Filomat 38:4 (2024), 1153–1183
https://doi.org/10.2298/FIL2404153B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Linear relations generated by integral equations with Nevanlinna operator measures

Vladislav M. Bruk[a]

[a] *Saratov State Technical University, Saratov, Russia*

Abstract. We consider a family of minimal relations $\mathcal{L}_0(\lambda)$ generated by an integral equation with a Nevanlinna operator measure and give a description the families $\mathcal{L}_0(\lambda)$, $\mathcal{L}_0^*(\overline{\lambda})$, where $\lambda \in \mathbb{C}$. We prove that the families $\mathcal{L}_0(\lambda)$, $\mathcal{L}_0^*(\overline{\lambda})$ are holomorphic and give a description of families relations $T(\lambda)$ such that $\mathcal{L}_0(\lambda) \subset T(\lambda) \subset \mathcal{L}_0^*(\overline{\lambda})$ and $T^{-1}(\lambda)$ are bounded everywhere defined operators. The results obtained are applied to the proof of the existence of a characteristic operator for the integral equation.

1. Introduction

In this paper, we consider the integral equation

$$y(t) = x_0 - iJ\int_a^t d\mathbf{r}_\lambda(s)y(s) - iJ\int_a^t d\mathbf{m}(s)f(s), \tag{1}$$

where y is an unknown function, $a \leqslant t \leqslant b$; J is an operator in a separable Hilbert space H, $J = J^*$, $J^2 = E$ (E is the identical operator); $x_0 \in H$, $f \in L_2(H, d\mathbf{m}; a, b)$, $\lambda \in \mathbb{C}$; $\mathbf{r}_\lambda$ is a family of operator-valued measures defined on Borel sets $\Delta \subset [a, b]$ and taking values in the set of linear bounded operators acting in H; $\mathbf{m} = (\mathrm{Im}\lambda_1)^{-1}\mathrm{Im}\,\mathbf{r}_{\lambda_1}$ for some fixed λ_1. We assume that the measures $\mathbf{r}_\lambda$ are Nevanlinna measures, i.e., the function $\lambda \to \mathbf{r}_\lambda(\Delta)$ is holomorphic; $\mathbf{r}^*_\lambda(\Delta) = \mathbf{r}_{\overline{\lambda}}(\Delta)$; $(\mathrm{Im}\lambda)^{-1}\mathrm{Im}\,\mathbf{r}_\lambda(\Delta) \geqslant 0$ for all Borel sets Δ and all λ such that $\mathrm{Im}\lambda \neq 0$ (see a more detailed description of the properties of $\mathbf{r}_\lambda$ and $\mathbf{m}$ in section 2).

We define a family of minimal relations $\mathcal{L}_0(\lambda)$ generated by equation (1) and prove that the families $\mathcal{L}_0(\lambda)$, $\mathcal{L}_0^*(\overline{\lambda})$ are holomorphic. If $\mathbf{r}_\lambda = \mathbf{p} + \lambda\mathbf{m}$ ($\mathbf{p}$, $\mathbf{m}$ are self-adjoint operator measures and $\mathbf{m}$ is non-negative), then $\mathcal{L}_0(\lambda) = L_0 - \lambda E$, $\mathcal{L}_0^*(\overline{\lambda}) = L_0^* - \lambda E$, where L_0 is a minimal relation generated by the equation

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}(s)y(s) - iJ\int_a^t d\mathbf{m}(s)f(s). \tag{2}$$

Linear relations generated by equation (2) were studied in [14], [15], [16], [17].

2020 *Mathematics Subject Classification*. 47A10; 46G12; 45N05/
Keywords. Hilbert space; Integral equation; Operator measure; Linear relation; Holomorphic family of relation; characteristic operator.
Received: 08 June 2023; Accepted: 10 August 2023
Communicated by Dragan S. Djordjević
Email address: vladislavbruk@mail.ru (Vladislav M. Bruk)

In equation (1), suppose that the measures $\mathbf{r}_\lambda$ are absolutely continuous (i.e., $\mathbf{r}_\lambda(\Delta) = \int_\Delta r_\lambda(t)dt$ for all Borel sets $\Delta \subset [a,b]$, where the functions $\|r_\lambda(t)\|$ belong to $L_1(a,b)$). Then integral equation (1) is transformed to a differential equation with the Nevanlinna operator function r_λ and the non-negative weight operator function $m = (\mathrm{Im}\lambda_1)^{-1}\mathrm{Im}\, r_{\lambda_1}$. Linear relations generated by such a differential equation were studied in [27], [21], [22], [23], [8] (also see the bibliography in these articles).

In equation (2), suppose that the measures $\mathbf{p}$, $\mathbf{m}$ are absolutely continuous. Then integral equation (2) is transformed to a differential equation with a non-negative weight operator function. Linear relations generated by such differential equations were considered in many works (see [26], [6], [7], further detailed bibliography can be found, for example, in [3], [4], [23].

The study of integral equations (1), (2) differs essentially from the study of differential equations by the presence of the following features: i) a representation of solutions of equations (1), (2) using an evolutional family of operators is possible if the measures $\mathbf{r}_\lambda$, $\mathbf{p}$, $\mathbf{m}$ have not common single-point atoms (see [11]); ii) the Lagrange formula contains summands relating to single-point atoms of the measures $\mathbf{p}$, $\mathbf{m}$ (see [12]). Note that this work corrects the errors made in the articles [9], [10].

This paper generalizes the results of the work [15], [16], [17] to the case in which the integral equation contains the Nevanlinna measure. We study the properties of families of linear relations $\mathcal{L}_0(\lambda)$, $\mathcal{L}_0^*(\overline{\lambda})$ and apply the obtained results to a describing relations $T(\lambda)$ such that $\mathcal{L}_0(\lambda) \subset T(\lambda) \subset \mathcal{L}_0^*(\overline{\lambda})$ and $T^{-1}(\lambda)$ are bounded everywhere defined operators and give an explicit form of the operators $T^{-1}(\lambda)$.

In [29], A.V. Straus introduced the definition of the characteristic function for a generalized resolvent of a symmetric operator generated by a formally self-adjont differential expression of even order in the scalar case. The notion of a characteristic operator generalizes the notion of the characteristic function. In the present paper, we define a characteristic operator for equation (1) and give a description of the linear relations that generate the characteristic operator. For differential equations with a Nevanlinna function, the definition of the characteristic operator is given in [21], [22].

2. Main assumptions, designations and preliminary assertions

Let $\mathbf{B}_1$, $\mathbf{B}_2$ be Banach spaces. A linear relation T is understood as any linear manifold $T \subset \mathbf{B}_1 \times \mathbf{B}_2$. The terminology on the linear relations can be found, for example, in [19], [18], [2], [3], [4]. In what follows we make use of the following notations: $\{\cdot,\cdot\}$ is an ordered pair; $\mathcal{D}(T)$ is the domain of T; $\mathcal{R}(T)$ is the range of T; $\ker T$ is a set of elements $x \in \mathbf{B}_1$ such that $\{x, 0\} \in T$; T^{-1} is the relation inverse for T, i.e., the relation formed by the pairs $\{x', x\}$, where $\{x, x'\} \in T$. A relation T is called surjective if $\mathcal{R}(T) = \mathbf{B}_2$. A relation T is called invertible or injective if $\ker T = \{0\}$ (i.e., the relation T^{-1} is an operator); it is called continuously invertible if it is closed, invertible, and surjective (i.e., T^{-1} is a bounded everywhere defined operator).

Suppose $\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{B}$ is a Hilbert space. A relation T^* is called adjoint for T if T^* consists of all pairs $\{y_1, y_2\}$ such that the equality $(x_2, y_1) = (x_1, y_2)$ holds for all pairs $\{x_1, x_2\} \in T$. A linear relation T is called dissipative (accumulative, symmetric) if for any $\{x, x'\} \in T$ we have $\mathrm{Im}(x', x) \geqslant 0$ (respectively, $\mathrm{Im}(x', x) \leqslant 0$, or $\mathrm{Im}(x', x) = 0$). A dissipative (accumulative, symmetric) relation T is called maximal dissipative (accumulative, symmetric) if it has no dissipative (accumulative, symmetric) extensions $T_1 \supset T$ such that $T_1 \neq T$. A symmetric relation is called self-adjoint if it is maximal dissipative and maximal accumulative at the same time. As know, a relation T is symmetric if and only if $T \subset T^*$ and T is self-adjoint if and only if $T = T^*$.

It is known (see, for example, [20, ch.3], [19, ch.1]) that the graph of an operator $T : \mathcal{D}(T) \to \mathbf{B}_2$ is the set of pairs $\{x, Tx\} \in \mathbf{B}_1 \times \mathbf{B}_2$, where $x \in \mathcal{D}(T) \subset \mathbf{B}_1$. Consequently, the linear operators can be treated as linear relations; this is why the notation $\{x_1, x_2\} \in T$ is used also for an operator T. Since all considered relations are linear, we shall often omit word "linear".

A family of linear relations is understood as a function $\lambda \to T(\lambda)$ ($\lambda \in \mathcal{D} \subset \mathbb{C}$), where $T(\lambda)$ is a linear relation, $T(\lambda) \subset \mathbf{B}_1 \times \mathbf{B}_2$ ($\mathbf{B}_1$, $\mathbf{B}_2$ are Banach spaces). A family of closed relations $T(\lambda)$ is called holomorphic at a point $\lambda_0 \in \mathbb{C}$ if there exist a Banach space $\mathbf{B}_0$ and a family of bounded linear operators $\mathcal{K}(\lambda) : \mathbf{B}_0 \to \mathbf{B}_1 \times \mathbf{B}_2$ such that the operator $\mathcal{K}(\lambda)$ bijectively maps $\mathbf{B}_0$ onto $T(\lambda)$ for any fixed λ from some neighborhood of λ_0 and the family $\lambda \to \mathcal{K}(\lambda)$ is holomorphic in this neighborhood of λ_0. A family of relations is called holomorphic on the domain $\mathcal{D}$ if it is holomorphic at all points belonging to $\mathcal{D}$. These definitions generalize the corresponding definitions of holomorphic families of closed operators [20, ch. 7].

In what follows, we use the following notions from measure theory.

Let H be a separable Hilbert space with a scalar product $(\cdot,\cdot)$ and a norm $\|\cdot\|$. By $\mathcal{B}$ denote a set of Borel subsets $\Delta\subset[a,b]$. We consider a function $\Delta\to\mathbf{P}(\Delta)$ defined on $\mathcal{B}$ and taking values in the set of linear bounded operators acting in H. The function $\mathbf{P}$ is called an operator measure on $[a,b]$ (see, for example, [5, ch. 5]) if it is zero on the empty set and the equality $\mathbf{P}\left(\bigcup_{n=1}^{\infty}\Delta_n\right)=\sum_{n=1}^{\infty}\mathbf{P}(\Delta_n)$ holds for disjoint Borel sets Δ_n, where the series converges weakly. Further, we extend any measure $\mathbf{P}$ on $[a,b]$ to a segment $[a,b_0]$ $(b_0>b)$ letting $\mathbf{P}(\Delta)=0$ for each Borel sets $\Delta\subset(b,b_0]$.

By $\mathbf{V}_\Delta(\mathbf{P})$ we denote $\mathbf{V}_\Delta(\mathbf{P})=\rho_{\mathbf{P}}(\Delta)=\sup\sum_n\|\mathbf{P}(\Delta_n)\|$, where the supremum is taken over finite sums of disjoint Borel sets $\Delta_n\subset\Delta$. The number $\mathbf{V}_\Delta(\mathbf{P})$ is called the variation of the measure $\mathbf{P}$ on the Borel set Δ. Suppose that the measure $\mathbf{P}$ has the bounded variation on $[a,b]$. Then for $\rho_{\mathbf{P}}$-almost all $\xi\in[a,b]$ there exists an operator function $\xi\to\Psi_{\mathbf{P}}(\xi)$ such that $\Psi_{\mathbf{P}}$ possesses the values in the set of linear bounded operators acting in H, $\|\Psi_{\mathbf{P}}(\xi)\|=1$, and the equality

$$\mathbf{P}(\Delta)=\int_\Delta\Psi_{\mathbf{P}}(s)d\rho_{\mathbf{P}}\tag{3}$$

holds for each set $\Delta\in\mathcal{B}$. The function $\Psi_{\mathbf{P}}$ is uniquely determined up to values on a set of zero $\rho_{\mathbf{P}}$-measure. Integral (3) converges in the sense of the usual operator norm ([5, ch. 5]).

Further, $\int_{t_0}^{t}$ stands for $\int_{[t_0t)}$ if $t_0<t$, for $-\int_{[t,t_0)}$ if $t_0>t$, and for 0 if $t_0=t$. This implies that $y(a)=x_0$ in equations (1), (2). A function h is integrable with respect to the measure $\mathbf{P}$ on a set $\Delta\in\mathcal{B}$ if there exists the Bochner integral $\int_\Delta\Psi_{\mathbf{P}}(t)h(t)d\rho_{\mathbf{P}}=\int_\Delta(d\mathbf{P})h(t)$. Then the function $y(t)=\int_{t_0}^{t}(d\mathbf{P})h(s)$ is continuous from the left.

By $\mathcal{S}_{\mathbf{P}}$ denote a set of single-point atoms of the measure $\mathbf{P}$ (i.e., a set $t\in[a,b]$ such that $\mathbf{P}(\{t\})\neq0$). The set $\mathcal{S}_{\mathbf{P}}$ is at most countable. The measure $\mathbf{P}$ is continuous if $\mathcal{S}_{\mathbf{P}}=\varnothing$, it is self-adjoint if $(\mathbf{P}(\Delta))^*=\mathbf{P}(\Delta)$ for each Borel set $\Delta\in\mathcal{B}$, it is non-negative if $(\mathbf{P}(\Delta)x,x)\geqslant0$ for all Borel sets $\Delta\in\mathcal{B}$ and for all elements $x\in H$.

In addition, we use the following notation. We construct a continuous measure $\mathbf{P}_0$ from the measure $\mathbf{P}$ in the following way. We set $\mathbf{P}_0(\{t_k\})=0$ for $t_k\in\mathcal{S}_{\mathbf{P}}$ and we set $\mathbf{P}_0(\Delta)=\mathbf{P}(\Delta)$ for all Borel sets such that $\Delta\cap\mathcal{S}_{\mathbf{P}}=\varnothing$. We denote $\widehat{\mathbf{P}}=\mathbf{P}-\mathbf{P}_0$. Then $\widehat{\mathbf{P}}(\{t_k\})=\mathbf{P}(\{t_k\})$ for all $t_k\in\mathcal{S}_{\mathbf{P}}$ and $\widehat{\mathbf{P}}(\Delta)=0$ for all Borel sets Δ such that $\Delta\cap\mathcal{S}_{\mathbf{P}}=\varnothing$.

Let $\{\Delta,\lambda\}\to\mathbf{r}_\lambda(\Delta)$ be a function with values in the set of linear bounded operators acting in H, where $\Delta\in\mathcal{B}$, $\lambda\in\mathbb{C}_0$, $\mathbb{C}_0\supset\mathbb{C}\backslash\mathbb{R}$, $\mathbb{C}$ ($\mathbb{R}$) is the set of complex (of real) numbers, respectively. We assume that this function is the Nevanlinna function for each fixed Δ, i.e., the following conditions hold: (a) each point from $\mathbb{C}_0$ has a neighborhood (independent of Δ) such that the function $\lambda\to\mathbf{r}_\lambda(\Delta)$ is holomorphic in this neighborhood; (b) $\mathbf{r}^*_{\lambda}(\Delta)=\mathbf{r}_{\overline{\lambda}}(\Delta)$; (c) $(\mathrm{Im}\lambda)^{-1}\mathrm{Im}\,\mathbf{r}_\lambda(\Delta)\geqslant0$ for all $\Delta\in\mathcal{B}$ and all λ such that $\mathrm{Im}\lambda\neq0$. Moreover, this function satisfies condition (d). Before the formulation of condition (d), we introduce following designations. We put $\mathbf{m}_\lambda(\Delta)=(\mathrm{Im}\lambda)^{-1}\mathrm{Im}\,\mathbf{r}_\lambda(\Delta)$. Then for all $\nu\in\mathbb{C}_0\cap\mathbb{R}$ there exists (at least in the weak sense) $\lim\limits_{\lambda\to\nu\pm i0}\mathbf{m}_\lambda(\Delta)=\mathbf{m}_\nu(\Delta)$.

In [22], it was shown that conditions (a) – (c) imply

$$k_1(\mathbf{m}_\lambda(\Delta)g,g)\leqslant(\mathbf{m}_\mu(\Delta)g,g)\leqslant k_2(\mathbf{m}_\lambda(\Delta)g,g).\tag{4}$$

It follows from [22], [8] that

$$\left|(\lambda-\mu)^{-1}((\mathbf{r}_\lambda(\Delta)-\mathrm{Re}\,\mathbf{r}_\mu(\Delta))g,h)\right|\leqslant k\left\|\mathbf{m}_\zeta^{1/2}(\Delta)g\right\|\left\|\mathbf{m}_\eta^{1/2}(\Delta)h\right\|,\tag{5}$$

where (i.e., in inequalities (4), (5)) $g,h\in H$; $\lambda,\mu,\zeta,\eta\in\mathbb{C}_0$; the constants $k,k_1,k_2>0$ are independent of $\Delta\in\mathcal{B}$, $\lambda,\mu,\zeta,\eta\in\mathrm{K}$ (K is an arbitrary fixed compact, $\mathrm{K}\subset\mathbb{C}_0$).

Condition (d): the function $\Delta\to\mathbf{r}_\lambda(\Delta)$ is an operator measure on $[a,b]$ for all $\lambda\in\mathbb{C}_0$ such that the measures $\mathrm{Re}\,\mathbf{r}_i$, $\mathbf{m}_{\lambda_1}$ (for some point $\lambda_1\in\mathbb{C}_0$) have the bounded variation on $[a,b]$.

It follows from condition (d) and (4), (5) that the measures $\mathbf{m}_\lambda$, $\mathbf{r}_\lambda-\mathrm{Re}\,\mathbf{r}_\mu$, $\mathbf{r}_\lambda$ ($\lambda,\mu\in\mathbb{C}_0$) have the bounded variation on $[a,b]$. We fix some $\lambda_1\in\mathbb{C}_0$ and put $\mathbf{m}=\mathbf{m}_{\lambda_1}$.

We denote $\mathbf{p} = \operatorname{Re}\mathbf{r}_i$, $\mathbf{n}_\lambda = \mathbf{r}_\lambda - \operatorname{Re}\mathbf{r}_i = \mathbf{r}_\lambda - \mathbf{p}$ and note that the measures $\mathbf{p}$, $\mathbf{p}_0$, $\widehat{\mathbf{p}}$, $\mathbf{m}$, $\mathbf{m}_0$, $\widehat{\mathbf{m}}$ are self-adjoint, the measures $\mathbf{m}$, $\mathbf{m}_0$, $\widehat{\mathbf{m}}$, $\operatorname{Im}\mathbf{n}_\lambda$, $\operatorname{Im}\mathbf{n}_{0\lambda}$, $\operatorname{Im}\widehat{\mathbf{n}}_\lambda$, $\mathbf{m}_\lambda$, $\mathbf{m}_{0\lambda}$ are non-negative. Condition (b) implies that $\mathbf{n}_\lambda^* = \mathbf{n}_{\overline{\lambda}}$. Moreover,

$$\mathbf{m}_\lambda = (\operatorname{Im}\lambda)^{-1}\operatorname{Im}\mathbf{n}_\lambda, \quad \mathbf{m}_{0\lambda} = (\operatorname{Im}\lambda)^{-1}\operatorname{Im}\mathbf{n}_{0\lambda}. \tag{6}$$

Using (4), (5), we get

$$k_1(\mathbf{m}(\Delta)g, g) \leqslant (\mathbf{m}_\lambda(\Delta)g, g) \leqslant k_2(\mathbf{m}(\Delta)g, g), \tag{7}$$

$$\left|(\mathbf{n}_\lambda(\Delta)g, h)\right| \leqslant k\left\|\mathbf{m}^{1/2}(\Delta)g\right\|\left\|\mathbf{m}^{1/2}(\Delta)h\right\|, \tag{8}$$

where $k, k_1, k_2 > 0$ are independent of $\Delta \in \mathcal{B}$, $g, h \in H$. We note that $\mathcal{S}_\mathbf{m} = \mathcal{S}_{\mathbf{m}_\lambda}$.

