$. Then $(a^*_\alpha, 0) \to (a^*, 0)$ and since V is closed, $(a^*, 0) \in V$, i.e. $a^* \in V_A$ and so V_A is closed. Also

$$(a^*a, 0) = (a^*, 0).(a, 0) \subseteq VG \subseteq V,$$

for each $a^* \in V_A$. Thus $a^*a \in V_A$, for each $a \in A$, and similarly $aa^* \in V_A$, for each $a \in A$.
On the other hand $A^{**} = V_A^* \oplus V_A^{\perp}$ and $G^{**} = V^* \oplus V^{\perp}$, where

$$\begin{aligned} V_A^{\perp} &= \{a^{**} \in A^{**} : \quad a^{**}(a^*) = 0, \quad \forall a^* \in V_A\} \\ &= \{a^{**} \in A^{**} : \quad (a^{**}, 0)(a^*, b^*) = 0, \quad \forall (a^*, b^*) \in V\} \\ &= \{a^{**} \in A^{**} : \quad (a^{**}, 0) \in V^{\perp}\}. \end{aligned}$$

Therefore, $V_A^* = \{a^{**} \in A^{**} : \quad (a^{**}, 0) \in V^* = G\} = A$.
Similarly $V_X^* = X$ and V_X is closed. Also if π_ℓ and π_r are zero, then V_X is a submodule of X^*. □

Remark 4.5. *Note that in the proof of Lemma 2.2 in [26], V_{*A} is not submodule unless $A = 0$ or $\theta = 0$; indeed if $V_{*_A} = 0$, then $A = 0$ and it is A-submodule. If $V_{*_A} \neq 0$ and it is A-submodule, then for each $a \in A$ and $b \in B$ and nonzero $v \in V_{*_A}$ we have*

$$\begin{aligned} 0 = & \langle (0, b), av \rangle \\ = & \langle (0, b), (a, 0)v \rangle \\ = & \langle (0, b)(a, 0), v \rangle \\ = & \langle (\theta(b)a, 0), v \rangle \\ = & \langle \theta(b)a, f \rangle \\ = & \theta(b)\langle a, f \rangle. \end{aligned}$$

Where similar to the proof of Lemma 2.2 in [26], we consider $v = (f, 0) \in A^* \times 0$. This implies that $\theta = 0$ or $A = 0$.

The following results generalize Theorem 2.4 in [26].

Theorem 4.6. *Let $G = A \boxtimes_\Theta X$ be a dual Banach algebra and Connes-amenable. Then we have the following statements.*

(i) A is Connes-amenable if for each w^*-w^*-continuous derivation d from A to a normal dual Banach A-module, $d \circ \Theta = 0$.

(ii) Let π_ℓ and π_r be the left and right module actions of A on X, respectively. Then X is Connes-amenable if for each normal dual Banach X-module E, we can consider E as an A-module with the left and right module actions π_ℓ^A and π_r^A such that for each w^*-w^*-continuous derivation d from X to E, we have $d \circ \pi_\ell = \pi_\ell^A \circ (i_A \times d)$ and $d \circ \pi_r = \pi_r^A \circ (d \times i_A)$.

Proof. From Lemma 4.4, A is a dual Banach algebra. Let E be a normal dual Banach A-module and let $d : A \to E$ be a w^*-w^*-continuous derivation. Consider E as a G-bimodule by the actions $e(a, x) = ea$, $(a, x)e = ae$. Define $D : G \to E$ by $D(a, x) = d(a)$. Then D is a w^*-w^*-continuous derivation, and so $d(a) = D((a, x)) = e(a, x) - (a, x)e = ea - ae$, for some $e \in E$. Hence A is Connes-amenable. Similarly, let $d : X \to E$ be a w^*-w^*-continuous derivation. By defining the actions $e(a, x) = \pi_r^A(e, a) + ex$ and $(a, x)e = \pi_\ell^A(a, e) + xe$, E is a G-bimodule. Now define $D : G \to E$ by $D(a, x) = d(x)$, which is a w^*-w^*-continuous derivation. Then similar above we can conclude that X is Connes-amenable. □

Corollary 4.7. *If, in Theorem 4.6, we put $\Theta = 0$, then $A \boxtimes_\Theta X = A \bowtie X$ and Connes-amenability of $A \bowtie X$ implies Connes-amenability of A. Moreover if π_ℓ and π_r are zero or $\pi_\ell(a, x) = \theta(a)x = \pi_r(x, a)$, for some $\theta \in \sigma(A)$, then Connes-amenability of $A \boxtimes_\Theta X$ implies Connes-amenability of X.*