Example 2.1. *Suppose*

$$\mathbf{r}_\lambda = \mathbf{p} + \lambda\mathbf{m}, \tag{9}$$

where $\mathbf{p}$, $\mathbf{m}$ *are self-adjoint operator measures having bounded variations and* $\mathbf{m}$ *is non-negative. Then* $\operatorname{Re}\mathbf{r}_i = \mathbf{p}$ *since* $\operatorname{Re}(i\mathbf{m}) = 0$. *Therefore,* $\mathbf{n}_\lambda = \lambda\mathbf{m}$. *Obviously, the measure* $\mathbf{r}_\lambda$ *satisfies conditions (a), (b), (c), (d). Moreover,* $\mathbb{C}_0 = \mathbb{C}$, $\mathbf{m}_\lambda = \mathbf{m}$ *for all* $\lambda \in \mathbb{C}$.

In following Lemma 2.2, $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{q}$ are operator measures having bounded variations and taking values in the set of linear bounded operators acting in H. Suppose that the measure $\mathbf{q}$ is self-adjoint and assume that these measures are extended on the segment $[a, b_0] \supset [a, b_0) \supset [a, b]$ in the manner described above.

Lemma 2.2. [12] *Let* f, g *be functions integrable on* $[a, b_0]$ *with respect to the measure* $\mathbf{q}$ *and* $y_0, z_0 \in H$. *Then any functions*

$$y(t) = y_0 - iJ\int_{t_0}^{t} d\mathbf{p}_1(s)y(s) - iJ\int_{t_0}^{t} d\mathbf{q}(s)f(s), \quad z(t) = z_0 - iJ\int_{t_0}^{t} d\mathbf{p}_2(s)z(s) - iJ\int_{t_0}^{t} d\mathbf{q}(s)g(s) \quad (a \leqslant t_0 < b_0,\ t_0 \leqslant t \leqslant b_0)$$

satisfy the following formula (*analogous to the Lagrange one*):

$$\begin{aligned}
&\int_{c_1}^{c_2}(d\mathbf{q}(t)f(t), z(t)) - \int_{c_1}^{c_2}(y(t), d\mathbf{q}(t)g(t)) = (iJy(c_2), z(c_2)) - (iJy(c_1), z(c_1)) + \int_{c_1}^{c_2}(y(t), d\mathbf{p}_2(t)z(t)) - \\
&- \int_{c_1}^{c_2}(d\mathbf{p}_1(t)y(t), z(t)) - \sum_{t \in \mathcal{S}_{\mathbf{p}_1} \cap \mathcal{S}_{\mathbf{p}_2} \cap [c_1, c_2)} (iJ\mathbf{p}_1(\{t\})y(t), \mathbf{p}_2(\{t\})z(t)) - \sum_{t \in \mathcal{S}_\mathbf{q} \cap \mathcal{S}_{\mathbf{p}_2} \cap [c_1, c_2)} (iJ\mathbf{q}(\{t\})f(t), \mathbf{p}_2(\{t\})z(t)) - \\
&\quad - \sum_{t \in \mathcal{S}_{\mathbf{p}_1} \cap \mathcal{S}_\mathbf{q} \cap [c_1, c_2)} (iJ\mathbf{p}_1(\{t\})y(t), \mathbf{q}(\{t\})g(t)) - \sum_{t \in \mathcal{S}_\mathbf{q} \cap [c_1, c_2)} (iJ\mathbf{q}(\{t\})f(t), \mathbf{q}(\{t\})g(t)), \quad t_0 \leqslant c_1 < c_2 \leqslant b_0.
\end{aligned} \tag{10}$$

We consider equation

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}(s)y(s) - iJ\int_a^t d\mathbf{n}_\lambda(s)y(s) - iJ\int_a^t d\mathbf{m}(s)f(s). \tag{11}$$

If $\mathbf{r}_\lambda = \mathbf{p} + \lambda\mathbf{m}$ (see Example 2.1, equality (9)), then equation (11) will take the form

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}(s)y(s) - iJ\lambda\int_a^t d\mathbf{m}(s)y(s) - iJ\int_a^t d\mathbf{m}(s)f(s). \tag{12}$$

Equations (11), (12) have unique solutions (see [11]).

By $W(t,\lambda)$ denote an operator solution of the equation

$$W(t,\lambda)x_0 = x_0 - iJ\int_a^t d\mathbf{p}_0(s)W(s,\lambda)x_0 - iJ\int_a^t d\mathbf{n}_{0\lambda}(s)W(s,\lambda)x_0, \tag{13}$$

where $x_0 \in H$, $\lambda \in \mathbb{C}_0$. Using Lemma 2.2, we get

$$W^*(t,\overline{\lambda})JW(t,\lambda) = J \tag{14}$$

by the standard method (see, for example, [13]). The functions $t \to W(t,\lambda)$ and $t \to W^{-1}(t,\lambda) = JW^*(t,\overline{\lambda})J$ are continuous with respect to the uniform operator topology. Consequently, there exist constants $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that the inequality

$$\varepsilon_1 \|x\|^2 \leqslant \|W(t,\lambda)x\|^2 \leqslant \varepsilon_2 \|x\|^2 \tag{15}$$

holds for all $x \in H$, $t \in [a,b_0]$, $\lambda \in C \subset \mathbb{C}_0$ (C is a compact set).

Lemma 2.3. *Suppose that a function f is integrable with respect to the measure $\mathbf{m}$. A function y is a solution of the equation*

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}_0(s)y(s) - iJ\int_a^t d\mathbf{n}_{0\lambda}(s)y(s) - iJ\int_a^t d\mathbf{m}(s)f(s), \quad x_0 \in H, \quad a \leqslant t \leqslant b_0, \tag{16}$$

if and only if y has the form

$$y(t) = W(t,\lambda)x_0 - W(t,\lambda)iJ\int_a^t W^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi). \tag{17}$$

Proof. We denote $\widetilde{\mathbf{p}}_0 = \mathbf{p}_0 + \mathbf{n}_{0\lambda}$. The measure $\widetilde{\mathbf{p}}_0$ is continuous. Equation (16) has a unique solution (see [11]). It is enough to prove that if we substitute the function from the right side (17) instead y in equation (16), then we get the identity. With this substitution, the right side (16) takes the form

$$\begin{aligned} x_0 - iJ\int_a^t d\widetilde{\mathbf{p}}_0(s)\left(W(s,\lambda)x_0 - W(s,\lambda)iJ\int_a^s W^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi)\right) - iJ\int_a^t d\mathbf{m}(s)f(s) = \\ = x_0 - iJ\int_a^t d\widetilde{\mathbf{p}}_0(s)W(s,\lambda)x_0 - J\int_a^t d\widetilde{\mathbf{p}}_0(s)W(s,\lambda)J\int_a^s W^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi) - iJ\int_a^t d\mathbf{m}(s)f(s). \end{aligned} \tag{18}$$

We change the limits of integration in the third term of the right-hand side (18). Then the third term takes the form

$$\begin{aligned} J\int_a^t d\widetilde{\mathbf{p}}_0(s)W(s,\lambda)J\int_a^s W^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi) = J\int_{[a,t)}\left(\int_{(\xi,t)} d\widetilde{\mathbf{p}}_0(s)W(s,\lambda)\right)JW^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi) = \\ = J\int_{[a,t)}\left(\int_{[\xi,t)} d\widetilde{\mathbf{p}}_0(s)W(s,\lambda)\right)JW^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi) - J\int_{[a,t)}\left(\int_{\{\xi\}} d\widetilde{\mathbf{p}}_0(s)W(s,\lambda)\right)JW^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi). \end{aligned} \tag{19}$$

The last term in (19) is equal to zero since the measure $\widetilde{\mathbf{p}}_0$ is continuous. Using (13), we continue equality (18)

$$W(t,\lambda)x_0 - \int_a^t J\left(\int_\xi^t d\widetilde{\mathbf{p}}_0(s)W(s,\lambda)\right)JW^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi) - iJ\int_a^t d\mathbf{m}(s)f(s). \tag{20}$$

It follows from (13) that (20) is equal to

$$\begin{aligned} W(t,\lambda)x_0 - \int_a^t i((W(t,\lambda)-E) - (W(\xi,\lambda)-E))JW^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi) - iJ\int_a^t d\mathbf{m}(s)f(s) = \\ = W(t,\lambda)x_0 - i\int_a^t W(t,\lambda)JW^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi) + i\int_a^t W(\xi,\lambda)JW^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi) - iJ\int_a^t d\mathbf{m}(s)f(s). \end{aligned}$$

Taking into account (14), we continue the last equality

$$W(t,\lambda)x_0 - iW(t,\lambda)J\int_a^t W^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi) + iJ\int_a^t d\mathbf{m}(\xi)f(\xi) - iJ\int_a^t d\mathbf{m}(s)f(s) = y(t).$$

The lemma is proved. □

For the case in which $\mathbf{n}_\lambda = \lambda\mathbf{m}$ (see Example 2.1, equality (9)), this lemma is proved in [15].

Remark 2.4. *Lemma 2.3 remains valid if in formulas* (16), (17) *we replace* $\mathbf{m}$ *by* $\mathbf{m}_0$ *together.*

3. The space $L_2(H, d\mathbf{m}; a, b)$

We introduce the quasi-scalar product

$$(x,y)_{\mathbf{m}} = \int_a^{b_0} (d\mathbf{m}(t)x(t), y(t)) \tag{21}$$

on a set of step-like functions with values in H defined on the segment $[a, b_0]$. Identifying with zero functions y obeying $(y,y)_{\mathbf{m}} = 0$ and making the completion, we arrive at the Hilbert space denoted by $L_2(H, d\mathbf{m}; a, b) = \mathfrak{H}$. The elements of $\mathfrak{H}$ are the classes of functions identified with respect to the norm $\|y\|_{\mathbf{m}} = (y,y)_{\mathbf{m}}^{1/2}$. In order not to complicate the terminology, the class of functions with a representative y is indicated by the same symbol and we write $y \in \mathfrak{H}$. The equality of the functions in $\mathfrak{H}$ is understood as the equality for associated equivalence classes. We denote the scalar product (the norm) in $\mathfrak{H}$ by $(\cdot,\cdot)_{\mathfrak{H}}$ or $(\cdot,\cdot)_{\mathbf{m}}$ (by $\|\cdot\|_{\mathfrak{H}}$ or $\|\cdot\|_{\mathbf{m}}$, respectively).

Remark 3.1. *It follows from* (4) *that the space* $\mathfrak{H}$ *does not depend on the choice of the point* $\lambda_1 \in \mathbb{C}_0$ *in the following sense. If we change the measure* $\mathbf{m} = \mathbf{m}_{\lambda_1}$ *to* $\mathbf{m}_\lambda$ ($\lambda \in \mathbb{C}_0$) *in* (21)*, then we obtain the same set* $\mathfrak{H}$ *supplied with an equivalent norm.*

By $\mathfrak{X}_A = \mathfrak{X}_A(t)$ denote an operator characteristic function of a set A, i.e., $\mathfrak{X}_A(t) = E$ if $t \in A$ and $\mathfrak{X}_A(t) = 0$ if $t \notin A$. We shall often omit the argument t in the notation $\mathfrak{X}_A$.

Lemma 3.2. *The inequality*

$$\left|\int_a^{b_0} (d\mathbf{n}_\lambda(t)y(t), x(t))\right| \leqslant k\,\|y\|_{\mathbf{m}}\,\|x\|_{\mathbf{m}} \tag{22}$$

holds for all functions $y, x \in \mathfrak{H}$*, where* $k > 0$ *is independent of* $\lambda \in \mathrm{K}$ (K *is an arbitrary fixed compact,* $\mathrm{K} \subset \mathbb{C}_0$).

Proof. Using (8), we obtain that inequality (22) holds for step functions. Now the desired statement follows from the definition of the space $\mathfrak{H}$. The lemma is proved. □

It follows from Lemma 3.2 that the one-and-half-form in the left-hand side (22) is continuous on $\mathfrak{H} \times \mathfrak{H}$. Hence there exists a bounded operator $\Lambda_\lambda : \mathfrak{H} \to \mathfrak{H}$ such that

$$\int_a^{b_0} (d\mathbf{n}_\lambda(t)y(t), x(t)) = (\Lambda_\lambda y, x)_{\mathbf{m}}. \tag{23}$$

In (23), we take $x(t) = \mathfrak{X}_\Delta(t)x_0$ for any $x_0 \in H$, $\Delta \in \mathcal{B}$. Then we obtain $\int_\Delta (d\mathbf{n}_\lambda(t)y(t), x_0) = \int_\Delta (d\mathbf{m}(t)(\Lambda_\lambda y)(t), x_0)$. Consequently, the equality

$$\int_\Delta d\mathbf{n}_\lambda(t)y(t) = \int_\Delta d\mathbf{m}(t)(\Lambda_\lambda y)(t) \tag{24}$$

holds for each Borel set Δ and each function $y \in \mathfrak{H}$. The equality $\mathbf{n}_\lambda^* = \mathbf{n}_{\overline{\lambda}}$ implies that $\Lambda_\lambda^* = \Lambda_{\overline{\lambda}}$. Using (6), (23), (24), we get

$$\int_\Delta (d\mathbf{m}_\lambda(t)y(t), x(t)) = (\mathrm{Im}\lambda)^{-1}\int_\Delta (d\mathbf{m}(t)((\mathrm{Im}\Lambda_\lambda)y)(t), x(t)); \ \int_\Delta d\mathbf{m}_\lambda(t)y(t) = (\mathrm{Im}\lambda)^{-1}\int_\Delta d\mathbf{m}(t)((\mathrm{Im}\Lambda_\lambda)y)(t). \quad (25)$$

Remark 3.3. *It follows from* (24) *that*

$$\int_\Delta d\mathbf{n}_{0\lambda}(t)y(t) = \int_\Delta d\mathbf{m}_0(t)(\Lambda_\lambda y)(t). \quad (26)$$

Moreover, equality (24) *implies that* $\Lambda_\lambda(\mathfrak{X}_{\{\tau\}}x_0)(t) = 0$ *if* $t \neq \tau$. *Therefore,* $\mathfrak{X}_{\{\tau\}}\Lambda_\lambda(\mathfrak{X}_{\{\tau\}}x_0) = \Lambda_\lambda(\mathfrak{X}_{\{\tau\}}x_0)$ *and*

$$\int_{\{\tau\}} d\mathbf{n}_\lambda(t)\mathfrak{X}_{\{\tau\}}(t)x_0 = \int_{\{\tau\}} d\mathbf{m}(t)(\Lambda_\lambda(\mathfrak{X}_{\{\tau\}}x_0))(t) = \mathbf{m}(\{\tau\})(\Lambda_\lambda(\mathfrak{X}_{\{\tau\}}x_0))(\tau) \quad (27)$$

for any $x_0 \in H$ *and any* $\tau \in \mathcal{S}_\mathbf{m}$. *Hence,* $\mathbf{n}_\lambda(\{\tau\})x_0 = \mathbf{m}(\{\tau\})(\Lambda_\lambda(\mathfrak{X}_{\{\tau\}}x_0))(\tau)$. *To shorten the notation, we will denote* $\Lambda_\lambda(\mathfrak{X}_{\{\tau\}}x_0)(\tau) = \Lambda_\lambda x_0$.

Remark 3.4. *If* $\mathbf{r}_\lambda$ *has form* (9) (*see Example 2.1*), *then* $\Lambda_\lambda = \lambda E$.

Lemma 3.5. *The operator* $\mathrm{Im}\Lambda_\lambda : \mathfrak{H} \to \mathfrak{H}$ *has an everywhere defined bounded inverse* $(\mathrm{Im}\Lambda_\lambda)^{-1}$ *for any* λ *such that* $\mathrm{Im}\lambda \neq 0$.

Proof. First we prove that there exists a number $\gamma = \gamma(\lambda) > 0$ such that

$$(\mathrm{Im}\lambda)^{-1}(\mathrm{Im}\Lambda_\lambda y, y)_\mathfrak{H} \geqslant \gamma(y, y)_\mathfrak{H} \quad (28)$$

for all $y \in \mathfrak{H}$ and $\mathrm{Im}\lambda \neq 0$. Using (21), (24), we get

$$\int_a^{b_0} \mathrm{Im}(d\mathbf{n}_\lambda(t)y(t), y(t)) = \mathrm{Im}(\Lambda_\lambda y, y)_\mathbf{m}. \quad (29)$$

Therefore, $\int_a^{b_0}(d\mathbf{m}_\lambda(t)y(t), y(t)) = (\mathrm{Im}\lambda)^{-1}\mathrm{Im}(\Lambda_\lambda y, y)_\mathbf{m}$ or $(y, y)_{\mathbf{m}_\lambda} = (\mathrm{Im}\lambda)^{-1}\mathrm{Im}(\Lambda_\lambda y, y)_\mathbf{m}$. Now inequality (28) follows from (7), (8). The application of (28) and the equality $\Lambda_\lambda^* = \Lambda_{\overline{\lambda}}$ yields $\mathcal{R}(\mathrm{Im}\Lambda_\lambda) = \mathfrak{H}$, where $\mathcal{R}(\mathrm{Im}\Lambda_\lambda)$ is the range of $\mathrm{Im}\Lambda_\lambda$. This and (28) implies the desired statement. The lemma is proved. □

Theorem 3.6. *The operator* $\Lambda_\lambda : \mathfrak{H} \to \mathfrak{H}$ *has an everywhere defined bounded inverse* Λ_λ^{-1} *for any* λ *such that* $\mathrm{Im}\lambda \neq 0$.

Proof. Using (28), we obtain

$$\left|(\Lambda_\lambda y, y)_\mathfrak{H}\right|^2 = \left|((\mathrm{Re}\Lambda_\lambda)y, y)_\mathfrak{H}\right|^2 + \left|((\mathrm{Im}\Lambda_\lambda)y, y)_\mathfrak{H}\right|^2 \geqslant \gamma \left\|y\right\|_\mathfrak{H}^2 \quad (30)$$

for any $y \in \mathfrak{H}$. Suppose a sequence $\{\Lambda_\lambda y_n\}$ converges to zero in $\mathfrak{H}$ as $n \to \infty$. We claim that the sequence $\{y_n\}$ converges to zero. First we prove that the sequence $\{y_n\}$ is bounded. Assume the converse, let $\left\|y_n\right\|_\mathfrak{H} \to \infty$ as $n \to \infty$. We denote $\widetilde{y}_n = y_n \left\|y_n\right\|_\mathfrak{H}^{-1}$. Then $\Lambda_\lambda \widetilde{y}_n \to 0$ as $n \to \infty$. By (30), we get $\widetilde{y}_n \to 0$. But $\left\|\widetilde{y}_n\right\|_\mathfrak{H} = 1$. This contradiction proves that $\{y_n\}$ is bounded. Now (30) implies that $\{y_n\} \to 0$ in $\mathfrak{H}$ as $n \to \infty$. Finally, using the equality $\Lambda_\lambda^* = \Lambda_{\overline{\lambda}}$, we obtain $\mathcal{R}(\Lambda_\lambda) = \mathfrak{H}$. This completes the proof of theorem. □

Lemma 3.7. *The operator function* $\lambda \to \Lambda_\lambda$ *is holomorphic on* $\mathbb{C}_0$.

Proof. Taking into account condition (a) and equality (23), we obtain that the function $\lambda \to \Lambda_\lambda y$ is holomorphic for each step-like function y. Now the desired statement follows from Lemma 3.2 and the density of the set of step-like functions in $\mathfrak{H}$. The lemma is proved. □

We investigate the structure of the space $\mathfrak{H}$ in detail. It follows from condition (d) that there exists an operator function $\Psi_{\mathbf{m}}$ satisfying equality (3) in which the measure $\mathbf{P}$ is changed to the measure $\mathbf{m}$. In particular, the equality $\mathbf{m}(\Delta) = \int_\Delta \Psi_{\mathbf{m}}(\xi)d\rho_{\mathbf{m}}$ holds.