5. ψ-biprojectivity and (α, β)-biprojectivity of $A \boxtimes_\Theta X$

The concepts of biflatness and biprojectivity of Banach algebras were defined by A. Ya. Helemskii in [10]; see also [27] and [5] for more details. Using this concept, he showed that every Banach algebra A is amenable if and only if it is biflat and has a bounded approximate identity. The sufficient and necessary conditions for Biflatness and biprojectivity of many classes of Banach algebras such as C^*-algebras, the group algebra $L^1(G)$ of a locally compact group G and the second dual of Banach algebras have been obtained in [27], [28], and [21]. For the other approaches, see [25], [8], and references therein. Medghalchi and Sattari in [20] proved that any triangular Banach algebra $T = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$, is biflat (resp. biprojective) if and only if the corner Banach algebras A and B are biflat (resp. biprojective) and $M = 0$, where M is an essential (A, B)-module, that is, $\overline{AM} = M = \overline{MB}$. Afterward, Khodami and Vishki in [11] showed that each θ-Lau product of Banach algebra $A \times_\theta B$ is biflat (resp. biprojective) if and only if the Banach algebras A and B are biflat (resp. biprojective), where A is unital Banach algebra and $\theta \in \sigma(B)$. See also [1] and [9] for a generalization of this work.

In the following, we consider A, X and Θ as before and we characterize ϕ-biprojectivity of $A \boxtimes_\Theta X$.

Remark 5.1. *Note that the product of a Banach algebra A dose not effect on injectivity, projectivity and flatness of A as a C-module, so Theorems 3.4–3.7 in [26] are true for $G = A \boxtimes_{\Theta} X$. Indeed $G \cong A \otimes_{\theta} X$ for $\theta \in \sigma(A)$ as a Banach space. That is G is injective (respectively, projective, and flat) if and only if A and X have the corresponding properties with the module actions defined before the same Theorems in [26]. For the definition of these concepts see [10], [29] and [26].*

The generalized amalgamated Banach algebra $G = A \boxtimes_{\Theta} X$ is a Banach A-bimodule under the following module actions.

$c.(a, x) =: (c, 0).(a, x)$ and $(a, x).c =: (a, x).(c, 0)$, where $a, c \in A$ and $x \in X$. we can be made G into a Banach X-bimodule in a similar way.

We define the usual projections $P_A : G \to A$ by $P_A(a, x) = a$ and $P_X : G \to X$ by $P_X(a, x) = x$, $a \in A$, $x \in X$. Also, the usual injections $J_A : A \to G$ by $J_A(a) = (a, 0)$ and $J_X : X \to G$ by $J_X(x) = (0, x)$, $a \in A$, $x \in X$. The mappings P_A and J_A are A-bimodule. J_X is a X-bimodule if and only if $\Theta = 0$, and P_X is not X-bimodule in general. The unique induced mapping $P_X \otimes P_X$ from $G\hat{\otimes}G$ into $X\hat{\otimes}X$ is defined by $(P_X \otimes P_X)\big((a, x) \otimes (a', x')\big) = x \otimes x'$, and the unique induced mapping $J_X \otimes J_X$ from $X\hat{\otimes}X$ into $G\hat{\otimes}G$ is defined by $(J_X \otimes J_X)(x \otimes x') = (0, x) \otimes (0, x')$.

Definition 5.2. (i) *[27] A Banach algebra A is said to be biprojective if for $\Delta_A : A\hat{\otimes}A \to A$ there exists a bounded A-bimodule map $\lambda_A : A \to A\hat{\otimes}A$ which is a right inverse of Δ_A i.e. $\Delta_A \circ \lambda_A = id_A$, where $\Delta_A(a \otimes b) = ab$.*

(ii) *[30] Let $\psi \in \sigma(A)$. Then the Banach algebra A is called ψ-biprojective if there is a bounded A-bimodule map $\lambda_A : A \to A \otimes A$ such that $\psi \circ \Delta_A \circ \lambda_A(a) = \psi(a)$, for any $a \in A$.*

Note that it is easy to see that every biprojective Banach algebra is biflat; see 2.8.41(i)-[5], and also in Theorem 2.9.65 in [5] one can see that A is amenable Banach algebra if and only if it is biflat and has a bounded approximate identity.

Theorem 5.3. *Let $G = A \boxtimes_{\Theta} X$ be (α, β)-biprojective and there are A-bimodule maps $S : X \to A, L : A \to X, K : A \to A$ and $T : A \to A$ such that T is also a homomorphism and for each $a, b \in A$ and $x, y \in X$ we have*

(i) $S(x)S(y) = S(xy) + T(\Theta(x, y))$.

(ii) $S(x)T(a) = S(xa)$ *and* $T(a)S(x) = S(ax)$.

(iii) $\alpha \circ T = \alpha = \alpha \circ K + \beta \circ L$ *and* $\alpha \circ S = \beta$.

Then A is α-biprojective.

Proof. Consider the G-bimodule map $\lambda_G : G \to G \otimes G$ such that $(\alpha, \beta)\Delta_G\lambda_G = (\alpha, \beta)$. Define $P : G \to A$, $J : A \to G$ and $\lambda_A : A \to A \otimes A$ by $P((a, x)) = T(a) + S(x)$, $J(a) = (K(a), L(a))$ and $\lambda_A = (P \otimes P)\lambda_G J$. Then it is easy to verify that (i) and (ii) imply that $\Delta_A(P \otimes P) = P\Delta_G$ and then we conclude that $\alpha\Delta_A\lambda_A = \alpha$, by (iii). Note that since K, L, T and S are A-module maps, J and P are also A-bimodule maps and so λ_A is an A-bimodule map. □

Theorem 5.4. *Let $(\alpha, \beta) \in \sigma(G)$. If A is α-biprojective and there are bounded linear maps $S : X \to A, L : A \to X, K : A \to A$ and $T : A \to A$ such that for each $a, b \in A$ and $x, y \in X$ we have*

(i) $\theta(L(a), L(b)) + K(a)K(b) = K(ab)$.

(ii) $L(ab) = K(a)L(b) + L(a)K(b) + L(a)L(b)$.

(iii) $\alpha \circ T = \alpha = \alpha \circ K + \beta \circ L$ *and* $\alpha \circ S = \beta$.

(iv) *T is a homomorphism and* $S(x)S(y) = S(xy) + T(\Theta(x, y))$.

(v) $S(x)T(a) = S(xa)$ *and* $T(a)S(x) = S(ax)$.

(vi) $K(T(a)b) = aK(b), K(bT(a)) = K(b)a, K(S(x)b) = \Theta(x, L(b)), K(bS(x)) = \Theta(L(b), x)$.

(vii) $L(T(a)b) = aL(b), L(bT(a)) = L(b)a, L(S(x)b) = xK(b) + xL(b), L(bS(x)) = K(b)x + L(b)x.$

Then G is (α,β)-biprojective.

Proof. Consider P and J as the latter theorem and consider A-bimodule map $\lambda_A : A \to A \otimes A$ such that $\alpha\Delta_A\lambda_A = \alpha$. Define $\lambda_G : G \to G \otimes G$ by $\lambda_G = (J \otimes J)\lambda_A P$. Then it is easy to verify that by (i) and (ii) $\Delta_G(J \otimes J) = J\Delta_A$ and by (iii) $(\alpha,\beta)J = \alpha$ and $\alpha P = (\alpha,\beta)$. Therefore we conclude that $(\alpha,\beta)\Delta_G\lambda_G = (\alpha,\beta)$. Also (iv)–(v) imply that P is a homomorphism and so λ_G is a G-bimodule, by (vi)–(vii). □

Corollary 5.5. *Suppose that $G = A \bowtie X$ is $(\alpha,0)$-biprojective, where $\alpha \in \sigma(A)$. Then A is α-biprojective. Moreover, if both of π_ℓ and π_r are zero and A is α-biprojective, then G is $(\alpha,0)$-biprojective.*

Proof. In Theorem 5.3 Put $\Theta = 0, L = 0, S = 0, T = K = id_A$. □

Remark 5.6. *Corollary 5.5 is the modified and generalized form of Theorem 4.4 in [26]. Note that in the proof of that theorem, the mapping $\tilde{\mu}$ is not $A \times_\theta B$-module morphism; indeed, we have*

$$\begin{aligned}\tilde{\mu}((a,b)(c,d)) &= (q_B \otimes q_B)(\mu(bd))\\ &= (q_B \otimes q_B)(b\mu(d))\\ &= \sum (0, bb_i) \otimes (0, d_i).\end{aligned}$$

and

$$\begin{aligned}(a,b)\tilde{\mu}((c,d)) &= (a,b)(q_B \otimes q_B)(\mu(d))\\ &= \sum (a,b)(0,b_i) \otimes (0,d_i)\\ &= \sum (a\theta(b_i), bb_i) \otimes (0,d_i),\end{aligned}$$

where $\mu(d) = \sum b_i \otimes d_i$. Therefore if $\theta \neq 0$ and $A \neq 0$, then it may be T is not $A \times_\theta B$-bimodule morphism.