The inequality $\mathbf{m}(\Delta) \geqslant 0$ implies $\Psi_{\mathbf{m}}(\xi) \geqslant 0$ for $\rho_{\mathbf{m}}$-almost all $\xi \in [a, b_0)$. We use some constructions from [7]. We denote $G(t) = \ker \Psi_{\mathbf{m}}(t)$; $H(t) = H \ominus G(t)$. Let $\Psi_0(t)$ be the restriction of $\Psi_{\mathbf{m}}(t)$ to $H(t)$. Then the operator $\Psi_0(t)$ acting in $H(t)$ has the inverse $\Psi_0^{-1}(t)$ (which is unbounded, in general). Let $\{H_\tau(t)\}$ $(-\infty < \tau < \infty)$ be a Hilbert scale of spaces generated by the operator $\Psi_0^{-1}(t)$ [5, ch. 3], [19, ch. 2]. It follows from the definition of Hilbert scale that the operator $\Psi_0(t)$ can be extended to the operator $\widehat{\Psi}_0(t)$ that maps $H_{-\alpha}(t)$ onto $H_{1-\alpha}(t)$ $(\alpha \geqslant 0)$ continuously and bijectively. Below the case $\alpha = 1/2$ is considered. Then $H_{1/2}(t) \subset H(t) \subset H_{-1/2}(t)$. Let $\widehat{\Psi}_{\mathbf{m}}(t)$ denote the operator that is defined on $H_{-1/2}(t) \oplus G(t)$ and is equal to $\widehat{\Psi}_0(t)$ on $H_{-1/2}(t)$ and to zero on $G(t)$. The operator $\widehat{\Psi}_{\mathbf{m}}(t)$ is the extension of $\Psi_{\mathbf{m}}(t)$ to $H_{-1/2}(t) \oplus G(t)$ by continuity. The operator $\widehat{\Psi}_{\mathbf{m}}(t)$ maps $H_{-1/2}(t) \oplus G(t)$ onto $H_{1/2}(t)$ continuously.

The operator $\Psi_0^{1/2}(t)$ can be treated as an operator that maps $H(t)$ onto $H_{1/2}(t)$ continuously and bijectively. Then the adjoint operator $\widehat{\Psi}_0^{1/2}(t)$ maps continuously and bijectively $H_{-1/2}(t)$ onto $H(t)$ and it is an extension of $\Psi_0^{1/2}(t)$ (see [5, ch. 3], [19, ch. 2]). Hence, $\widehat{\Psi}_0(t) = \Psi_0^{1/2}(t)\widehat{\Psi}_0^{1/2}(t) : H_{-1/2}(t) \to H_{1/2}(t)$.

In [7], the case is considered in which the measure $\rho_{\mathbf{m}}$ is the usual Lebesgue measure, i.e., $\rho_{\mathbf{m}}([a_1, b_1)) = b_1 - a_1$. By the literal repetition of the argumentation from [7], it is proved that the spaces $H_{-1/2}(t)$ are $\rho_{\mathbf{m}}$-measurable with respect to the parameter t [24, ch. 1] whenever for measurable functions one takes functions of the form $t \to \widehat{\Psi}_0^{-1}(t)\Psi_{\mathbf{m}}^{1/2}(t)h(t)$, where h is an arbitrary $\rho_{\mathbf{m}}$-measurable function ranging in H. The space $\mathfrak{H}$ is a measurable sum of the spaces $H_{-1/2}(t)$ with respect to the measure $\rho_{\mathbf{m}}$ and $\mathfrak{H}$ consists of all functions of the form $t \to \widehat{\Psi}_0^{-1}(t)\Psi_{\mathbf{m}}^{1/2}(t)g(t)$, where g is an arbitrary $\rho_{\mathbf{m}}$-measurable function ranging in H such that $\int_a^{b_0} \|g(t)\|^2 d\rho_{\mathbf{m}} < \infty$. We note that the above description of the space $\mathfrak{H}$ follows also from [25]. Thus, we obtain that the equality

$$(x, y)_{\mathbf{m}} = \int_a^{b_0} (\widehat{\Psi}_{\mathbf{m}}(t)x(t), y(t))d\rho_{\mathbf{m}} = \int_a^{b_0} (\widehat{\Psi}_0^{1/2}(t)x(t), \widehat{\Psi}_0^{1/2}(t)y(t))d\rho_{\mathbf{m}}$$

holds for all functions $x, y \in \mathfrak{H}$. In general, if a function $y \in \mathfrak{H}$, then $y(t) \in H_{-1/2}(t)$, $\widehat{\Psi}_0^{1/2}(t)y(t) \in H(t)$, $\Psi_{\mathbf{m}}^{1/2}(t)\widehat{\Psi}_0^{1/2}(t)y(t) \in H_{1/2}(t)$ for $\rho_{\mathbf{m}}$-almost all t.

4. Linear relations generated by the integral equations

Let us define a family of relations $\mathcal{L}(\lambda)$ (a relation L) generated by equation (11) (equation (12) for $\lambda = 0$, respectively) in the following way. The relations $\mathcal{L}(\lambda)$, L consist of pairs $\{\widetilde{y}, \widetilde{f}\} \in \mathfrak{H} \times \mathfrak{H}$ satisfying the conditions: for each pair $\{\widetilde{y}, \widetilde{f}\}$ there exists a pair $\{y, f\}$ such that the pairs $\{\widetilde{y}, \widetilde{f}\}$, $\{y, f\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$ and $\{y, f\}$ satisfies equation (11) (equation (12) for $\lambda = 0$, respectively).

Now we define *a family of minimal relations* $\mathcal{L}_0(\lambda)$ (*a minimal relation* L_0) generated by equation (11) (equation (12) for $\lambda = 0$, respectively) in the following way. The relations $\mathcal{L}_0(\lambda)$, L_0 are restrictions of $\mathcal{L}(\lambda)$ (L, respectively) to a set of pairs $\{y, f\} \in \mathfrak{H} \times \mathfrak{H}$ satisfying equalities (31) (for $\mathcal{L}_0(\lambda)$), (32) (for L_0) (see below)

$$y(a) = y(b_0) = y(\alpha) = 0, \quad \alpha \in \mathcal{S}_{\mathbf{p}}; \quad \mathbf{n}_\lambda(\{\beta\})y(\beta) + \mathbf{m}(\{\beta\})f(\beta) = 0, \quad \beta \in \mathcal{S}_{\mathbf{m}}; \tag{31}$$

$$y(a) = y(b_0) = y(\alpha) = 0, \quad \alpha \in \mathcal{S}_{\mathbf{p}}; \quad \mathbf{m}(\{\beta\})f(\beta) = 0, \quad \beta \in \mathcal{S}_{\mathbf{m}}. \tag{32}$$

Further, without loss of generality, it can be assumed that if $\{y, f\} \in \mathcal{L}_0(\lambda)$, ($\{y, f\} \in L_0$), then equalities (11), (31) ((12) for $\lambda = 0$, (32), respectively) hold for this pair. In general, the relations $\mathcal{L}_0(\lambda)$, L_0 are not operators since a function y can happen to be identified with zero in $\mathfrak{H}$, while f is non-zero.

In Lemma 2.2, we set $c_1 = a$, $c_2 = b_0$, $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}$, $\mathbf{q} = \mathbf{m}$. Using (10), we obtain that the relation L_0 is symmetric and $L \subset L_0^*$. Now in Lemma 2.2, we set $\mathbf{p}_1 = \mathbf{p} + \mathbf{n}_\lambda$, $\mathbf{p}_2 = \mathbf{p} + \mathbf{n}_{\overline{\lambda}}$, $\mathbf{q} = \mathbf{m}$. Then (10) implies that $\mathcal{L}(\lambda) \subset \mathcal{L}_0^*(\overline{\lambda})$.

Lemma 4.1. *If the pair* $\{y, f\} \in \mathcal{L}_0(\lambda)$, *then*

$$y(t) = -iJ\int_a^t d\mathbf{p}_0(s)y(s) - iJ\int_a^t d\mathbf{n}_{0\lambda}(s)y(s) - iJ\int_a^t d\mathbf{m}_0(s)f(s). \tag{33}$$

Proof. Let $\{y, f\} \in \mathcal{L}_0(\lambda)$. It follows from the definition of the relation $\mathcal{L}_0(\lambda)$ that the pair $\{y, f\}$ satisfies the equation

$$y(t) = -iJ\int_a^t d\mathbf{p}(s)y(s) - iJ\int_a^t d\mathbf{n}_\lambda(s)y(s) - iJ\int_a^t d\mathbf{m}(s)f(s).$$

Therefore,

$$y(t) = -iJ\int_a^t d(\mathbf{p}_0(s) + \widehat{\mathbf{p}}(s))y(s) - iJ\int_a^t d(\mathbf{n}_{0\lambda}(s) + \widehat{\mathbf{n}}_\lambda(s))y(s) - iJ\int_a^t d(\mathbf{m}_0(s) + \widehat{\mathbf{m}}(s))f(s).$$

Now using (31), we obtain (33). The lemma is proved. □

Remark 4.2. *It follows from the proof of Lemma 4.1 that every pair* $\{y, f\} \in \mathcal{L}_0(\lambda)$ *satisfies the equation*

$$y(t) = -iJ\int_a^t d\mathbf{p}_0(s)y(s) - iJ\int_a^t d\mathbf{n}_\lambda(s)y(s) - iJ\int_a^t d\mathbf{m}(s)f(s), \tag{34}$$

Corollary 4.3. [15] *If the pair* $\{y, f\} \in L_0 - \lambda E$, *then*

$$y(t) = -iJ\int_a^t d\mathbf{p}_0(s)y(s) - iJ\lambda\int_a^t d\mathbf{m}_0(s)y(s) - iJ\int_a^t d\mathbf{m}_0(s)f(s).$$

Lemma 4.4. *A pair* $\{\widetilde{y}, \widetilde{f}\} \in \mathfrak{H} \times \mathfrak{H}$ *belongs to the relation* $\mathcal{L}_0(\lambda)$ *if and only if there exists a pair* $\{y, f\}$ *such that the pairs* $\{\widetilde{y}, \widetilde{f}\}$, $\{y, f\}$ *are identical in* $\mathfrak{H} \times \mathfrak{H}$ *and the equalities*

$$y(t) = -W(t,\lambda)iJ\int_a^t W^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s), \tag{35}$$

$$y(\alpha) = W(\alpha,\lambda)iJ\int_a^\alpha W^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s) = 0, \quad \alpha \in \mathcal{S}_\mathbf{p} \cup \{b_0\}, \tag{36}$$

$$\mathbf{n}_\lambda(\{\beta\})y(\beta) + \mathbf{m}(\{\beta\})f(\beta) = 0, \quad \beta \in \mathcal{S}_\mathbf{m}, \tag{37}$$

hold.

Proof. The desired assertion follows from (31) and Lemmas 2.3, 4.1. □

Corollary 4.5. *If* $y \in \mathcal{D}(\mathcal{L}_0(\lambda))$, *then* y *is continuous and* $y(b) = 0$.

Corollary 4.6. *Suppose a pair* $\{y, f\}$ *satisfies equality* (35). *The function* $f \in \mathfrak{H}$ *belongs to the range* $\mathcal{R}(\mathcal{L}_0(\lambda))$ *if and only if the function* f *satisfies the conditions*

$$\int_a^\alpha W^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s) = 0, \quad \mathbf{n}_\lambda(\{\beta\})y(\beta) + \mathbf{m}(\{\beta\})f(\beta) = 0,$$

where $\alpha \in \mathcal{S}_\mathbf{p} \cup \{b_0\}$, $\beta \in \mathcal{S}_\mathbf{m}$.

Remark 4.7. *Lemma 4.4, Corollary 4.5 imply that we can replace* b_0 *by* b *in* (36) *and Corollary 4.6.*

Remark 4.8. *It follows from* (23) *that the second condition in* (31) (*and in equality* (37)) *can be written as*

$$\mathbf{m}(\{\beta\})((\Lambda_\lambda y)(\beta) + f(\beta)) = 0, \quad \beta \in \mathcal{S}_{\mathbf{m}}. \tag{38}$$

Remark 4.9. *Equality* (38) *means that the function* $\mathfrak{X}_{\{\beta\}}((\Lambda_\lambda y)(\beta) + f(\beta))$ *is identified with zero in* $\mathfrak{H}$.

It follows from (24) and the definition of L_0, L that equation (11) can be written as

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}(s)y(s) - iJ\int_a^t d\mathbf{m}(t)(\Lambda_\lambda y)(t) - iJ\int_a^t d\mathbf{m}(s)f(s). \tag{39}$$

Using (39), we get

$$\mathcal{L}_0(\lambda) = L_0 - \Lambda_\lambda, \quad \mathcal{L}(\lambda) = L - \Lambda_\lambda. \tag{40}$$

It follows from (10), (40), and the equality $\Lambda^*_\lambda = \Lambda_{\overline{\lambda}}$ that

$$\mathcal{L}^*_0(\overline{\lambda}) = L^*_0 - \Lambda_\lambda \supset \mathcal{L}(\lambda). \tag{41}$$

Lemma 4.10. *The families of the relations* $\mathcal{L}_0(\lambda)$, $\mathcal{L}(\lambda)$, $\mathcal{L}^*_0(\overline{\lambda})$ *are holomorphic on* $\mathbb{C}_0$.

Proof. The desired statement follows from Lemma (3.7), equalities (40), (41). □

Let $\mathcal{S}_0$ be the set $t \in [a,b]$ such that $y(t) = 0$ for all $y \in \mathcal{D}(L_0)$. The set $\mathcal{S}_0$ is closed and $\overline{\mathcal{S}}_{\mathbf{p}} \cup \{a\} \cup \{b\} \subset \mathcal{S}_0$.

Lemma 4.11. [15] *Suppose* $\{y, f\} \in L_0$. *Then* $f(t) = 0$ *for* $\mathbf{m}$*-almost all* $t \in \mathcal{S}_0$.

By $\mathfrak{H}_0$ (by $\mathfrak{H}_1$) denote a subspace of functions that vanish on $[a,b] \setminus \mathcal{S}_0$ (on $\mathcal{S}_0$, respectively) with respect to the norm in $\mathfrak{H}$. The subspaces $\mathfrak{H}_0$, $\mathfrak{H}_1$ are orthogonal and $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$. We note that $\mathfrak{H}_0 = \{0\}$ if and only if $\mathbf{m}(\mathcal{S}_0) = 0$.

Lemma 4.12. $\mathfrak{H}_0 \perp \mathcal{D}(\mathcal{L}_0(\lambda))$ *and* $\mathfrak{H}_0 \perp \mathcal{R}(\mathcal{L}_0(\lambda))$, *i.e., the equalities* $(g_0, y)_{\mathfrak{H}} = 0$, $(g_0, f)_{\mathfrak{H}} = 0$ *hold for all* $g_0 \in \mathfrak{H}_0$, $y \in \mathcal{D}(\mathcal{L}_0(\lambda))$, $f \in \mathcal{R}(\mathcal{L}_0(\lambda))$.

Proof. It follows from (40) that $\mathcal{D}(\mathcal{L}_0(\lambda)) = \mathcal{D}(L_0)$. Suppose $y \in \mathcal{D}(L_0)$. Then $y(t) = 0$ for $t \in \mathcal{S}_0$. Consequently, $(g_0, y)_{\mathfrak{H}} = 0$. Suppose $f \in \mathcal{R}(\mathcal{L}_0(\lambda))$. It follows from (40) that $f = f_0 - \Lambda_\lambda y$, where $f_0 \in \mathcal{R}(L_0)$, $y \in \mathcal{D}(L_0)$, $\{y, f_0\} \in L_0$. Lemma 4.11 implies that $(f_0, g_0)_{\mathfrak{H}} = 0$. We claim that $(\Lambda_\lambda y, g_0)_{\mathfrak{H}} = 0$. Indeed, using (8), we obtain $\left|(\mathbf{n}_\lambda(\mathcal{S}_0)y(t), g_0(t))\right| \leqslant k\left\|\mathbf{m}^{1/2}(\mathcal{S}_0)y(t)\right\|\left\|\mathbf{m}^{1/2}(\mathcal{S}_0)g_0(t)\right\| = 0$ since $\left\|\mathbf{m}^{1/2}(\mathcal{S}_0)y(t)\right\| = 0$. By (24), so that $(\Lambda_\lambda y, g_0)_{\mathfrak{H}_0} = 0$. The lemma is proved. □

It follows from Lemmas 4.11, 4.12 that $L_0 \cap (\mathfrak{H}_0 \times \mathfrak{H}_0) = \{0, 0\}$, $\mathcal{L}_0(\lambda) \cap (\mathfrak{H}_0 \times \mathfrak{H}_0) = \{0, 0\}$. We denote $L_{10} = L_0 \cap (\mathfrak{H}_1 \times \mathfrak{H}_1)$, $\mathcal{L}_{10}(\lambda) = \mathcal{L}_0(\lambda) \cap (\mathfrak{H}_1 \times \mathfrak{H}_1)$. Lemma 4.12 implies that $\mathcal{D}(L_{10}) \subset \mathfrak{H}_1$, $\mathcal{R}(L_{10}) \subset \mathfrak{H}_1$, $\mathcal{D}(\mathcal{L}_{10}(\lambda)) \subset \mathfrak{H}_1$, $\mathcal{R}(\mathcal{L}_{10}(\lambda)) \subset \mathfrak{H}_1$. Therefore,

$$L^*_0 = (\mathfrak{H}_0 \times \mathfrak{H}_0) \oplus L^*_{10}, \quad \mathcal{L}^*_0(\lambda) = (\mathfrak{H}_0 \times \mathfrak{H}_0) \oplus \mathcal{L}^*_{10}(\lambda). \tag{42}$$

Using (40), (42), we get

$$\mathcal{L}_{10}(\lambda) = L_{10} - \Lambda_\lambda, \quad \mathcal{L}^*_{10}(\overline{\lambda}) = L^*_{10} - \Lambda_\lambda, \quad \mathcal{L}^*_{10}(\lambda) = L^*_{10} - \Lambda_{\overline{\lambda}}. \tag{43}$$

The set $\mathcal{T}_{\mathbf{p}} = (a,b) \setminus \mathcal{S}_0$ is open and it is the union of at most a countable number of disjoint open intervals $\mathcal{J}_k$, i.e., $\mathcal{T}_{\mathbf{p}} = \bigcup_{k=1}^{\Bbbk_1} \mathcal{J}_k$ and $\mathcal{J}_k \cap \mathcal{J}_j = \varnothing$ for $k \neq j$, where $\Bbbk_1$ is a natural number (equal to the number of intervals if this number is finite) or the symbol ∞ (if the number of intervals is infinite). By $\mathbb{J}$ denote the set of this intervals $\mathcal{J}_k$.

Remark 4.13. *The boundaries* α_k, β_k *of any interval* $\mathcal{J}_k = (\alpha_k, \beta_k) \in \mathbb{J}$ *belong to* $\mathcal{S}_0$. *This follows from* (32).

We denote

$$w_k(t,\lambda) = \mathfrak{X}_{[\alpha_k,\beta_k)}W(t,\lambda)W^{-1}(\alpha_k,\lambda), \tag{44}$$

where $(\alpha_k,\beta_k) = \mathcal{J}_k \in \mathbb{J}$. Using (14), we get

$$w_k^*(t,\overline{\lambda})Jw_k(t,\lambda) = J, \quad \alpha_k \leqslant t < \beta_k. \tag{45}$$

Lemma 4.14. *Let $g \in \mathfrak{H}_1$ and let a function $G_{\mathbf{o}}$ be given by the following equality*

$$G_{\mathbf{o}}(t) = -\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}w_k(t,\lambda)iJ\int_{\alpha_k}^{t}w_k^*(s,\overline{\lambda})d\mathbf{m}(s)g(s),$$

where $(\alpha_k,\beta_k) = \mathcal{J}_k \in \mathbb{J}$. Then the pair $\{G_{\mathbf{o}},g\} \in \mathcal{L}_{10}^*(\overline{\lambda})$ if g vanishes outside of $[\alpha_k,\beta_k)$.

Proof. We denote

$$G(t) = -w_k(t,\lambda)iJ\int_{\alpha_k}^{t}w_k^*(s,\overline{\lambda})d\mathbf{m}(s)g(s).$$

Equalities (44), (14) imply

$$G(t) = -\mathfrak{X}_{[\alpha_k,\beta_k)}W(t,\lambda)iJ\int_{\alpha_k}^{t}W^*(s,\overline{\lambda})d\mathbf{m}(s)g(s).$$

It follows from Lemma 2.3 that the function G is a solution of equation (16) in which $x_0 = 0$, $\alpha_k \leqslant t \leqslant \gamma$, $\gamma < \beta_k$, and a, y, f are replaced by α_k, G, g, respectively, i.e., the pair $\{G,g\}$ satisfies equation (16) on the segment $[\alpha_k,\gamma]$.

Suppose a pair $\{y,f\} \in \mathcal{L}_0(\overline{\lambda})$. Using the definition of $\mathcal{L}_0(\overline{\lambda})$, we obtain that this pair $\{y,f\}$ satisfies equation (34) and conditions (31) in which λ is replaced by $\overline{\lambda}$. We can apply formula (10) to the functions y, f, G, g for $c_1 = \alpha_k$, $c_2 = \gamma$, $\mathbf{q} = \mathbf{m}$, $\mathbf{p}_1 = \mathbf{p}_0 + \mathbf{n}_{\overline{\lambda}}$, $\mathbf{p}_2 = \mathbf{p}_0 + \mathbf{n}_{0\lambda}$. Since the measures $\mathbf{p}_0$, $\mathbf{n}_{0\lambda}$ are continuous, $\mathbf{p}_0$, $\mathbf{m}$ are self-adjoint, we obtain

$$\int_{\alpha_k}^{\gamma}(d\mathbf{m}(s)f(s),G(s)) - \int_{\alpha_k}^{\gamma}(y,d\mathbf{m}(s)g(s)) = (iJy(\gamma),G(\gamma)) - \int_{\alpha_k}^{\gamma}(d\widehat{\mathbf{n}}_{\overline{\lambda}}(s)y(s),G(s)).$$

Using the equality $G_{\mathbf{o}}(t) = G(t) - \mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}G(t)$ and (37), we get

$$\int_{\alpha_k}^{\gamma}(d\mathbf{m}(s)f(s),G_{\mathbf{o}}(s)) - \int_{\alpha_k}^{\gamma}(y,d\mathbf{m}(s)g(s)) = (iJy(\gamma),G(\gamma)) - \\ - \sum_{s\in\mathcal{S}_{\mathbf{m}}\cap[\alpha_k,\gamma)}(\widehat{\mathbf{n}}_{\overline{\lambda}}(\{s\})y(s),G(s)) - \sum_{s\in\mathcal{S}_{\mathbf{m}}\cap[\alpha_k,\gamma)}(\widehat{\mathbf{m}}(\{s\})f(s),G(s)) = (iJy(\gamma),G(\gamma)). \tag{46}$$

The function y is continuous from the left and $y(\beta_k) = 0$ (see (36) and Corollary 4.5). Hence passing to the limit as $\gamma \to \beta_k - 0$ in (46), we obtain

$$\int_{\alpha_k}^{\beta_k}(d\mathbf{m}(s)f(s),G_{\mathbf{o}}(s)) = \int_{\alpha_k}^{\beta_k}(y(s),d\mathbf{m}(s)g(s)).$$

This implies the desired statement. The lemma is proved. □

Lemma 4.15. *A pair* $\{\widetilde{y},\widetilde{f}\}\in\mathfrak{H}\times\mathfrak{H}$ *belongs to the relation* $\mathcal{L}_{10}(\lambda)$ *if and only if there exists a pair* $\{y,f\}$ *such that the pairs* $\{\widetilde{y},\widetilde{f}\}$, $\{y,f\}$ *are identical in* $\mathfrak{H}\times\mathfrak{H}$ *and the equalities*

$$y(t)=-\sum_{k=1}^{\Bbbk_1} w_k(t,\lambda)iJ\int_{\alpha_k}^{t} w_k^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s), \tag{47}$$

$$\int_{\alpha_k}^{\beta_k} w_k^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s)=0,\quad -\mathfrak{X}_{\mathcal{S}_\mathbf{m}\cap(\alpha_k,\beta_k)}w_k(t,\lambda)iJ\int_{\alpha_k}^{t} w_k^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s)+(\Lambda_\lambda^{-1}\,\mathfrak{X}_{\mathcal{S}_\mathbf{m}\cap(\alpha_k,\beta_k)}f)(t)=0 \tag{48}$$

hold for all k*, where* $k=1,...,\Bbbk_1$ *if* $\Bbbk_1$ *is finite and* k *is any natural number if* $\Bbbk_1$ *is infinite,* $\mathrm{Im}\lambda\neq 0$*.*

Proof. Using (44), (45), we obtain that equalities (35), (36) are equivalent to equality (47) and first equality (48). It follows from (38) and Remark 4.9 that the function $\mathfrak{X}_{\{\beta\}}y(\beta)+\Lambda_\lambda^{-1}\mathfrak{X}_{\{\beta\}}f(\beta)$ ($\beta\in\mathcal{S}_\mathbf{m}$) is identified with zero in $\mathfrak{H}$. This and (47) imply second equality (48). Now the desired statement follows from Lemma 4.4. □

Corollary 4.16. *The function* $f\in\mathfrak{H}_1$ *belongs to the range* $\mathcal{R}(\mathcal{L}_{10}(\lambda))$ *if and only if* f *satisfies conditions* (48)*.*

By $\mathfrak{H}_{10}$ (by $\mathfrak{H}_{11}$) denote a subspace of functions that belong to $\mathfrak{H}_1$ and vanish on $\mathcal{S}_\mathbf{m}$ (on $[a,b]\setminus\mathcal{S}_\mathbf{m}$, respectively) with respect to the norm in $\mathfrak{H}$. So, $\mathfrak{H}_{10}$ ($\mathfrak{H}_{11}$) consists of functions of the form $\mathfrak{X}_{[a,b]\setminus(\mathcal{S}_0\cup\mathcal{S}_\mathbf{m})}h$ (of the form $\mathfrak{X}_{\mathcal{S}_\mathbf{m}\setminus\mathcal{S}_0}h$, respectively), where $h\in\mathfrak{H}_1$ is an arbitrary function. Therefore,

$$\mathfrak{H}_1=\mathfrak{H}_{10}\oplus\mathfrak{H}_{11},\quad \mathfrak{H}=\mathfrak{H}_0\oplus\mathfrak{H}_{10}\oplus\mathfrak{H}_{11}.$$

Obviously, the space $\mathfrak{H}_{11}$ is the closure in $\mathfrak{H}$ of the linear span of functions that have the form $\mathfrak{X}_{\{\tau\}}(\cdot)x$, where $x\in H$, $\tau\in\mathcal{S}_\mathbf{m}\setminus\mathcal{S}_0$. By (32), it follows that $\mathfrak{H}_{11}\subset\ker L_{10}^*$.

We define an operator $\mathcal{U}_k(\lambda):\mathfrak{H}_1\to\mathfrak{H}_1$ by the equality

$$(\mathcal{U}_k(\lambda)f)(t)=-\mathfrak{X}_{[a,b]\setminus\mathcal{S}_\mathbf{m}}w_k(t,\lambda)iJ\int_a^t w_k^*(s,\overline{\lambda})d\mathbf{m}(s)(\Lambda_\lambda f)(s),\quad f\in\mathfrak{H}_1. \tag{49}$$

The operator $\mathcal{U}_k(\lambda)$ is bounded. Taking into account (44) and Lemma 4.14, we obtain that the pair $\{\mathcal{U}_k(\lambda)f,\mathfrak{X}_{[\alpha_k,\beta_k)}(\Lambda_\lambda f)\}\in\mathcal{L}_{10}^*(\overline{\lambda})$. Let $u_k(t,\lambda,\tau):H\to\mathfrak{H}_1$ be an operator acting by the formula

$$u_k(t,\lambda,\tau)x=(\mathcal{U}_k(\lambda)\mathfrak{X}_{\{\tau\}}x)(t)=-\mathfrak{X}_{[a,b]\setminus\mathcal{S}_\mathbf{m}}w_k(t,\lambda)iJ\int_a^t w_k^*(s,\overline{\lambda})d\mathbf{m}(s)(\Lambda_\lambda\mathfrak{X}_{\{\tau\}}x)(s), \tag{50}$$

where $x\in H$, $\tau\in(\alpha_k,\beta_k)\cap\mathcal{S}_\mathbf{m}$, $(\alpha_k,\beta_k)=\mathcal{J}_k\in\mathbb{J}$. Then the pair $\{u_k(\cdot,\lambda,\tau)x,\Lambda_\lambda\mathfrak{X}_{\{\tau\}}x\}\in\mathcal{L}_{10}^*(\overline{\lambda})$. The definition of L_0 implies that the function $\mathfrak{X}_{\{\tau\}}x\in\ker L_0^*$. It follows from (43) that the pair $\{\mathfrak{X}_{\{\tau\}}x,-\Lambda_\lambda\mathfrak{X}_{\{\tau\}}x\}\in\mathcal{L}_{10}^*(\overline{\lambda})$. Thus, for any $x\in H$, the function

$$u_k(\cdot,\lambda,\tau)x+\mathfrak{X}_{\{\tau\}}(\cdot)x\in\ker\mathcal{L}_{10}^*(\overline{\lambda}). \tag{51}$$

The linear span of functions of the form $\mathfrak{X}_{\{\tau\}}(\cdot)x$ ($x\in H$, $\tau\in\mathcal{S}_\mathbf{m}\setminus\mathcal{S}_0$) is dense in the space $\mathfrak{H}_{11}$. It follows from (50), (51) that for any the function $z_1\in\mathfrak{H}_{11}$

$$\mathcal{U}_k(\lambda)z_1+z_1\in\ker\mathcal{L}_{10}^*(\overline{\lambda}). \tag{52}$$

Lemma 4.17. *The linear span of functions of the form* $\mathfrak{X}_{[a,b]\setminus(\mathcal{S}_\mathbf{m}\cup\mathcal{S}_0)}w_k(\cdot,\lambda)x_0$ *is dense in* $\mathfrak{H}_{10}\cap\ker\mathcal{L}_{10}^*(\overline{\lambda})$*. Here* $x_0\in H$*;* $k=1,...,\Bbbk_1$ *if* $\Bbbk_1$ *is finite and* k *is any natural number if* $\Bbbk_1$ *is infinite.*

Proof. Suppose that $h_0\in\mathfrak{H}_{10}\cap\ker\mathcal{L}_{10}^*(\overline{\lambda})$ and

$$(h_0,\mathfrak{X}_{[a,b]\setminus\mathcal{S}_\mathbf{m}}w_k(\cdot,\lambda)x)_{\mathfrak{H}}=\int_a^b(d\mathbf{m}_0(s)h_0(s),\mathfrak{X}_{[a,b]\setminus\mathcal{S}_\mathbf{m}}w_k(s,\lambda)x)=0 \tag{53}$$

for any $x \in H$ and for all k. Let us prove that $h_0(t) = 0$ $\mathbf{m}_0$-almost everywhere. We set $h_0(t) = 0$ for $t \in \mathcal{S}_0$ and denote

$$y(t) = -\sum_{k=1}^{\Bbbk_1} w_k(t,\overline{\lambda}) iJ \int_{\alpha_k}^{t} w_k^*(s,\lambda) d\mathbf{m}_0(s) h_0(s). \tag{54}$$

We define the function h as follows. We put $h(t) = h_0(t)$ for $t \in [a,b] \setminus \mathcal{S}_{\mathbf{m}}$, and $h(\tau) = -(\Lambda_{\overline{\lambda}} y)(\tau)$ for $\tau \in \mathcal{S}_{\mathbf{m}} \setminus \mathcal{S}_0$. It follows from (24) for $\Delta = \{\tau\}$ that $\mathbf{n}_{\overline{\lambda}}(\{\tau\}) y(\tau) = \mathbf{m}(\{\tau\})(\Lambda_{\overline{\lambda}} y)(\tau)$. Therefore, $\mathbf{m}(\{\tau\}) h(\tau) + \mathbf{n}_{\overline{\lambda}}(\{\tau\}) y(\tau) = 0$. The function y will not change if h_0 is replaced by h in (54). Moreover, equality (53) will remain with this replacement. Then it follows from Lemma 4.15 that the pair $\{y,h\} \in \mathcal{L}_{10}(\overline{\lambda})$. Hence, $(h_0,h)_{\mathfrak{H}} = 0$ since $h_0 \in \ker \mathcal{L}_{10}^*(\overline{\lambda})$. On the other hand, $(h_0,h)_{\mathfrak{H}} = (h_0,h_0)_{\mathfrak{H}}$. This implies $h_0 = 0$. The lemma is proved. □

Lemma 4.18. *The linear span of functions of the form* $\mathfrak{X}_{[a,b]\setminus(\mathcal{S}_{\mathbf{m}} \cup \mathcal{S}_0)} w_k(\cdot,\lambda) x_0$ *and* $u_k(\cdot,\lambda,\tau) x_j + \mathfrak{X}_{\{\tau\}}(\cdot) x_j$ *is dense in* $\ker \mathcal{L}_{10}^*(\overline{\lambda})$. *Here* $x_0, x_j \in H$; $\tau \in (\alpha_k,\beta_k) \cap \mathcal{S}_{\mathbf{m}}$; $k = 1,\dots,\Bbbk_1$ *if* $\Bbbk_1$ *is finite and* k *is any natural number if* $\Bbbk_1$ *is infinite.*

Proof. Let $z \in \ker \mathcal{L}_{10}^*(\overline{\lambda})$. Then $z = z_0 + z_1$, where $z_0 \in \mathfrak{H}_{10}$, $z_1 \in \mathfrak{H}_{11}$. Suppose that the function z is orthogonal to the functions listed in the condition of this lemma. We claim that $z = 0$. Indeed, the pair $\{z_1, -\Lambda_\lambda z_1\} \in \mathcal{L}_{10}^*(\overline{\lambda})$ since $z_1 \in \ker L_{10}^*$ and (43) holds. Therefore, $\{z_0, \Lambda_\lambda z_1\} \in \mathcal{L}_{10}^*(\overline{\lambda})$. We denote $z_k = \mathfrak{X}_{[\alpha_k,\beta_k)} z$, $z_{0k} = \mathfrak{X}_{[\alpha_k,\beta_k)} z_0$, $z_{1k} = \mathfrak{X}_{[\alpha_k,\beta_k)} z_1$. Using Lemma 4.14, we get

$$z_{0k}(t) = -\mathfrak{X}_{[a,b]\setminus \mathcal{S}_{\mathbf{m}}} w_k(t,\lambda) iJ \int_a^t w_k^*(s,\overline{\lambda}) d\mathbf{m}(s)(\Lambda_\lambda z_{1k})(s) + h_0(t), \tag{55}$$

where $h_0 \in \ker \mathcal{L}_{10}^*(\overline{\lambda})$. Moreover, $h_0 \in \mathfrak{H}_{10}$ since $z_{0k} \in \mathfrak{H}_{10}$ and the first term in (55) belongs to $\mathfrak{H}_{10}$. It follows from Lemma 4.17 that h_0 belongs to the closure of the linear span of functions that have the form $\mathfrak{X}_{[\alpha_k,\beta_k)\setminus \mathcal{S}_{\mathbf{m}}} w_k(\cdot,\lambda) x'$, $x' \in H$. Using (49), (55), we obtain $z_k = \mathcal{U}_k(\lambda) z_{1k} + z_{1k} + h_0$. By assumption, $(z_k, \mathcal{U}_k(\lambda) z_{1k} + z_{1k})_{\mathfrak{H}} = 0$ and $(z_k, h_0)_{\mathfrak{H}} = 0$. Hence, $(z_k,z_k)_{\mathfrak{H}} = 0$ for all k. Therefore, $(z,z)_{\mathfrak{H}} = 0$. The lemma is proved. □

Remark 4.19. *Lemma 4.18 remains true if the functions of the form* $u_k(\cdot,\lambda,\tau) x_j + \mathfrak{X}_{\{\tau\}}(\cdot) x_j$ *are replaced by the functions* $u_k(\cdot,\lambda,\tau) B_\tau(\lambda) x_j + \mathfrak{X}_{\{\tau\}}(\cdot) B_\tau(\lambda) x_j$, *where* $B_\tau(\lambda)\colon H \to H$ *is a bounded continuously invertible operator,* $x_j \in H$.

Let $\widetilde{W}(t)$ be an operator solution of the equation

$$\widetilde{W}(t) x_0 = x_0 - iJ \int_a^t d\mathbf{p}_0(s) \widetilde{W}(s) x_0, \tag{56}$$

where $x_0 \in H$. We denote

$$\widetilde{w}_k(t) = \mathfrak{X}_{[\alpha_k,\beta_k)} \widetilde{W}(t) \widetilde{W}^{-1}(\alpha_k), \tag{57}$$

where $(\alpha_k,\beta_k) = \mathcal{J}_k \in \mathbb{J}$. The following lemma is proved in [15].

Lemma 4.20. *The linear span of functions of the form* $\mathfrak{X}_{[a,b]\setminus \mathcal{S}_{\mathbf{m}}} \widetilde{w}_k(\cdot) x_0$ *and* $\mathfrak{X}_{\{\tau\}}(\cdot) x_j$ *is dense in* $\ker L_{10}^*$. *Here* $x_0, x_j \in H$; $\tau \in (\alpha_k,\beta_k) \cap \mathcal{S}_{\mathbf{m}}$; $k = 1,\dots,\Bbbk_1$ *if* $\Bbbk_1$ *is finite and* k *is any natural number if* $\Bbbk_1$ *is infinite.*

Let $\mathbb{M}$ be a set consisting of intervals $\mathcal{J} \in \mathbb{J}$ and single-point sets $\{\tau\}$, where $\tau \in \mathcal{S}_{\mathbf{m}} \setminus \mathcal{S}_0$. The set $\mathbb{M}$ is at most countable. Let $\Bbbk$ be the number of elements in $\mathbb{M}$. We arrange the elements of $\mathbb{M}$ in the form of a finite or infinite sequence and denote these elements by $\mathcal{E}_k$, where k is any natural number if the number of elements in $\mathbb{M}$ is infinite, and $1 \leqslant k \leqslant \Bbbk$ if the number of elements in $\mathbb{M}$ is finite.

To each element $\mathcal{E}_k \in \mathbb{M}$ assign operator functions ϑ_k, $\widetilde{\vartheta}_k$ in the following way. If $\mathcal{E}_k$ is the interval, $\mathcal{E}_k = \mathcal{J}_k = (\alpha_k,\beta_k) \in \mathbb{J}$, then

$$\vartheta_k(t,\lambda) = \mathfrak{X}_{[\alpha_k,\beta_k)\setminus \mathcal{S}_{\mathbf{m}}} w_k(t,\lambda), \quad \widetilde{\vartheta}_k(t) = \mathfrak{X}_{[\alpha_k,\beta_k)\setminus \mathcal{S}_{\mathbf{m}}} \widetilde{w}_k(t). \tag{58}$$

If $\mathcal{E}_k$ is the single-point set, $\mathcal{E}_k = \{\tau_k\}$, then

$$\vartheta_k(t,\lambda) = (U_n(\lambda)\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}})(t) + \mathfrak{X}_{\{\tau_k\}}(t)(\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}})(t), \quad \widetilde{\vartheta}_k(t) = \mathfrak{X}_{\{\tau_k\}}(t), \tag{59}$$

where $\tau_k \in (\mathcal{S}_\mathbf{m} \setminus \mathcal{S}_0) \cap \mathcal{J}_n$, $\mathcal{J}_n = (\alpha_n, \beta_n) \in \mathbb{J}$, $n = 1, ..., \mathbb{k}_1$ if $\mathbb{k}_1$ is finite and k is any natural number if $\mathbb{k}_1$ is infinite. In case (59), using (49) and Remark 3.3, we get

$$\vartheta_k(t,\lambda)x = -\mathfrak{X}_{[a,b]\setminus(\mathcal{S}_\mathbf{m}\cup\mathcal{S}_0)} w_n(t,\lambda) iJ \int_a^t w_n^*(s,\overline{\lambda}) d\mathbf{m}(s)(\mathfrak{X}_{\{\tau_k\}}x)(s) + (\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}x)(t). \tag{60}$$

Remark 4.21. *It follows from* (44),(57) *that equalities* (58) *are equivalent to the following:* $\vartheta_k(t,\lambda) = \mathfrak{X}_{[a,b]\setminus\mathcal{S}_\mathbf{m}} w_k(t,\lambda)$, $\widetilde{\vartheta}_k(t) = \mathfrak{X}_{[a,b]\setminus\mathcal{S}_\mathbf{m}} \widetilde{w}_k(t)$.

Lemma 4.22. *The linear span of the functions* $t \to \vartheta_k(t,\lambda)\xi_j$ ($t \to \widetilde{\vartheta}_k(t)\xi_j$) *is dense in* $\ker \mathcal{L}_{10}^*(\overline{\lambda})$ (*in* $\ker L_{10}^*$, *respectively*). (*Here* $\xi_j \in H$, $k \in \mathbb{N}$ *if* $\mathbb{k} = \infty$, *and* $1 \leqslant k \leqslant \mathbb{k}$ *if* $\mathbb{k}$ *is finite.*)

Proof. The required statement follows from Remark 4.19 and Lemmas 4.18, 4.20 immediately. □

Corollary 4.23. *A function* $f \in \mathfrak{H}_1$ *belongs to the range* $\mathcal{R}(\mathcal{L}_{10}(\lambda))$ (*the range* $\mathcal{R}(L_{10})$) *if and only if the equality* $(f, \vartheta_k(\cdot,\overline{\lambda})\xi_j)_\mathfrak{H} = 0$ (*the equality* $(f, \widetilde{\vartheta}_k(\cdot)\xi_j)_\mathfrak{H} = 0$, *respectively*) *holds for all k and all for* $\xi_j \in H$. (*Here* $k \in \mathbb{N}$ *if* $\mathbb{k} = \infty$, *and* $1 \leqslant k \leqslant \mathbb{k}$ *if* $\mathbb{k}$ *is finite.*)

Proof. The proof follows from the equalities $\mathcal{R}(\mathcal{L}_{10}(\lambda)) \oplus \ker \mathcal{L}_{10}^*(\lambda) = \mathfrak{H}_1$, $\mathcal{R}(L_{10}) \oplus \ker L_{10}^* = \mathfrak{H}_1$ and Lemma 4.22. □

Lemma 4.24. *There exist constants* $\gamma_{1k} = \gamma_{1k}(\lambda)$, $\gamma_{2k} = \gamma_{2k}(\lambda) > 0$ *such that the inequality*

$$\gamma_{1k} \left\| \widetilde{\vartheta}_k(\cdot)x \right\|_\mathfrak{H} \leqslant \|\vartheta_k(\cdot,\lambda)x\|_\mathfrak{H} \leqslant \gamma_{2k} \left\| \widetilde{\vartheta}_k(\cdot)x \right\|_\mathfrak{H} \tag{61}$$

holds for all $x \in H$.