Theorem 5.7. *Let X be unital and β-biprojective and there are bounded linear maps $R : X \to A$ and $T : X \to X$ and $\alpha \in \sigma(A)$ such that for each $x, y \in X$,*

(i) $\alpha \circ \Theta = 0, \beta(ax) = \alpha(a)\beta(x) = \beta(xa).$

(ii) $R(xy) = R(x)R(y) + \Theta(T(x)T(y)), T(xy) = R(x)T(y) + T(x)R(y) + T(x)T(y).$

(iii) $\beta \circ T + \alpha \circ R = \beta.$

Then there is a left G-module $\lambda_G : G \to G \otimes G$ such that $(\alpha,\beta)\Delta_G\lambda_G = (\alpha,\beta)$.

Proof. Consider X-bimodule $\lambda_X : X \to X \otimes X$ such that $\beta\Delta_X\lambda_X = \beta$. Define $U : X \to G$ with $U(x) = (R(x), T(x))$ and $\lambda_G : G \to G \otimes G$ by $\lambda_G((a,x)) = (a,x)(U \otimes U)\lambda_X(1_X)$. Then (i) implies that $(\alpha,\beta) \in \sigma(G)$ and (ii) implies that $\Delta_G(U \otimes U) = U\Delta_X$. Combining with (iii) we conclude that

$$\begin{aligned}(\alpha,\beta)\Delta_G\lambda_G(a,x) &= (\alpha,\beta)\Delta_G((a,x)(U \otimes U)\lambda_X(1_X))\\ &= (\alpha,\beta)((a,x)(U\Delta_X\lambda_X(1_X))\\ &= (\alpha,\beta)((a,x))(\alpha,\beta)(U\Delta_X\lambda_X(1_X))\\ &= (\alpha,\beta)((a,x))\beta(\Delta_X\lambda_X(1_X))\\ &= (\alpha,\beta)((a,x))\beta(1_X)\\ &= (\alpha,\beta)((a,x)).\end{aligned}$$

Obviously λ_G is a left G-module. □

Theorem 5.8. *Let G be (α,β)-biprojective and $\alpha\Theta = 0$. Then X is β-biprojective if there are bounded linear maps $M : A \to X, N : X \to X, R : X \to A$ and $T : X \to X$ such that for each $x, y \in X$ and $a \in A$,*

(i) M *is a homomorphism,* $N(ax) = M(a)N(x), N(xa) = N(x)M(a)$ *and* $N(x)N(y) = N(xy) + M(\Theta(x, y))$.

(ii) $\beta \circ M = \alpha, \beta \circ n = \beta = \alpha \circ R + \beta \circ T$.

(iii) $\Theta(T(x), y) = R(xy) = \Theta(x, T(y)), xR(y) + xT(y) = T(xy) = R(x)y + T(x)y$.

(iv) $N(x)y = M(xy) + N(xy) = xN(y), N(xa) = xM(a), N(ax) = M(a)x$.

Proof. Since $\alpha\circ\Theta = 0$, we have $\beta \in \sigma(X)$ by Theorem 3.2. Now consider the G-bimodule map $\lambda_G : G \to G\otimes G$ such that $(\alpha, \beta)\Delta_G\lambda_G = (\alpha, \beta)$. Define $\Phi : G \to X$ by $\Phi((a, x)) = M(a) + N(x)$, $U : X \to G$ as before theorem and $\lambda_X : X \to X \otimes X$ by $\lambda_X = (\Phi \otimes \Phi)\lambda_G U$. Then (i) implies that $\Delta_X(\Phi \otimes \Phi) = \Phi\Delta_G$ and (ii) implies that $\beta\Phi = (\alpha, \beta)$ and $(\alpha, \beta)U = \beta$. Therefore $\beta\Delta_X\lambda_X = \beta$. Now (iii) implies that $U(x)(0, y) = U(xy) = (0, x)U(y)$ and (iv) implies that $\Phi((0, x)g) = x\Phi(g)$ and $\Phi(g(0, x)) = \Phi(g)x$, for each $g \in G$ and $x \in X$. This implies that λ_X is an X-bimodule. □

Corollary 5.9. *Let $G = A \bowtie X$, X is unital and $1_X a = a1_X$. Suppose that $(\alpha, \beta) \in \sigma(G)$. Then X is β-biprojective if and only if G is (α, β)-biprojective.*

Proof. In Theorems 5.7 and 5.8 define $\Theta = 0, R = 0, T = N = id_X$ and $M(a) = a1_X$, for each $a \in A$. □

Theorem 5.10. *Suppose $(\alpha, \beta) \in \sigma(G)$ and $\alpha \circ \Theta = 0$. Let A be α-biprojective and X be β-biprojective and unital. Then there is a map $\lambda_G : G \to G\otimes G$ with $\Delta_G\lambda_G = id_G$, if there exist bounded linear maps $S : X \to X$ and $T : X \to A$ and a homomorphism $K : A \to A$ such that for each $a, b \in A$ and $x, y \in X$, we have*

(i) $\Theta \circ (1_X K(a), 1_X K(b)) = 0$.