Proof. Using (13), (56), and Lemma 2.3, we obtain

$$W(t,\lambda)x_0 = \widetilde{W}(t)x_0 - \widetilde{W}(t)iJ \int_a^t \widetilde{W}^*(s) d\mathbf{n}_{0\lambda}(s) W(s,\lambda)x_0, \quad x_0 \in H, \tag{62}$$

$$\widetilde{W}(t)x_0 = W(t,\lambda)x_0 + W(t,\lambda)iJ \int_a^t W^*(s,\lambda) d\mathbf{n}_{0\lambda}(s) \widetilde{W}(s)x_0, \quad x_0 \in H. \tag{63}$$

Suppose that ϑ_k, $\widetilde{\vartheta}_k$ have form (58). It follows from (26), (44), (57), (62), (63) that

$$\vartheta_k(t,\lambda)x_0 = \widetilde{\vartheta}_k(t)x_0 - \widetilde{\vartheta}_k(t)iJ \int_{\alpha_k}^t \widetilde{\vartheta}_k^*(s) d\mathbf{m}_0(s)(\Lambda_\lambda \vartheta_k(\cdot,\lambda)x_0)(s), \quad x_0 \in H, \tag{64}$$

$$\widetilde{\vartheta}_k(t)x_0 = \vartheta_k(t,\lambda)x_0 + \vartheta_k(t,\lambda)iJ \int_{\alpha_k}^t \vartheta_k^*(s,\lambda) d\mathbf{m}_0(s)(\Lambda_\lambda \widetilde{\vartheta}_k x_0)(s), \quad x_0 \in H. \tag{65}$$

Equalities (15), (64), (65) imply (61) in the case when ϑ_k, $\widetilde{\vartheta}_k$ have form (58).

Suppose that ϑ_k, $\widetilde{\vartheta}_k$ have form (59). It follows from Remark 3.3 and (49) that

$$\|\vartheta_k(\cdot,\lambda)x\|_\mathfrak{H}^2 = \left\| U_n(\lambda)\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}x \right\|^2 + \left\| \Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}x \right\|_\mathfrak{H}^2 \geqslant \left\| \Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}x \right\|_\mathfrak{H}^2 \geqslant \gamma_3 \left\| \widetilde{\vartheta}_k(\cdot)x \right\|_\mathfrak{H}^2,$$

where $\gamma_3 = \gamma_3(\lambda) > 0$. On the other hand, using Remark 3.3 and (49), we obtain

$$\|\vartheta_k(\cdot,\lambda)x\|_\mathfrak{H} \leqslant \left\| U_n(\lambda)\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}x \right\|_\mathfrak{H} + \left\| \mathfrak{X}_{\{\tau_k\}}\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}x \right\|_\mathfrak{H} \leqslant \gamma_4 \left\| \Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}x \right\|_\mathfrak{H} \leqslant \gamma_5 \left\| \widetilde{\vartheta}_k(\cdot)x \right\|_\mathfrak{H},$$

where $\gamma_4 = \gamma_4(\lambda) > 0$, $\gamma_5 = \gamma_5(\lambda) > 0$. The lemma is proved. □

Let $Q_{k,0}$ be a set $x \in H$ such that the functions $t \to \widetilde{\vartheta}_k(t)x$ are identical with zero in $\mathfrak{H}$. We put $Q_k = H \ominus Q_{k,0}$. On the linear space Q_k, we introduce a norm $\|\cdot\|_-$ by the equality

$$\|\xi_k\|_- = \left\|\widetilde{\vartheta}_k(\cdot)\xi_k\right\|_{\mathfrak{H}}, \quad \xi_k \in Q_k. \tag{66}$$

By Q_k^- denote the completion of Q_k with respect to norm (66). This norm (66) is generated by the scalar product $(\xi_k, \eta_k)_- = (\widetilde{\vartheta}_k(\cdot)\xi_k, \widetilde{\vartheta}_k(\cdot)\eta_k)_{\mathfrak{H}}$, where $\xi_k, \eta_k \in Q_k$. From formula (3) in which the measure $\mathbf{P}$ is replaced by $\mathbf{m}$, it follows that

$$\|\xi_k\|_- \leqslant \gamma \|\xi_k\|, \quad \xi_k \in Q_k, \tag{67}$$

where $\gamma > 0$ is independent of $\xi_k \in Q_k$. It follows from (67) that the space Q_k^- can be treated as a space with a negative norm with respect to Q_k ([5, ch. 1], [19, ch.2]). By Q_k^+ denote the associated space with a positive norm. The definition of spaces with positive and negative norms implies that $Q_k^+ \subset Q_k \subset Q_k^-$. By $(\cdot,\cdot)_+$ and $\|\cdot\|_+$ we denote the scalar product and the norm in Q_k^+, respectively.

Remark 4.25. *By* (61)*, it follows that the set $Q_{k,0}$ will not change if the function $\widetilde{\vartheta}_k(\cdot)$ is replaced by $\vartheta_k(\cdot,\lambda)$ in the definition of $Q_{k,0}$. Moreover, with this replacement,the space Q_k^- will not change in the following sense: the set Q_k^- will not change, and the norm in it will be replaced by the equivalent one. The similar statement holds for the space Q_k^+.*

Suppose that a sequence $\{x_{kn}\}$, $x_{kn} \in Q_k$, converges in the space Q_k^- to $x_0 \in Q_k^-$ as $n \to \infty$. It follows from Lemma 4.24 that a sequence $\{\vartheta_k(\cdot,\lambda)x_{kn}\}$ is fundamental in $\mathfrak{H}$. Therefore, this sequence converges to some element in $\mathfrak{H}$. By $\vartheta_k(\cdot,\lambda)x_0$ we denote this element.

Let $\widetilde{Q}_N^- = Q_1^- \times \dots \times Q_N^-$ ($\widetilde{Q}_N^+ = Q_1^+ \times \dots \times Q_N^+$) be the Cartesian product of the first N sets Q_k^- (Q_k^+, respectively) and let $V_N(t,\lambda) = (\vartheta_1(t,\lambda), \dots, \vartheta_N(t,\lambda))$ be the operator one-row matrix. It is convenient to treat elements from $\widetilde{Q}_N^-$ as one-column matrices and to assume that $V_N(t,\lambda)\widetilde{\xi}_N = \sum_{k=1}^N \vartheta_k(t,\lambda)\xi_k$, where we denote $\widetilde{\xi}_N = \mathrm{col}(\xi_1, \dots, \xi_N) \in \widetilde{Q}_N^-$, $\xi_k \in Q_k^-$.

Let $\ker_k(\lambda)$ be a linear space of functions $t \to \vartheta_k(t,\lambda)\xi_k$, $\xi_k \in Q_k^-$. By (66) and Lemma 4.24, it follows that $\ker_k(\lambda)$ is closed in $\mathfrak{H}$. We denote $\mathcal{K}_N(\lambda) = \ker_1(\lambda) \dotplus \dots \dotplus \ker_N(\lambda)$. Obviously, $\mathcal{K}_{N_1}(\lambda) \subset \mathcal{K}_{N_2}(\lambda)$ for $N_1 < N_2$.

Lemma 4.26. *The set $\cup_N \mathcal{K}_N(\lambda)$ is dense in* $\ker \mathcal{L}_{10}^*(\overline{\lambda})$.

Proof. The required statement follows from Lemma 4.22 immediately. □

By $\mathbf{V}_N(\lambda)$ denote the operator $\widetilde{\xi}_N \to V_N(\cdot,\lambda)\widetilde{\xi}_N$, where $\widetilde{\xi}_N \in \widetilde{Q}_N^-$. The operator $\mathbf{V}_N(\lambda)$ maps continuously and one-to-one $\widetilde{Q}_N^-$ onto $\mathcal{K}_N(\lambda) \subset \mathfrak{H}_1 \subset \mathfrak{H}$. Hence the adjoint operator $\mathbf{V}_N^*(\lambda)$ maps $\mathfrak{H}$ onto $\widetilde{Q}_N^+$ continuously. We find the form of the operator $\mathbf{V}_N^*$. For all $\widetilde{\xi}_N \in \widetilde{Q}_N = Q_1 \times \dots Q_N$, $f \in \mathfrak{H}$, we have

$$(f, \mathbf{V}_N(\lambda)\,\widetilde{\xi}_N)_{\mathfrak{H}} = \int_a^{b_0} (d\mathbf{m}(s)f(s), V_N(s,\lambda)\widetilde{\xi}_N) = \int_a^{b_0} (V_N^*(s,\lambda)d\mathbf{m}(s)f(s), \widetilde{\xi}_N) = (\mathbf{V}_N^*(\lambda)f, \widetilde{\xi}_N).$$

Since $\widetilde{Q}_N$ is dense in $\widetilde{Q}_N^-$, we obtain

$$\mathbf{V}_N^*(\lambda)f = \int_a^{b_0} V_N^*(s,\lambda)d\mathbf{m}(s)f(s). \tag{68}$$

Thus, we have proved the following statement.

Lemma 4.27. *The operator $\mathbf{V}_N(\lambda)$ maps continuously and one-to-one $\widetilde{Q}_N^-$ onto $\mathcal{K}_n(\lambda)$. The adjoint operator $\mathbf{V}_N^*(\lambda)$ maps continuously $\mathfrak{H}$ onto $\widetilde{Q}_N^+$ and acts by formula* (68)*. Moreover, $\mathbf{V}_N^*(\lambda)$ maps one-to-one $\mathcal{K}_N(\lambda)$ onto $\widetilde{Q}_N^+$.*

Let Q_-, Q_+, Q be linear spaces of sequences, respectively, $\widetilde{\eta} = \{\eta_k\}$, $\widetilde{\varphi} = \{\varphi_k\}$, $\widetilde{\xi} = \{\xi_k\}$, where $\eta_k \in Q_k^-$, $\varphi_k \in Q_k^+$, $\xi_k \in Q_k$; $k \in \mathbb{N}$ if $\mathbb{k} = \infty$, and $1 \leqslant k \leqslant \mathbb{k}$ if $\mathbb{k}$ is finite; $\mathbb{k}$ is the number of elements in $\mathbb{M}$. We assume that the series $\sum_{k=1}^{\infty} \|\eta_k\|_-^2$, $\sum_{k=1}^{\infty} \|\varphi_k\|_+^2$, $\sum_{k=1}^{\infty} \|\xi_k\|^2$ converge if $\mathbb{k} = \infty$. These spaces become Hilbert spaces if we introduce scalar products by the formulas

$$(\widetilde{\eta}, \widetilde{\zeta})_- = \sum_{k=1}^{\mathbb{k}} (\eta_k, \zeta_k)_-, \quad \widetilde{\eta}, \widetilde{\zeta} \in Q_-; \quad (\widetilde{\varphi}, \widetilde{\psi})_+ = \sum_{k=1}^{\mathbb{k}} (\varphi_k, \psi_k)_+, \quad \widetilde{\varphi}, \widetilde{\psi} \in Q_+; \quad (\widetilde{\xi}, \widetilde{\sigma}) = \sum_{k=1}^{\mathbb{k}} (\xi_k, \sigma_k), \quad \widetilde{\xi}, \widetilde{\sigma} \in Q.$$

In these spaces, the norms are defined by the equalities

$$\|\widetilde{\eta}\|_-^2 = \sum_{k=1}^{\mathbb{k}} \|\eta_k\|_-^2, \quad \|\widetilde{\varphi}\|_+^2 = \sum_{k=1}^{\mathbb{k}} \|\varphi_k\|_+^2, \quad \left\|\widetilde{\xi}\right\|^2 = \sum_{k=1}^{\mathbb{k}} \|\xi_k\|^2 .$$

The spaces Q_+, Q_- can be treated as spaces with positive and negative norms with respect to Q ([5, ch. 1], [19, ch.2]). So, $Q_+ \subset Q \subset Q_-$ and $\gamma_1 \|\widetilde{\varphi}\|_- \leqslant \|\widetilde{\varphi}\| \leqslant \gamma_2 \|\widetilde{\varphi}\|_+$, where $\widetilde{\varphi} \in Q_+$, $\gamma_1, \gamma_2 > 0$. The "scalar product" $(\widetilde{\eta}, \widetilde{\varphi})$ is defined for all $\widetilde{\varphi} \in Q_+$, $\widetilde{\eta} \in Q_-$. If $\widetilde{\eta} \in Q$, then $(\widetilde{\eta}, \widetilde{\varphi})$ coincides with the scalar product in Q.

Let $\mathcal{M} \subset Q_-$ be a set of sequences vanishing starting from a certain number (its own for each sequence). The set $\mathcal{M}$ is dense in the space Q_-. The operator $\mathbf{V}_N(\lambda)$ is the restriction of $\mathbf{V}_{N+1}(\lambda)$ to $\widetilde{Q}_N^-$. By $\mathbf{V}'(\lambda)$ denote an operator in $\mathcal{M}$ such that $\mathbf{V}'(\lambda)\widetilde{\eta} = \mathbf{V}_N(\lambda)\widetilde{\eta}_N$ for all $N \in \mathbb{N}$, where $\widetilde{\eta} = (\widetilde{\eta}_N, 0, \ldots)$, $\widetilde{\eta}_N \in \widetilde{Q}_N^-$. It follows from (66), (61) that $\mathbf{V}'(\lambda)$ admits an extension by continuity to the space Q_-. By $\mathbf{V}(\lambda)$ denote the extended operator. This operator maps continuously and one-to-one Q_- onto $\ker \mathcal{L}_{10}^*(\overline{\lambda}) \subset \mathfrak{H}_1 \subset \mathfrak{H}$. Moreover, we denote $\widetilde{\mathrm{V}}(t, \lambda)\widetilde{\eta} = (\mathbf{V}(\lambda)\widetilde{\eta})(t)$, where $\widetilde{\eta} = \{\eta_k\} \in Q_-$.

The adjoint operator $\mathbf{V}^*(\lambda)$ maps continuously $\mathfrak{H}$ onto Q_+. Let us find the form of $\mathbf{V}^*(\lambda)$. Suppose $f \in \mathfrak{H}$, $\widetilde{\eta} \in \mathcal{M}$, $\widetilde{\eta} = \{\widetilde{\eta}_N, 0, \ldots\}$. Then

$$(\widetilde{\eta}, \mathbf{V}^*(\lambda) f) = (\mathbf{V}(\lambda)\widetilde{\eta}, f)_{\mathfrak{H}} = \int_a^{b_0} (d\mathbf{m}(t)\widetilde{\mathrm{V}}(t, \lambda)\widetilde{\eta}, f(t)) = \int_a^{b_0} (\widetilde{\eta}, \widetilde{\mathrm{V}}^*(t, \lambda) d\mathbf{m}(t) f(t)).$$

Since $\mathbf{V}^*(\lambda) f \in Q_+$ and the set $\mathcal{M}$ is dense in Q_-, we get

$$\mathbf{V}^*(\lambda) f = \int_a^{b_0} \widetilde{\mathrm{V}}^*(t, \lambda) d\mathbf{m}(t) f(t). \tag{69}$$

Taking into account Lemmas 4.26,4.27, we obtain the following statement.

Lemma 4.28. *The operator $\mathbf{V}(\lambda)$ maps Q_- onto $\ker \mathcal{L}_{10}^*(\overline{\lambda})$ continuously and one to one. A function z belongs to $\ker \mathcal{L}_{10}^*(\overline{\lambda})$ if and only if there exists an element $\widetilde{\eta} = \{\eta_k\} \in Q_-$ such that $z(t) = (\mathbf{V}(\lambda)\widetilde{\eta})(t) = \widetilde{\mathrm{V}}(t, \lambda)\widetilde{\eta}$. The operator $\mathbf{V}^*(\lambda)$ maps $\mathfrak{H}$ onto Q_+ continuously, and acts by formula (69), and $\ker \mathbf{V}^*(\lambda) = \mathfrak{H}_0 \oplus \mathcal{R}(\mathcal{L}_{10}(\overline{\lambda}))$. Moreover, $\mathbf{V}^*(\lambda)$ maps $\ker \mathcal{L}_{10}^*(\overline{\lambda})$ onto Q_+ one to one.*

The following theorem is proved in [15] for the relation $L_0^* - \lambda E$. We have changed some designations from [15] to shorten the record.

Theorem 4.29. *A pair $\{\widetilde{y}, \widetilde{f}\} \in \mathfrak{H} \times \mathfrak{H}$ belongs to $\mathcal{L}_0^*(\overline{\lambda})$ if and only if there exist a pair $\{\widehat{y}, \widehat{f}\} \in \mathfrak{H} \times \mathfrak{H}$, functions $y_0, y_0' \in \mathfrak{H}_0$, $y, f \in \mathfrak{H}_1$, and an element $\widetilde{\eta} \in Q_-$ such that the pairs $\{\widetilde{y}, \widetilde{f}\}$, $\{\widehat{y}, \widehat{f}\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$ and the equalities*

$$\widehat{y} = y_0 + y, \ \widehat{f} = y_0' + f, \tag{70}$$

$$y(t) = \widetilde{\mathrm{V}}(t, \lambda)\widetilde{\eta} - \sum_{k=1}^{\mathbb{k}_1} \mathfrak{X}_{[a,b]\setminus \mathcal{S}_{\mathrm{m}}} w_k(t, \lambda) iJ \int_a^t w_k^*(s, \overline{\lambda}) d\mathbf{m}(s) f(s) \tag{71}$$

hold, where the series in (71) converges in $\mathfrak{H}_1$, $\mathbb{k}_1$ is the number of intervals $\mathcal{J}_k \in \mathbb{J}$.

Proof. Equalities (70) follow from (42). Let us prove that equality (71) holds. It follows from Lemma 4.28 that $\mathbf{V}(\lambda)\widetilde{\eta} \in \ker \mathcal{L}_{10}^*(\overline{\lambda})$. We prove that if the functions y, f satisfy equality (71), then the pair $\{y, f\} \in \mathcal{L}_{10}^*(\overline{\lambda})$. If $\Bbbk_1$ is finite, then this statement follows from Lemmas 4.14, 4.28. We assume that $\Bbbk_1 = \infty$ and first prove that the series in (71) converges in $\mathfrak{H}_1$ for each function $f \in \mathfrak{H}_1$. We denote $f_k(t) = \mathfrak{X}_{[\alpha_k,\beta_k)} f(t)$. The function

$$v_k(t) = -\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,\lambda) iJ \int_a^t w_k^*(s,\overline{\lambda}) d\mathbf{m}(s) f_k(s) = -\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,\lambda) iJ \int_{\alpha_k}^t w_k^*(s,\overline{\lambda}) \Psi_{\mathbf{m}}(s) f_k(s) d\rho_{\mathbf{m}}(s) \tag{72}$$

vanishes outside the interval $[\alpha_k, \beta_k)$. (Here $\Psi_{\mathbf{m}}$, $\rho_{\mathbf{m}}$ are functions from formula (3) in which the measure $\mathbf{P}$ is replaced by $\mathbf{m}$.) Using (72), (15), (3), we get

$$\|v_k(t)\| \leqslant \varepsilon_1 \|w_k(t,\lambda)\| \int_{\alpha_k}^{\beta_k} \left\|w_k^*(s,\overline{\lambda})\right\| \left\|\Psi_{\mathbf{m}}^{1/2}(s) f_k(s)\right\| d\rho_{\mathbf{m}}(s) \leqslant \varepsilon \left(\int_{\alpha_k}^{\beta_k} \left\|\Psi_{\mathbf{m}}^{1/2}(s) f_k(s)\right\|^2 d\rho_{\mathbf{m}}(s) \right)^{1/2} = \varepsilon \left\|f_k\right\|_{\mathfrak{H}}, \ \varepsilon_1, \varepsilon > 0.$$

This implies

$$\|v_k\|_{\mathfrak{H}}^2 = \int_{\alpha_k}^{\beta_k} (\Psi_{\mathbf{m}}(t) v_k(t), v_k(t)) d\rho_{\mathbf{m}}(t) \leqslant \varepsilon^2 \rho_{\mathbf{m}}([\alpha_k, \beta_k)) \left\|f_k\right\|_{\mathfrak{H}}^2. \tag{73}$$

We denote $S_n(t) = \sum_{k=1}^n v_k(t)$ and prove that the sequence $\{S_n\}$ converges in $\mathfrak{H}_1$. From (73), we obtain

$$\|S_n\|_{\mathfrak{H}}^2 = \sum_{k=1}^n \|v_k\|_{\mathfrak{H}}^2 \leqslant \varepsilon^2 \sum_{k=1}^n \rho_{\mathbf{m}}([\alpha_k, \beta_k)) \left\|f_k\right\|_{\mathfrak{H}}^2 \leqslant \varepsilon^2 \rho_{\mathbf{m}}([a,b]) \left\|f\right\|_{\mathfrak{H}}^2.$$

Hence the sequence $\{S_n\}$ converges to some function $S \in \mathfrak{H}_1$ and

$$S(t) = -\sum_{k=1}^{\infty} \mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,\lambda) iJ \int_a^t w_k^*(s,\overline{\lambda}) d\mathbf{m}(s) f(s), \quad \|S\|_{\mathfrak{H}_1} \leqslant \varepsilon_2 \left\|f\right\|_{\mathfrak{H}_1}, \ \ \varepsilon_2 > 0. \tag{74}$$

It follows from Lemma 4.14 that the pair $\{S_n, \sum_{k=1}^n f_k\} \in \mathcal{L}_{10}^*(\overline{\lambda})$. The relation $\mathcal{L}_{10}^*(\overline{\lambda})$ is closed. Consequently, the pair $\{S, f\} \in \mathcal{L}_{10}^*(\overline{\lambda})$ and the pair $\{y, f\} \in \mathcal{L}_{10}^*(\overline{\lambda})$, where $y = \mathbf{V}(\lambda)\widetilde{\eta} + S$.

Now we assume that a pair $\{y, f\} \in \mathcal{L}_{10}^*(\overline{\lambda})$. For the function f, we find a function S by formula (74). Then $\{S, f\} \in \mathcal{L}_{10}^*(\overline{\lambda})$. Hence $y - S \in \ker \mathcal{L}_{10}^*(\overline{\lambda})$. By Lemma 4.28, it follows that there exists an element $\widetilde{\eta} \in \mathcal{Q}_-$ such that $y - S = \mathbf{V}(\lambda)\widetilde{\eta}$. Therefore, y has form (71). Now (42) implies the desired assertion. The theorem is proved. □

5. Continuously invertible extensions of the relation $\mathcal{L}_{10}(\lambda)$

Lemma 5.1. *Equality* (71) *holds if and only if*

$$y(t) = \widetilde{\mathrm{V}}(t,\lambda)\widetilde{\zeta} + 2^{-1} \sum_{k=1}^{\Bbbk_1} \left[-\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,\lambda) iJ \int_{\alpha_k}^t w_k^*(s,\overline{\lambda}) d\mathbf{m}_0(s) f(s) - (\Lambda_\lambda^{-1} \mathfrak{X}_{\mathcal{S}_{\mathbf{m}} \cap (\alpha_k,\beta_k)} f)(t) \right] +$$
$$+ 2^{-1} \sum_{k=1}^{\Bbbk_1} \left[\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,\lambda) iJ \int_t^{\beta_k} w_k^*(s,\overline{\lambda}) d\mathbf{m}_0(s) f(s) - (\Lambda_\lambda^{-1} \mathfrak{X}_{\mathcal{S}_{\mathbf{m}} \cap (\alpha_k,\beta_k)} f)(t) \right], \tag{75}$$

where $\widetilde{\zeta} \in \mathcal{Q}_-$; $y, f \in \mathfrak{H}_1$.

Proof. By standard transformations, equality (71) is reduced to the form

$$y(t)=\widetilde{\mathrm{V}}(t,\lambda)\widetilde{\theta}-2^{-1}\sum_{k=1}^{\Bbbk_1}\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}w_k(t,\lambda)iJ\int_a^t w_k^*(s,\overline{\lambda})d\mathbf{m}(s)f(s)+$$

$$+2^{-1}\sum_{k=1}^{\Bbbk_1}\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}w_k(t,\lambda)iJ\int_t^b w_k^*(s,\overline{\lambda})d\mathbf{m}(s)f(s), \quad (76)$$

where $\widetilde{\theta}=\{\theta_k\}\in\mathcal{Q}_-$, and $\theta_k=\eta_k$ if ϑ_k has form (60), and $\theta_k=\eta_k-2^{-1}iJ\int_{\alpha_k}^{\beta_k}w_k^*(s,\overline{\lambda})d\mathbf{m}(s)f(s)$ if ϑ_k has form (58).

Let us write the function

$$\mathfrak{w}_k(t,\lambda)=-\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}w_k(t,\lambda)iJ\int_a^t w_k^*(s,\overline{\lambda})d\mathbf{m}(s)f(s) \quad (77)$$

in a different form. Using (77), (49), and Remark 3.3, we get

$$\mathfrak{w}_k(t,\lambda)=-\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}w_k(t,\lambda)iJ\int_{\alpha_k}^t w_k^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s)-\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}w_k(t,\lambda)iJ\int_{\alpha_k}^t w_k^*(s,\overline{\lambda})d\mathbf{m}(s)\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}(s)f(s)=$$

$$=-\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}w_k(t,\lambda)iJ\int_{\alpha_k}^t w_k^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s)-(\Lambda_\lambda^{-1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}f)(t)+$$

$$+[(\Lambda_\lambda^{-1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}f)(t)+\mathcal{U}_k(\lambda)(\Lambda_\lambda^{-1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}f)(t)].$$

We denote

$$\mathfrak{r}_k=\Lambda_\lambda^{-1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}f+\mathcal{U}_k(\lambda)\Lambda_\lambda^{-1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}f.$$

Using (27), (52), we get $\mathfrak{r}_k\in\ker\mathcal{L}_{10}^*(\overline{\lambda})$. Therefore,

$$\mathfrak{w}_k(t,\lambda)=-\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}w_k(t,\lambda)iJ\int_{\alpha_k}^t w_k^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s)-(\Lambda_\lambda^{-1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}f)(t)+\mathfrak{r}_k(t). \quad (78)$$

Similarly, we transform the function

$$\widehat{\mathfrak{w}}_k(t,\lambda)=\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}w_k(t,\lambda)iJ\int_t^b w_k^*(s,\overline{\lambda})d\mathbf{m}(s)f(s)$$

to the form

$$\widetilde{\mathfrak{w}}_k(t,\lambda)=\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}w_k(t,\lambda)iJ\int_t^{\beta_k} w_k^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s)-(\Lambda_\lambda^{-1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}f)(t)+$$

$$+[(\Lambda_\lambda^{-1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}f)(t)+\mathcal{U}_k(\lambda)(\Lambda_\lambda^{-1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}f)(t)]+\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}w_k(t,\lambda)iJ\int_{\alpha_k}^{\beta_k} w_k^*(s,\overline{\lambda})d\mathbf{m}(s)\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}(s)f(s).$$

By Lemma 4.18, Remark 4.19, and (27), (52), it follows that here the sum of the last three terms belongs to $\ker\mathcal{L}_{10}^*(\overline{\lambda})$. Consequently,

$$\widetilde{\mathfrak{w}}_k(t,\lambda)=-\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}w_k(t,\lambda)iJ\int_t^{\beta_k} w_k^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s)-(\Lambda_\lambda^{-1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}f)(t)+\widetilde{\mathfrak{r}}_k(t), \quad (79)$$

where $\widetilde{\mathfrak{r}}_k\in\ker\mathcal{L}_{10}^*(\overline{\lambda})$. Now the desired statement follows from (76), (78), (79), and Lemma 4.28. The lemma is proved. □

Theorem 5.2. *Let $T(\lambda)$ be a linear relation such that $\mathcal{L}_{10}(\lambda)\subset T(\lambda)\subset\mathcal{L}^*_{10}(\overline{\lambda})$. The relation $T(\lambda)$ is continuously invertible in the space $\mathfrak{H}_1$ if and only if there exists a bounded operator $M(\lambda):Q_+\to Q_-$ such that the following equality holds for any pair $\{y,f\}\in T(\lambda)$ and for any λ, $\mathrm{Im}\lambda\neq 0$,*

$$y(t)=\int_a^b\widetilde{\mathrm{V}}(t,\lambda)M(\lambda)\widetilde{\mathrm{V}}^*(s,\overline{\lambda})d\mathbf{m}(s)f(s)+$$
$$+2^{-1}\sum_{k=1}^{\Bbbk_1}\int_a^b\mathfrak{X}_{[\alpha_k,\beta_k)\setminus\mathcal{S}_\mathbf{m}}(t)w_k(t,\lambda)\mathrm{sgn}(s-t)iJw_k^*(s,\overline{\lambda})d\mathbf{m}(s)\mathfrak{X}_{[\alpha_k,\beta_k)\setminus\mathcal{S}_\mathbf{m}}(s)f(s)-\sum_{k=1}^{\Bbbk_1}(\Lambda_\lambda^{-1}\mathfrak{X}_{\mathcal{S}_\mathbf{m}\cap(\alpha_k,\beta_k)}f)(t).\quad(80)$$

Proof. First note that the range $\mathcal{R}(\mathcal{L}_{10}(\lambda))$ is closed and $\ker\mathcal{L}_{10}(\lambda)=\{0\}$. This follows from Lemma 4.4 and Corollary 4.6. Suppose that the relation $T^{-1}(\lambda)$ is a boundary everywhere defined operator and $y=T^{-1}(\lambda)f$. Then y has form (75). In this equality, $\widetilde{\zeta}\in Q_-$ is uniquely determined by f and λ, i.e., $\widetilde{\zeta}=\widetilde{\zeta}(f,\lambda)$. Indeed, if $f=0$, then $\widetilde{\mathrm{V}}(t,\lambda)\widetilde{\zeta}=T^{-1}(\lambda)0=0$. It follows from Lemma 4.28 that $\widetilde{\zeta}=0$. Moreover, $\widetilde{\zeta}$ depends on f linearly. Consequently, $\widetilde{\zeta}=S(\lambda)f$, where $S(\lambda):\mathfrak{H}_1\to Q_-$ is a linear operator for fixed λ. We claim that the operator $S(\lambda)$ is bounded. Indeed, if a sequence $\{f_n\}$ converges to zero in the space $\mathfrak{H}_1$ as $n\to\infty$, then the sequence $\{y_n\}$=$\{T^{-1}(\lambda)f_n\}$ converges to zero in $\mathfrak{H}_1$. Hence the sequence $\{\mathbf{V}(\lambda)\widetilde{\zeta}_n\}$ (where $\widetilde{\zeta}_n=S(\lambda)f_n$) converges to zero in $\mathfrak{H}_1$. By Lemma 4.28, it follows that the sequence $\{S(\lambda)f_n\}$ converges to zero in the space Q_-. Therefore, $S(\lambda)$ is the bounded operator.

Now we prove that $\widetilde{\zeta}(f,\lambda)$ is uniquely determined by the element $\mathbf{V}^*(\overline{\lambda})f\in Q_+$. Suppose $\mathbf{V}^*(\overline{\lambda})f=0$. The application of Lemma 4.28 and Corollary 4.16 yields that $f\in\mathcal{R}(\mathcal{L}_{10}(\lambda))$ and $\int_{\alpha_k}^{\beta_k}w_k^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s)=0$. Therefore, the second and third terms coincide in equality (75). Then equality (75) takes the form

$$y(t)=\widetilde{\mathrm{V}}(t,\lambda)\widetilde{\zeta}+\sum_{k=1}^{\Bbbk_1}\left[-\mathfrak{X}_{[a,b]\setminus\mathcal{S}_\mathbf{m}}w_k(t,\lambda)iJ\int_{\alpha_k}^t w_k^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s)-(\Lambda_\lambda^{-1}\mathfrak{X}_{\mathcal{S}_\mathbf{m}\cap(\alpha_k,\beta_k)}f)(t)\right].\quad(81)$$

We denote

$$u_k(t)=-w_k(t,\lambda)iJ\int_{\alpha_k}^t w_k^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s)$$

and continue equality (81)

$$y(t)=\widetilde{\mathrm{V}}(t,\lambda)\widetilde{\zeta}+\sum_{k=1}^{\Bbbk_1}\left[u_k(t)-\left(\mathfrak{X}_{\mathcal{S}_m\cap(\alpha_k,\beta_k)}(t)u_k(t)+(\Lambda_\lambda^{-1}\mathfrak{X}_{\mathcal{S}_\mathbf{m}\cap(\alpha_k,\beta_k)}f)(t)\right)\right].$$

Using Lemma 4.15, Corollary 4.16, and equality (38), we get

$$\mathfrak{X}_{\mathcal{S}_m\cap(\alpha_k,\beta_k)}(t)u_k(t)+(\Lambda_\lambda^{-1}\mathfrak{X}_{\mathcal{S}_\mathbf{m}\cap(\alpha_k,\beta_k)}f)(t)=0$$

and the pair $\{\sum_{k=1}^{\Bbbk_1}u_k,f\}\in\mathcal{L}_{10}(\lambda)$. This and the invertibility of $T(\lambda)$ imply that $\widetilde{\zeta}(f,\lambda)=0$.

Thus, $S(\lambda)f=M(\lambda)\mathbf{V}^*(\overline{\lambda})f$, where $M(\lambda):Q_+\to Q_-$ is an everywhere defined operator. Let $\mathbf{V}_0^*(\overline{\lambda})$ be a restriction of $\mathbf{V}^*(\overline{\lambda})$ to $\ker\mathcal{L}^*_{10}(\lambda)$. By Lemma 4.28, it follows that $M(\lambda)=S(\lambda)(\mathbf{V}_0^*(\overline{\lambda}))^{-1}$. Hence $M(\lambda)$ is the bounded operator. Thus, equality (75) takes the form

$$y(t)=\widetilde{\mathrm{V}}(t,\lambda)M(\lambda)\mathbf{V}^*(\overline{\lambda})f+2^{-1}\sum_{k=1}^{\Bbbk_1}\left[-\mathfrak{X}_{[a,b]\setminus\mathcal{S}_\mathbf{m}}w_k(t,\lambda)iJ\int_{\alpha_k}^t w_k^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s)-(\Lambda_\lambda^{-1}\mathfrak{X}_{\mathcal{S}_\mathbf{m}\cap(\alpha_k,\beta_k)}f)(t)\right]+$$
$$+2^{-1}\sum_{k=1}^{\Bbbk_1}\left[\mathfrak{X}_{[a,b]\setminus\mathcal{S}_\mathbf{m}}w_k(t,\lambda)iJ\int_t^{\beta_k}w_k^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s)-(\Lambda_\lambda^{-1}\mathfrak{X}_{\mathcal{S}_\mathbf{m}\cap(\alpha_k,\beta_k)}f)(t)\right],\quad(82)$$

where the pair $\{y,f\}\in T(\lambda)$.

Conversely, assume that equality (82) holds. Then $y = 0$ if $f = 0$ in (82). Therefore, $T^{-1}(\lambda)$ is an operator. We claim that the operator $T^{-1}(\lambda)$ is bounded. Indeed, suppose that pairs $\{y_n, f_n\}$ satisfy equality (82) and the sequence $\{f_n\}$ converges to zero in $\mathfrak{H}$. It follows from Lemma 4.28 and equality (82) that the sequence $\{y_n\}$ converges to zero. So, $T^{-1}(\lambda)$ is the boundary everywhere defined operator. To conclude the proof, it remains to note that equality (80) is the another form of equality (82). The theorem is proved. □

The proof of the next lemma repeats verbatim the proof of the analogous lemma from [17].

Lemma 5.3. *In Theorem 5.2, the function $\lambda \to T^{-1}(\lambda)f$ is holomorphic for any $f \in \mathfrak{H}$ at a point λ_1 ($\mathrm{Im}\lambda_1 \neq 0$) if and only if the function $\lambda \to M(\lambda)\widetilde{x}$ is holomorphic for any element $\widetilde{x} \in \mathcal{Q}_+$ at the same point λ_1.*

Remark 5.4. *Let $\widetilde{T}(\lambda) \subset \mathfrak{H} \times \mathfrak{H}$ be a linear relation and $L_0(\lambda) \subset \widetilde{T}(\lambda) \subset L_0^*(\overline{\lambda})$. It follows from (42) that $\widetilde{T}(\lambda)$ is continuously invertible in the space $\mathfrak{H}$ if and only if $\widetilde{T}(\lambda)$ has the form $\widetilde{T}(\lambda) = T_0(\lambda) \oplus T(\lambda)$, where $T_0(\lambda) \subset \mathfrak{H}_0 \times \mathfrak{H}_0$, $T(\lambda) \subset \mathfrak{H}_1 \times \mathfrak{H}_1$ are linear relations, $L_{10}(\lambda) \subset T(\lambda) \subset L_{10}^*(\overline{\lambda})$, $T(\lambda)$ is continuously invertible in $\mathfrak{H}_1$, $T_0(\lambda)$ is any continuously invertible relation in $\mathfrak{H}_0$.*

6. The characteristic operator

Let T be a closed symmetric relation, $T \subset \mathbf{B} \times \mathbf{B}$, and let $\widetilde{T}$ be a self-adjoint extension of T to $\widetilde{\mathbf{B}}$, where $\mathbf{B}$, $\widetilde{\mathbf{B}}$ are Hilbert spaces, $\widetilde{\mathbf{B}} \supset \mathbf{B}$, and scalar products coincide in $\mathbf{B}$ and $\widetilde{\mathbf{B}}$. By P denote an orthogonal projection of $\widetilde{\mathbf{B}}$ onto $\mathbf{B}$. A function $\lambda \to R(\lambda)$ defined by the formula $R(\lambda) = P(\widetilde{T} - \lambda E)^{-1}|_{\mathbf{B}}$, $\mathrm{Im}\lambda \neq 0$, is called the generalized resolvent of the relation T (see, for example, [1, ch.9], [18]).

A.V. Straus (see [28]) obtained a formula for all generalized resolvents of a symmetric operator. It is shown in [18] that this formula remains true for symmetric relations also. By $\mathfrak{N}_\lambda$ denote a defect subspace of the closed symmetric relation T, i.e., the orthogonal complement in $\mathbf{B}$ to the range of the relation $T - \lambda E$. We fix some number λ_0 ($\mathrm{Im}\lambda_0 \neq 0$). Let $\lambda \to \mathcal{F}(\lambda)$ be a holomorphic operator function, where $\mathcal{F}(\lambda) : \mathfrak{N}_{\lambda_0} \to \mathfrak{N}_{\overline{\lambda}_0}$ is a bounded operator, $\|\mathcal{F}(\lambda)\| \leqslant 1$, $\mathrm{Im}\lambda \cdot \mathrm{Im}\lambda_0 > 0$. Let $T_{\mathcal{F}(\lambda)}$ be the relation consisting of all pairs of the form $\{y_0 + \mathcal{F}(\lambda)u - u, y_1 + \lambda_0\mathcal{F}(\lambda)u - \overline{\lambda}_0 u\}$, where $\{y_0, y_1\} \in T$, $u \in \mathfrak{N}_{\lambda_0}$. Then $T_{\mathcal{F}(\lambda)} \subset T^*$. The articles [28], [18] prove that the family of operators $R(\lambda)$ is a generalized resolvent of T if and only if $R(\lambda)$ can be represented in the form

$$R(\lambda) = (T_{\mathcal{F}(\lambda)} - \lambda E)^{-1}, \quad \mathrm{Im}\lambda \cdot \mathrm{Im}\lambda_0 > 0.$$

In [18], [28], it is established that a function $\lambda \to R(\lambda)$ is a generalized resolvent if and only if this function is holomorphic on the half-planes $\mathrm{Im}\lambda \neq 0$, $R(\lambda) = R^*(\overline{\lambda})$, and the inequality

$$(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(R(\lambda)f, f)_{\mathbf{B}} - (R(\lambda)f, R(\lambda)f)_{\mathbf{B}} \geqslant 0.$$

holds. In the article [29], A.V. Straus described the generalized resolvents of a symmetric operator generated by a formally self-adjoint differential expression of even order in the scalar case. In such a description, a function $M(\lambda)$ ($\lambda \in \mathbb{C}$) plays an essential role. This function has the property $(\mathrm{Im}\lambda)^{-1}\mathrm{Im}M(\lambda) \geqslant 0$. In [29], the function $M(\lambda)$ is called the characteristic function of the generalized resolvent. The article [17] describes the generalized resolvents of the symmetric relation L_{10} generated by integral equation (2). This description uses the characteristic function.