(ii) $T(xy) = T(x)T(y) + \Theta(S(x), S(y))$ *and* $S(xy) + S(x)S(y) + S(x)T(y) + T(x)S(y)$.

(iii) $\alpha \circ T + \beta \circ S = \beta$.

Proof. Theorem 3.2 says that since $\alpha \circ \Theta = 0$ we have $\beta \in \sigma(X)$. Consider the A-bimodule $\lambda_A : A \to A \otimes A$ and the X-bimodule $\lambda_X : X \to X \otimes X$ such that $\Delta_A\lambda_A = id_A$ and $\Delta_X\lambda_X = id_X$. Define $\xi : A \to G$ by $\xi(a) = (K(a), -1_X K(a))$, for each $a \in A$, and $U : X \to G$ by $U(x) = (T(x), S(x))$ Put $\lambda_G(a, x) = (\xi \otimes \xi) \circ \lambda_A(a) + (a, x)(U \otimes U) \circ \lambda_X(1_X)$. Then $\Delta_G \circ (\xi \otimes \xi) = \xi \circ \Delta_A$, $\Delta_G \circ (U \otimes U) = U \circ \Delta_X$ and $(\alpha, \beta)U = \beta$, by (i), (ii) and (iii), respectively. Also we have for each $a \in A$,

$$(\alpha, \beta)\xi(a) = \alpha(K(a)) + \beta(-1_X K(a)) = \alpha(K(a))(1 - \beta(1_X)) = 0.$$

Therefore we have

$$\begin{aligned}(\alpha, \beta)\Delta_G \circ \lambda_G((a, x)) &= (\alpha, \beta)\Delta_G\big((\xi \otimes \xi) \circ \lambda_A(a) + (a, x) \circ (U \otimes U) \circ \lambda_X(1_X)\big)\\ &= (\alpha, \beta)(\xi \circ \Delta_A \circ \lambda_A(a) + (a, x)U \circ \Delta_X \circ \lambda_X(1_X))\\ &= (\alpha, \beta)((a, x)U \circ \Delta_X \circ \lambda_X(1_X))\\ &= (\alpha, \beta)((a, x))(\alpha, \beta)(U \circ \Delta_X \circ \lambda_X(1_X))\\ &= (\alpha, \beta)((a, x))\beta(1_X)\\ &= (a, x).\end{aligned}$$

□

Remark 5.11. *In the latter theorem, let we have also for each $a \in A$, $\Theta(x, 1_X K(a)) = 0$ and $\lambda_A \circ \Theta = 0$. Then we have for each $a, b \in A$ and $x, y \in X$*

$$(0, x)(\xi \otimes \xi)\lambda_A = 0,$$

and

$$\begin{aligned}(\xi \otimes \xi) \circ \lambda_A(ab + \Theta(x, y)) &= (a, 0)((\xi \otimes \xi) \circ \lambda_A(b) \\ &= (a, 0)((\xi \otimes \xi) \circ \lambda_A(b) + (0, x)(\xi \otimes \xi)\lambda_A(b) \\ &= (a, x)((\xi \otimes \xi) \circ \lambda_A(b).\end{aligned}$$

Therefore

$$\begin{aligned}\lambda_G((a, x)(b, y)) &= (\xi \otimes \xi) \circ \lambda_A(ab + \Theta(x, y)) + (a, x)(b, y)(U \otimes U) \circ \lambda_X(1_X) \\ &= (a, 0)((\xi \otimes \xi) \circ \lambda_A(b) + (b, y) \circ (U \otimes U) \circ \lambda_X(1_X)) \\ &= (a, x)\lambda_G((b, y)).\end{aligned}$$

So we get λ_G is a left G-module map. Also, If in addition $\Theta(1K(a), x) = 0$ and $(U \otimes U) \circ \lambda_X(1_X)$ commutes with elements of G, we can show similarly it is a right G-module map.

Corollary 5.12. *Let $(\alpha, \beta) \in \sigma(A \bowtie X)$, A be α-biprojective and X be β-biprojective and unital. Then $G = A \bowtie X$ is (α, β)-biprojective.*

Proof. In Theorem 5.10 and Remark 5.11 put $K = id_A, T = 0, S = id_X$ and $\Theta = 0$. □

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