The definition of the characteristic operator is given in [21], [22] for a differential equation with a Nevanlinna operator function. Using this definition and Theorem 1.1 from [22], we introduce the notion of a characteristic operator for integral equation (11). An essential role in the study of the characteristic operator is played by Theorem 6.1 that is formulated below.

It follows from Lemma 4.28 that for any element $\widetilde{x} \in \mathcal{Q}_+$ there exists a function $f \in \mathfrak{H}_1$ such that $\mathbf{V}^*(\overline{\lambda})f = \widetilde{x}$. We denote

$$z(t) = z(t, f, \lambda) = \widetilde{\mathrm{V}}(t, \lambda)M(\lambda)\widetilde{x} - \sum_{n=1}^{\Bbbk_1} 2^{-1}\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t, \lambda)iJ\int_{\alpha_n}^{\beta_n} w_n^*(s, \overline{\lambda})d\mathbf{m}_0(s)f(s), \tag{83}$$

where $\widetilde{x} = \mathbf{V}^*(\overline{\lambda})f$. By Lemmas 4.18, 4.22, it follows that $z \in \ker \mathcal{L}_{10}^*(\overline{\lambda})$.

Theorem 6.1. *Suppose $T(\lambda)$ is a linear relation such that $\mathcal{L}_{10}(\lambda) \subset T(\lambda) \subset \mathcal{L}_{10}^*(\overline{\lambda})$, and $T(\lambda)$ is continuously invertible in the space $\mathfrak{H}_1$ for $\mathrm{Im}\lambda \neq 0$, and $R(\lambda) = T^{-1}(\lambda)$. Then the equality*

$$(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(M(\lambda)\widetilde{x},\widetilde{x}) - (z,z)_{\mathbf{m}_\lambda} = (\mathrm{Im}\lambda)^{-1}\mathrm{Im}(R(\lambda)f,f)_{\mathfrak{H}} - \mathrm{Im}(R(\lambda)f,R(\lambda)f)_{\mathbf{m}_\lambda} \tag{84}$$

holds for all $f \in \mathfrak{H}_1$ and $\widetilde{x} = \mathbf{V}^*(\overline{\lambda})f \in \mathcal{Q}_+$, where $M(\lambda)$ is the function from Theorem 5.2.

We recall that the symbols $(\cdot,\cdot)_{\mathfrak{H}}$, $(\cdot,\cdot)_{\mathbf{m}}$ denote the scalar product in $\mathfrak{H}$. The symbol $\mathbf{m}_\lambda$ is defined in equality (6). *The proof* of Theorem 6.1 is given in section 7.

Corollary 6.2. *If $\mathbf{r}_\lambda$ has form* (9) *(i.e., $\Lambda_\lambda = \lambda E$, see Example 2.1), then equality* (84) *has the form*

$$(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(M(\lambda)\widetilde{x},\widetilde{x}) - (z,z)_{\mathbf{m}} = (\mathrm{Im}\lambda)^{-1}\mathrm{Im}(R(\lambda)f,f)_{\mathbf{m}} - (R(\lambda)f,R(\lambda)f)_{\mathbf{m}}.$$

Suppose $T(\lambda)$ is a linear relation such that $\mathcal{L}_{10}(\lambda) \subset T(\lambda) \subset \mathcal{L}_{10}^*(\overline{\lambda})$ and $T(\lambda)$ is continuously invertible. Let $R(\lambda) = T^{-1}(\lambda)$. By $\mathbb{R}(\lambda)$ denote an operator in $\mathfrak{H}_1$ defined by the equality

$$\mathbb{R}(\lambda)f = (\mathrm{Im}\lambda)^{-1}R(\lambda)(\mathrm{Im}\Lambda_\lambda)f \tag{85}$$

for all $f \in \mathfrak{H}_1$ and for all λ such that $\mathrm{Im}\lambda \neq 0$. Using (25), (80), and the equality $\mathcal{S}_{\mathbf{m}} = \mathcal{S}_{\mathbf{m}_\lambda}$, we get

$$\begin{aligned}(\mathbb{R}(\lambda)f)(t) = &\int_a^b \widetilde{\mathrm{V}}(t,\lambda)M(\lambda)\widetilde{\mathrm{V}}^*(s,\overline{\lambda})d\mathbf{m}_\lambda(s)f(s)+\\ &+2^{-1}\sum_{k=1}^{\Bbbk_1}\int_a^b \mathfrak{X}_{[\alpha_k,\beta_k)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_k(t,\lambda)\mathrm{sgn}(s-t)iJw_k^*(s,\overline{\lambda})d\mathbf{m}_\lambda(s)\mathfrak{X}_{[\alpha_k,\beta_k)\setminus\mathcal{S}_{\mathbf{m}}}(s)f(s)-\\ &-(\mathrm{Im}\lambda)^{-1}\Lambda_\lambda^{-1}\sum_{k=1}^{\Bbbk_1}(\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}(\mathrm{Im}\Lambda_\lambda)f)(t).\end{aligned} \tag{86}$$

Definition 6.3. *Let $\lambda \to M(\lambda) = M^*(\overline{\lambda})$ be a function holomorphic for $\mathrm{Im}\lambda \neq 0$ whose values are bounded linear operators and $\mathcal{D}(M(\lambda)) = \mathcal{Q}_+$, $\mathcal{R}(M(\lambda)) \subset \mathcal{Q}_-$. This function M is called the characteristic operator of equation* (11) *if the operator $\mathbb{R}(\lambda)$* (86) *satisfies the inequality*

$$(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(\mathbb{R}(\lambda)f,f)_{\mathbf{m}_\lambda} - (\mathbb{R}(\lambda)f,\mathbb{R}(\lambda)f)_{\mathbf{m}_\lambda} \geqslant 0$$

for all functions $f \in \mathfrak{H}_1$ and for all λ such that $\mathrm{Im}\lambda \neq 0$.

We note that if $\mathcal{S}_{\mathbf{m}} = \varnothing$, then Definition 6.3 is the same as the definition of the characteristic operator from [21], [22] (see Definition 1.1 , Theorem 1.1 in [22]).

By $z_\lambda(t) = z_\lambda(t,f)$ denote $z(t,(\mathrm{Im}\lambda)^{-1}(\mathrm{Im}\Lambda_\lambda)f,\lambda)$, where $z(t,f,\lambda)$ is defined by equality (83). Using (25), we get

$$\begin{aligned}z_\lambda(t) = &\widetilde{\mathrm{V}}(t,\lambda)M(\lambda)\int_a^{b_0}\widetilde{\mathrm{V}}^*(t,\lambda)d\mathbf{m}(t)((\mathrm{Im}\lambda)^{-1}(\mathrm{Im}\Lambda_\lambda)f)(t)-\\ &-\sum_{n=1}^{\Bbbk_1}2^{-1}\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^{\beta_n}w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)((\mathrm{Im}\lambda)^{-1}(\mathrm{Im}\Lambda_\lambda)f)(s) =\\ &=\widetilde{\mathrm{V}}(t,\lambda)M(\lambda)\widetilde{x}_\lambda - \sum_{n=1}^{\Bbbk_1}2^{-1}\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^{\beta_n}w_n^*(s,\overline{\lambda})d\mathbf{m}_{0\lambda}(s)f(s),\end{aligned}$$

where $\widetilde{x}_\lambda = \int_a^{b_0}\widetilde{\mathrm{V}}^*(t,\lambda)d\mathbf{m}_\lambda(t)f(t)$.

Lemma 6.4. *Let the operator* $\mathbb{R}(\lambda)$ *be defined by equality* (86) (*or* (85)) . *Then the equality*

$$(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(M(\lambda)\widetilde{x}_\lambda,\widetilde{x}_\lambda\,)-(z_\lambda,z_\lambda)_{\mathbf{m}_\lambda}=(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(\mathbb{R}(\lambda)f,f)_{\mathbf{m}_\lambda}-\mathrm{Im}(\mathbb{R}(\lambda)f,\mathbb{R}(\lambda)f)_{\mathbf{m}_\lambda} \tag{87}$$

holds for all functions $f\in\mathfrak{H}_1$ *and for all* λ *such that* $\mathrm{Im}\lambda\neq 0$.

Proof. Substituting $(\mathrm{Im}\lambda)^{-1}(\mathrm{Im}\Lambda_\lambda)f$ for f in (84) and using equalities (25), (85), and the equality $z_\lambda(t,f)=z(t,(\mathrm{Im}\lambda)^{-1}(\mathrm{Im}\Lambda_\lambda)f,\lambda)$, we get (87). The lemma is proved. □

Lemma 6.5. *The inequality*

$$(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(R(\lambda)f,f)_{\mathbf{m}}-(R(\lambda)f,R(\lambda)f)_{\mathbf{m}_\lambda}\geqslant 0 \tag{88}$$

holds for all $f\in\mathfrak{H}_1$ *if and only if the inequality*

$$(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(\mathbb{R}(\lambda)g,g)_{\mathbf{m}_\lambda}-(\mathbb{R}(\lambda)g,\mathbb{R}(\lambda)g)_{\mathbf{m}_\lambda}\geqslant 0 \tag{89}$$

holds for all $g\in\mathfrak{H}_1$ *and for all* λ *such that* $\mathrm{Im}\lambda\neq 0$.

Proof. Replace the function f by $(\mathrm{Im}\lambda)^{-1}(\mathrm{Im}\Lambda_\lambda)g$ in (88). Then we obtain

$$(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(R(\lambda)(\mathrm{Im}\lambda)^{-1}(\mathrm{Im}\Lambda_\lambda)g,(\mathrm{Im}\lambda)^{-1}(\mathrm{Im}\Lambda_\lambda)g)_{\mathbf{m}}-(R(\lambda)(\mathrm{Im}\lambda)^{-1}(\mathrm{Im}\Lambda_\lambda)g,R(\lambda)(\mathrm{Im}\lambda)^{-1}(\mathrm{Im}\Lambda_\lambda)g)_{\mathbf{m}_\lambda}\geqslant 0.$$

Using (25), (85), we get (89). According to Lemma 3.5, the operator $\mathrm{Im}\Lambda_\lambda$ has an everywhere defined bounded inverse. If we replace the function g by $\mathrm{Im}\lambda(\mathrm{Im}\Lambda_\lambda)^{-1}f$ in (89), then we obtain (88). The lemma is proved. □

We fix some number λ_0 ($\mathrm{Im}\lambda_0\neq 0$). Let $\lambda\to F(\lambda)$ be an operator function such that $F(\lambda):\mathfrak{N}_{\lambda_0}\to\mathfrak{N}_{\overline{\lambda}_0}$ is a bounded operator, $\|F(\lambda)\|\leqslant 1$, where $\mathrm{Im}\lambda\cdot\mathrm{Im}\lambda_0>0$; $\mathfrak{N}_\lambda$ is a defect subspace of the symmetric relation L_{10}. Let $L_{F(\lambda)}$ be the relation consisting of all pairs of the form $\{y_0+F(\lambda)u-u,y_1+\lambda_0F(\lambda)u-\overline{\lambda}_0u\}$, where $\{y_0,y_1\}\in L_{10}$, $u\in\mathfrak{N}_{\lambda_0}$. We set $F(\overline{\lambda})=F^*(\lambda)$ and denote $\mathcal{L}_{F(\lambda)}=L_{F(\lambda)}-\Lambda_\lambda$. By (43), so that $\mathcal{L}_{10}(\lambda)\subset\mathcal{L}_{F(\lambda)}\subset\mathcal{L}^*_{10}(\overline{\lambda})$.

Lemma 6.6. *The family of relations* $\mathcal{L}_{F(\lambda)}=L_{F(\lambda)}-\Lambda_\lambda$ *is holomorphic if and only if the function* $F(\lambda)$ *is holomorphic* ($\mathrm{Im}\lambda\neq 0$).

Proof. The relation $\mathcal{L}_{F(\lambda)}=L_{F(\lambda)}-\Lambda_\lambda$ consists of all pairs of the form

$$\{y_0+F(\lambda)u-u,\;y_1+\lambda_0F(\lambda)u-\overline{\lambda}_0u-\Lambda_\lambda(y_0+F(\lambda)u-u)\}, \tag{90}$$

where $\{y_0,y_1\}\in L_{10}$, $u\in\mathfrak{N}_{\lambda_0}$. It is known (see, for example, [18]) that $L^*_{10}=L_{10}\oplus\mathfrak{N}_{\lambda_0}\oplus\mathfrak{N}_{\overline{\lambda}_0}\subset\mathfrak{H}_1\times\mathfrak{H}_1$ (the norm of space $\mathfrak{H}_1\times\mathfrak{H}_1$ is considered on L^*_{10}). We denote $\mathbf{B}_0=L_{10}\oplus\mathfrak{N}_{\lambda_0}$. Let the operator $\mathcal{K}(\lambda):\mathbf{B}_0\to\mathcal{L}_{F(\lambda)}$ take each element $\{y_0,y_1,u\}\in\mathbf{B}_0$ to element (90) belonging to $\mathcal{L}_{F(\lambda)}$. Then the operator $\mathcal{K}(\lambda)$ bijectively maps $\mathbf{B}_0$ onto $\mathcal{L}_{F(\lambda)}$ for any fixed λ. Since the function $\lambda\to\Lambda_\lambda$ is holomorphic, we see that the function $\lambda\to\mathcal{K}(\lambda)$ is holomorphic if the function $\lambda\to F(\lambda)$ is holomorphic. Hence, the family $\lambda\to\mathcal{L}_{F(\lambda)}(\lambda)$ is holomorphic.

Conversely, suppose that the family $\lambda\to\mathcal{L}_{F(\lambda)}(\lambda)$ is holomorphic. The operator $F(\lambda)$ is bounded. Therefore, it follows from (90) that the function $\lambda\to F(\lambda)$ is holomorphic. □

Lemma 6.7. *There exists a number* $\gamma=\gamma(\lambda)>0$ *such that the inequality*

$$-(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(y_1,y)_{\mathfrak{H}}\geqslant\gamma(y,y)_{\mathfrak{H}} \tag{91}$$

holds for all pairs $\{y,y_1\}\in\mathcal{L}_{F(\lambda)}$.

Proof. If a pair $\{y,y_1\}\in\mathcal{L}_{F(\lambda)}$, then there exists a pair $\{y,v_1\}\in L_{F(\lambda)}$ such that $y_1=v_1-\Lambda_\lambda y$. Note that $(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(v_1,v)_{\mathfrak{H}}\leqslant 0$ for all pairs $\{v,v_1\}\in L_{F(\lambda)}$ (see, for example, [18]). Therefore, using (28), we get

$$-(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(y_1,y)_{\mathfrak{H}}=-(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(v_1,y)_{\mathfrak{H}}+(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(\Lambda_\lambda y,y)_{\mathfrak{H}}\geqslant\gamma(y,y)_{\mathfrak{H}}.$$

The lemma is proved. □

Theorem 6.8. *The relation $\mathcal{L}_{F(\lambda)}$ has an everywhere defined bounded inverse operator $(\mathcal{L}_{F(\lambda)})^{-1} = (L_{F(\lambda)} - \Lambda_\lambda)^{-1}$ in $\mathfrak{H}_1$ for any λ such that* $\mathrm{Im}\lambda \neq 0$.

Proof. Using (91), we obtain

$$\left|(y_1, y)_{\mathfrak{H}}\right|^2 = \left|\mathrm{Re}(y_1, y)_{\mathfrak{H}}\right|^2 + \left|\mathrm{Im}(y_1, y)_{\mathfrak{H}}\right|^2 \geqslant \gamma \left\|y\right\|_{\mathfrak{H}}^2 \tag{92}$$

for any pairs $\{y, y_1\} \in \mathcal{L}_{F(\lambda)}$. Consider a sequence $\{\{y_n, y_{1n}\}\}$ of pairs $\{y_n, y_{1n}\} \in \mathcal{L}_{F(\lambda)}$ such that the sequence $\{y_{1n}\}$ converges to zero in $\mathfrak{H}$ as $n \to \infty$. We claim that the sequence $\{y_n\}$ converges to zero. First we prove that the sequence $\{y_n\}$ is bounded. Assume the converse, let $\left\|y_n\right\|_{\mathfrak{H}} \to \infty$ as $n \to \infty$. We denote $\widetilde{y_n} = y_n \left\|y_n\right\|_{\mathfrak{H}}^{-1}$. Then $\widetilde{y_{1n}} = y_{1n} \left\|y_n\right\|_{\mathfrak{H}}^{-1} \to 0$ as $n \to \infty$. By (92), we get $\widetilde{y_n} \to 0$. But $\left\|\widetilde{y_n}\right\|_{\mathfrak{H}} = 1$. This contradiction proves that $\{y_n\}$ is bounded. Now (92) implies that $\{y_n\} \to 0$ in $\mathfrak{H}_1$ as $n \to \infty$. Finally, using the equalities $\Lambda_\lambda^* = \Lambda_{\overline{\lambda}}$ and $L^*_{F(\lambda)} = L_{F(\overline{\lambda})}$, we obtain $\mathcal{R}(\mathcal{L}_{F(\lambda)}) = \mathfrak{H}_1$. This completes the proof of theorem. □

Theorem 6.9. *Inequality* (88) *holds if and only if $R(\lambda)$ has the form $R(\lambda) = (\mathcal{L}_{F(\lambda)})^{-1} = (L_{F(\lambda)} - \Lambda_\lambda)^{-1}$.*

Proof. First we assume that $R(\lambda) = (L_{F(\lambda)} - \Lambda_\lambda)^{-1}$. We denote $y = R(\lambda)f$. Then there exists an element $y_1 \in \mathcal{R}(L_{F(\lambda)})$ such that the pair $\{y, y_1\} \in L_{F(\lambda)}$ and $y_1 - \Lambda_\lambda y = f$. Hence, $(y_1, y)_{\mathbf{m}} - (\Lambda_\lambda y, y)_{\mathbf{m}} = (f, y)_{\mathbf{m}}$. Using (25), we obtain

$$-(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(y_1, y)_{\mathbf{m}} = (\mathrm{Im}\lambda)^{-1}\mathrm{Im}(y, f)_{\mathbf{m}} - (y, y)_{\mathbf{m}_\lambda}.$$

By $(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(y_1, y)_{\mathbf{m}} \leqslant 0$ (see, for example, [18]) , so that inequality (88) holds.

Now we assume that inequality (88) holds. According to (43), we have $L_{10} - \Lambda_\lambda \subset T(\lambda) \subset L^*_{10} - \Lambda_\lambda$, where $T^{-1}(\lambda) = R(\lambda)$. Then $T(\lambda) = L(\lambda) - \Lambda_\lambda$, where $L_{10} \subset L(\lambda) \subset L^*_{10}$. Consequently, there exists an element $y_1 \in \mathcal{R}(L(\lambda))$ such that the pair $\{y, y_1\} \in L(\lambda)$ and $y_1 - \Lambda_\lambda y = f$, $R(\lambda)f = y$. Hence, $(y_1, y)_{\mathbf{m}} - (\Lambda_\lambda y, y)_{\mathbf{m}} = (f, y)_{\mathbf{m}}$. Using (25), (88), we obtain

$$-(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(y_1, y)_{\mathbf{m}} = (\mathrm{Im}\lambda)^{-1}\mathrm{Im}(R(\lambda)f, f)_{\mathbf{m}} - (R(\lambda)f, R(\lambda)f)_{\mathbf{m}_\lambda} \geqslant 0.$$

This implies that the relation $(\mathrm{Im}\lambda)^{-1}L(\lambda)$ is accumulative, i.e., $(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(y_1, y)_{\mathbf{m}} \leqslant 0$ for all pairs $\{y_1, y\} \in (\mathrm{Im}\lambda)^{-1}L(\lambda)$. Since the range $\mathcal{R}(T(\lambda)) = \mathfrak{H}$, it follows that the relation $T(\lambda)$ is maximal accumulative. Therefore, the relation $(\mathrm{Im}\lambda)^{-1}L(\lambda)$ is maximal accumulative. Consequently, there exists an operator $F(\lambda)$: $\mathfrak{N}_{\lambda_0} \to \mathfrak{N}_{\overline{\lambda}_0}$ ($\mathrm{Im}\lambda \neq 0$) such that $\|F(\lambda)\| \leqslant 1$ and $L(\lambda) = L_{F(\lambda)}$ (see, for example, [18], [19]). This implies that $T(\lambda) = L_{F(\lambda)} - \Lambda_\lambda$. The theorem is proved. □

We denote $\mathbb{L}_{F(\lambda)} = (\mathrm{Im}\lambda)(\mathrm{Im}\Lambda_\lambda)^{-1}(L_{F(\lambda)} - \Lambda_\lambda)$. By Theorem 6.8 and Lemma 3.5, it follows that the relation $\mathbb{L}_{F(\lambda)}$ is continuously invertible and

$$(\mathbb{L}_{F(\lambda)})^{-1} = (\mathrm{Im}\lambda)^{-1}(L_{F(\lambda)} - \Lambda_\lambda)^{-1}(\mathrm{Im}\Lambda_\lambda) = (\mathrm{Im}\lambda)^{-1}(\mathcal{L}_{F(\lambda)})^{-1}\mathrm{Im}\Lambda_\lambda.$$

Theorem 6.10. *Inequality* (89) *holds if and only if $\mathbb{R}(\lambda)$ has the form $\mathbb{R}(\lambda) = (\mathrm{Im}\lambda)^{-1}(L_{F(\lambda)} - \Lambda_\lambda)^{-1}(\mathrm{Im}\Lambda_\lambda) = (\mathbb{L}_{F(\lambda)})^{-1}$.*

Proof. First we assume that $\mathbb{R}(\lambda) = (\mathbb{L}_{F(\lambda)})^{-1}$. Using (85), we get $R(\lambda) = (L_{F(\lambda)} - \Lambda_\lambda)^{-1}$. It follows from Theorem 6.9 that inequality (88) holds. Taking into account Lemma 6.5, we obtain inequality (89). Now we assume that inequality (89) holds. Arguing as above in reverse order, we see that $\mathbb{R}(\lambda) = (\mathbb{L}_{F(\lambda)})^{-1}$. The theorem is proved. □

We note that the function $M(\lambda)$ is the same in equalities (80), (86). Let $M(\lambda)$ be holomorphic and $M^*(\lambda) = M(\overline{\lambda})$ ($\mathrm{Im}\lambda \neq 0$). It follows from Lemma 6.5 and Definition 6.3 that the function $M(\lambda)$ is the characteristic operator in equalities (80), (86) if and only if inequalities (88), (89), respectively, hold. According to Lemma 6.5, the inequalities (88), (89) hold together. Moreover, $M(\lambda)$ is the characteristic operator if and only if $R(\lambda) = (L_{F(\lambda)} - \Lambda_\lambda)^{-1}$ or $\mathbb{R}(\lambda) = (\mathrm{Im}\lambda)^{-1}(L_{F(\lambda)} - \Lambda_\lambda)^{-1}(\mathrm{Im}\Lambda_\lambda)$, where $F(\lambda)$ is the holomorphic function and $F^*(\lambda) = F(\overline{\lambda})$.

7. The proof of Theorem 6.1

Proof. For the case in which the measure $\mathbf{r}_\lambda$ has form (9), this theorem is proved in [17]. Below is a more detailed proof. By standard transformations, equality (82) (or (80)) is reduced to the form

$$y(t)=\widetilde{\mathrm{V}}(t,\lambda)M(\lambda)\mathbf{V}^*(\overline{\lambda})f-\sum_{n=1}^{\Bbbk_1}\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_\mathbf{m}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^{t}w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s)+$$
$$+\sum_{n=1}^{\Bbbk_1}2^{-1}\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_\mathbf{m}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^{\beta_n}w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s)-\sum_{n=1}^{\Bbbk_1}(\Lambda_\lambda^{-1}\mathfrak{X}_{\mathcal{S}_\mathbf{m}\cap(\alpha_n,\beta_n)}f)(t). \quad (93)$$

We fix an interval $(\alpha_n,\beta_n)\in\mathbb{J}$ and introduce some notation. Let $\mathbb{M}_n\subset\mathbb{M}$ be a set consisting of the interval $(\alpha_n,\beta_n)\in\mathbb{J}$ and single-point sets $\{\tau\}$, where $\tau\in\mathcal{S}_\mathbf{m}\cap(\alpha_n,\beta_n)$. Suppose $\widetilde{\Bbbk}_n$ is a natural number (equal to the number of elements in $\mathbb{M}_n$ if this number is finite) or the symbol ∞ (if the number of elements in $\mathbb{M}_n$ is infinite). We arrange the elements of $\mathbb{M}_n$ in the form of a finite or infinite sequence and denote these elements by $\mathcal{E}_{nk}$, where k is any natural number if the number of elements in $\mathbb{M}_n$ is infinite, and $1\leqslant k\leqslant\widetilde{\Bbbk}_n$ if the number of elements in $\mathbb{M}_n$ is finite; $\mathcal{E}_{n1}$ is the interval (α_n,β_n), $\mathcal{E}_{nk}$ $(k\geqslant 2)$ is the single-point set $\{\tau_k\}$, $\tau_k\in\mathcal{S}_\mathbf{m}\cap(\alpha_n,\beta_n)$.

To each element $\mathcal{E}_{nk}\in\mathbb{M}_n$ assign the operator function ϑ_{nk} in the following way (see also (44), (58), (59)). If $k=1$, then

$$\vartheta_{n1}(t,\lambda)=\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_\mathbf{m}}w_n(t,\lambda),\quad w_n(t,\lambda)=\mathfrak{X}_{[\alpha_n,\beta_n)}W(t,\lambda)W^{-1}(\alpha_n,\lambda); \quad (94)$$

if $k\geqslant 2$, then

$$\vartheta_{nk}(t,\lambda)=U_n(\lambda)(\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}})(t)+(\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}})(t)=-\mathfrak{X}_{[\alpha_n,\beta)\setminus\mathcal{S}_\mathbf{m}}w_n(t,\lambda)iJ\int_{\alpha_n}^{t}w_n^*(s,\overline{\lambda})d\mathbf{m}(s)\mathfrak{X}_{\{\tau_k\}}(s)+(\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}})(t)=$$
$$=\begin{cases}0 \text{ for } t<\tau_k,\\(\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}})(t) \text{ for } t=\tau_k,\\-\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_\mathbf{m}}w_n(t,\lambda)iJw_n^*(\tau_k,\overline{\lambda})\mathbf{m}(\{\tau_k\}) \text{ for } t>\tau_k.\end{cases} \quad (95)$$

Using (94), (95), and the equality $\Lambda_\lambda^*=\Lambda_{\overline{\lambda}}$, we get

$$\vartheta_{n1}^*(t,\overline{\lambda})=\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}\mathbf{m}}w_n^*(t,\overline{\lambda});\quad \vartheta_{nk}^*(t,\overline{\lambda})=\begin{cases}0 \text{ for } t<\tau_k,\\(\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}})(t) \text{ for } t=\tau_k,\\\mathbf{m}(\{\tau_k\})w_n(\tau_k,\lambda)iJ\mathfrak{X}_{(\tau_k,b]}\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_\mathbf{m}}w_n^*(t,\overline{\lambda}) \text{ for } t>\tau_k.\end{cases}$$

Suppose $p_j:\mathbf{Q}_-\to Q_j^-$ is an operator defined by the formula $p_j\widetilde{\eta}=\eta_j$, where $\widetilde{\eta}=\{\eta_j\}\in\mathbf{Q}_-$. Then by $p_{j_{n1}}$ (by $p_{j_{nk}}$ for $k\geqslant 2$) we denote the projection of $\mathbf{Q}_-$ onto the space Q_{n1}^- supplied with the norm $\|\xi_1\|_-=\|\vartheta_{n1}(\cdot,0)\xi_1\|_{\mathfrak{H}}$ (onto the space Q_{nk}^- supplied with the norm $\|\xi_k\|_-=\|\vartheta_k(\cdot,0)\xi_k\|_{\mathfrak{H}}$ for $k\geqslant 2$, respectively)(see equality (66)).

We denote $f_n=\mathfrak{X}_{[\alpha_n,\beta_n)}f$, and $\mathbf{V}^*(\overline{\lambda})f_n=\widetilde{x}_n$, and

$$x_{n1}=\int_{\alpha_n}^{\beta_n}\vartheta_{n1}^*(s,\overline{\lambda})d\mathbf{m}(s)f_n(s)=\int_{\alpha_n}^{\beta_n}w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s), \quad (96)$$

$$x_{nk}=\int_{\alpha_n}^{\beta_n}\vartheta_{nk}^*(s,\overline{\lambda})d\mathbf{m}(s)f_n(s)=$$
$$=\int_{\{\tau_k\}}d\mathbf{m}(s)(\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}f_n(\tau_k))(s)+\mathbf{m}(\{\tau_k\})w_n(\tau_k,\lambda)iJ\int_{\tau_k}^{\beta_n}w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s),\ k\geqslant 2. \quad (97)$$

By (69) and Lemma 4.28, it follows that $\widetilde{x}_n\in\mathbf{Q}_+\subset\mathbf{Q}\subset\mathbf{Q}_-$ is a sequence $\widetilde{x}_n=\{x_j\}$ with elements x_{n1}, x_{nk} and zeros. It follows from the definition of adjoint operators in the spaces with the negative and positive norms that $x_{n1}\in Q_{n1}^+\subset Q_{n1}^-$, $x_{nk}\in Q_{nk}^+\subset Q_{nk}^-$.

Let r be a natural number such that $1 \leqslant r \leqslant \Bbbk_1$ if $\Bbbk_1$ is finite and r any natural number if $\Bbbk_1$ is infinite ($\Bbbk_1$ is the number of intervals $(\alpha_r, \beta_r) \in \mathbb{J}$). We denote $\mu_{nr1} = p_{j_{r1}} M(\lambda)\widetilde{x}_n$ and $\mu_{nrk} = p_{j_{rk}} M(\lambda)\widetilde{x}_n$ for $k \geqslant 2$. Using (94), (95), we get

$$\widetilde{V}(t,\lambda)M(\lambda)\widetilde{x}_n = \sum_{r=1}^{\Bbbk_1}\left(\vartheta_{r1}\mu_{nr1} + \sum_{k=2}^{\widetilde{\Bbbk}_r}\vartheta_{rk}\mu_{nrk}\right) =$$

$$= \sum_{r=1}^{\Bbbk_1}\left(\mathfrak{X}_{[\alpha_r,\beta_r)\setminus\mathcal{S}_{\mathbf{m}}} w_r(t,\lambda)\mu_{nr1} + \sum_{k=2}^{\widetilde{\Bbbk}_r}\left(-\mathfrak{X}_{[\alpha_r,\beta_r)\setminus\mathcal{S}_{\mathbf{m}}} w_r(t,\lambda)iJ\int_{\alpha_r}^{t} w_r^*(s,\overline{\lambda})d\mathbf{m}(s)\mathfrak{X}_{\{\tau_k\}}(s)\mu_{nrk} + (\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}\mu_{nrk})(t)\right)\right), \quad (98)$$

where $\tau_k \in \mathcal{S}_{\mathbf{m}} \cap (\alpha_r, \beta_r)$. Moreover,

$$(\widetilde{x}, M(\lambda)\widetilde{x}) = \sum_{r=1}^{\Bbbk_1}\left(x_{r1}, \sum_{n=1}^{\Bbbk_1}\mu_{nr1}\right) + \sum_{r=1}^{\Bbbk_1}\sum_{k=2}^{\widetilde{\Bbbk}_r}\left(x_{rk}, \sum_{n=1}^{\Bbbk_1}\mu_{nrk}\right). \quad (99)$$

Let $y_n = R(\lambda)f_n$. It follows from (93) that the equality

$$y_n(t) = \widetilde{\mathrm{V}}(t,\lambda)M(\lambda)\widetilde{x}_n - \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^{t} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s) +$$
$$+ 2^{-1}\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^{\beta_n} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s) - \Lambda_\lambda^{-1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_n,\beta_n)}f_n(t) \quad (100)$$

holds. We denote

$$z_n(t) = \widetilde{\mathrm{V}}(t,\lambda)M(\lambda)\widetilde{x}_n - 2^{-1}\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^{\beta_n} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s). \quad (101)$$

By (83), so that $z(t) = \sum_{n=1}^{\Bbbk_1} z_n(t)$. We note that $z_n \in \ker(L_{10}^* - \Lambda_\lambda) = \ker \mathcal{L}_{10}^*(\overline{\lambda})$.

Using (98), (100), (101), and (96), we obtain

$$y_n(t) - z_n(t) = -\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^{t} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s) + \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJx_{n1} - \sum_{k=2}^{\widetilde{\Bbbk}_n}\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}f_n(t);$$

$$y_n(t) + z_n(t) =$$
$$= 2\sum_{r=1}^{\Bbbk_1}\left(\mathfrak{X}_{[\alpha_r,\beta_r)\setminus\mathcal{S}_{\mathbf{m}}} w_r(t,\lambda)\mu_{nr1} + \sum_{k=2}^{\widetilde{\Bbbk}_r}\left(-\mathfrak{X}_{[\alpha_r,\beta_r)\setminus\mathcal{S}_{\mathbf{m}}} w_r(t,\lambda)iJ\int_{\alpha_r}^{t} w_r^*(s,\overline{\lambda})d\mathbf{m}(s)\mathfrak{X}_{\{\tau_k\}}(s)\mu_{nrk} + (\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}\mu_{nrk})(t)\right)\right) -$$
$$- \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^{t} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s) - \sum_{k=2}^{\widetilde{\Bbbk}_n}\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}f_n(t).$$

We decompose the functions $y_n - z_n$, $y_n + z_n$ into terms to which Lagrange formula (10) is applicable. Let us introduce the following designations:

$$\varphi_{1n}(t) = -\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^{t} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s) + \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJx_{n1};$$

$$\varphi_{2n}(t) = -\sum_{k=2}^{\widetilde{\Bbbk}_n}\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}(t)f_n(t); \quad (102)$$

$$\psi_{1n}(t) = 2\sum_{r=1}^{\Bbbk_1}\mathfrak{X}_{[\alpha_r,\beta_r)\setminus\mathcal{S}_{\mathbf{m}}} w_r(t,\lambda)\mu_{nr1} - \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^{t} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s);$$

$$\psi_{2n}(t) = 2\sum_{r=1}^{\Bbbk_1}\sum_{k=2}^{\widetilde{\Bbbk}_r}(\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_r,\beta_r)}\mu_{nrk})(t) - \sum_{k=2}^{\widetilde{\Bbbk}_n}\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}(t)f_n(t); \tag{103}$$

$$\psi_{3n}(t) = -2\sum_{r=1}^{\Bbbk_1}\sum_{k=2}^{\widetilde{\Bbbk}_r}\mathfrak{X}_{[\alpha_r,\beta_r)\setminus\mathcal{S}_{\mathbf{m}}}w_r(t,\lambda)iJ\int_{\alpha_r}^{t}w_r^*(s,\overline{\lambda})d\mathbf{m}(s)\mathfrak{X}_{\{\tau_k\}\cap(\alpha_r,\beta_r)}(s)\mu_{nrk} = \sum_{r=1}^{\Bbbk_1}\sum_{k=2}^{\widetilde{\Bbbk}_r}\psi_{3nrk}(t),$$

where

$$\psi_{3nrk}(t)=-2\mathfrak{X}_{[\alpha_r,\beta_r)\setminus\mathcal{S}_{\mathbf{m}}}w_r(t,\lambda)iJ\int_{\alpha_r}^{t}w_r^*(s,\overline{\lambda})d\mathbf{m}(s)\mathfrak{X}_{\{\tau_k\}}(s)\mu_{nrk}=\begin{cases}0 \ \text{ for } \ t\leqslant\tau_k,\\ -2\mathfrak{X}_{[\alpha_r,\beta_r)\setminus\mathcal{S}_{\mathbf{m}}}w_r(t,\lambda)iJw_r^*(\tau_k,\overline{\lambda})\mathbf{m}(\{\tau_k\})\mu_{nrk}\text{for } t>\tau_k.\end{cases}$$

Then

$$y_n - z_n = \varphi_{1n} + \varphi_{2n}, \quad y_n + z_n = \psi_{1n} + \psi_{2n} + \psi_{3n}. \tag{104}$$

We denote $\Phi_1 = \sum_{n=1}^{\Bbbk_1}\varphi_{1n}$, $\Phi_2 = \sum_{n=1}^{\Bbbk_1}\varphi_{2n}$, $\Psi_1 = \sum_{n=1}^{\Bbbk_1}\psi_{1n}$, $\Psi_2 = \sum_{n=1}^{\Bbbk_1}\psi_{2n}$, $\Psi_3 = \sum_{n=1}^{\Bbbk_1}\psi_{3n}$. Using (104), we get

$$y - z = \sum_{n=1}^{\Bbbk_1}(y_n - z_n) = \Phi_1 + \Phi_2, \quad y + z = \sum_{n=1}^{\Bbbk_1}(y_n + z_n) = \Psi_1 + \Psi_2 + \Psi_3. \tag{105}$$

Since $z \in \ker\mathcal{L}_{10}^*(\overline{\lambda})$, $\{y, f\} \in \mathcal{L}_{10}^*(\overline{\lambda})$, it follows that $\{y - z, f\} \in \mathcal{L}_{10}^*(\overline{\lambda})$, $\{y + z, f\} \in \mathcal{L}_{10}^*(\overline{\lambda})$. Theorem 4.29 implies that $\{\Phi_1, \mathfrak{X}_{[a,b]\setminus(\mathcal{S}_{\mathbf{m}}\cup\mathcal{S}_0)}f\} \in \mathcal{L}_{10}^*(\overline{\lambda})$, $\{\Psi_1, \mathfrak{X}_{[a,b]\setminus(\mathcal{S}_{\mathbf{m}}\cup\mathcal{S}_0)}f\} \in \mathcal{L}_{10}^*(\overline{\lambda})$. By (27), (32), so that $\Phi_2 \in \ker L_{10}^*$, $\Psi_2 \in \ker L_{10}^*$. Using (43), we obtain

$$\{y - z, \Lambda_\lambda(y - z) + f\} \in L_{10}^*, \ \ \{y + z, \Lambda_\lambda(y + z) + f\} \in L_{10}^*,$$

$$\{\Phi_1, \Lambda_\lambda\Phi_1 + \mathfrak{X}_{[a,b]\setminus(\mathcal{S}_{\mathbf{m}}\cup\mathcal{S}_0)}f\} \in L_{10}^*, \ \ \{\Psi_1, \Lambda_\lambda\Psi_1 + \mathfrak{X}_{[a,b]\setminus(\mathcal{S}_{\mathbf{m}}\cup\mathcal{S}_0)}f\} \in L_{10}^*.$$

We denote

$$u_- = \Lambda_\lambda(y - z) + f, \ \ u_+ = \Lambda_\lambda(y + z) + f. \tag{106}$$

Then $\{y - z, u_-\} \in L_{10}^*$, $\{y + z, u_+\} \in L_{10}^*$. We note that for all functions $g, h \in \mathfrak{H}$ the equalities

$$(\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}g, \mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}h)_{\mathbf{m}} = 0; \ \ (\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}g, h)_{\mathbf{m}} = (\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}g, \mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}h)_{\mathbf{m}} = (g, h)_{\mathbf{m}_0}; \ \ (\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}g, h)_{\mathbf{m}} = (\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}g, \mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}h)_{\mathbf{m}}$$

hold. Therefore, using (105), (106), we get

$$\begin{aligned}(u_-, y + z)_{\mathbf{m}} - (y - z, u_+)_{\mathbf{m}} &= (\Lambda_\lambda(\Phi_1 + \Phi_2) + f, \Psi_1 + \Psi_2 + \Psi_3)_{\mathbf{m}} - (\Phi_1 + \Phi_2, \Lambda_\lambda(\Psi_1 + \Psi_2 + \Psi_3) + f)_{\mathbf{m}} = \\ &= (\Lambda_\lambda\Phi_1, \Psi_1)_{\mathbf{m}_0} + (f, \Psi_1)_{\mathbf{m}_0} + (\Lambda_\lambda\Phi_2, \Psi_2)_{\mathbf{m}} + (f, \Psi_2)_{\mathbf{m}} + (\Lambda_\lambda\Phi_1, \Psi_3)_{\mathbf{m}_0} + (f, \Psi_3)_{\mathbf{m}_0} - \\ &- (\Phi_1, \Lambda_\lambda\Psi_1)_{\mathbf{m}_0} - (\Phi_1, \Lambda_\lambda\Psi_3)_{\mathbf{m}_0} - (\Phi_1, f)_{\mathbf{m}_0} - (\Phi_2, \Lambda_\lambda\Psi_2)_{\mathbf{m}} - (\Phi_2, f)_{\mathbf{m}}.\end{aligned} \tag{107}$$

It follows from (102), (103) that

$$(\Lambda_\lambda\Phi_2, \Psi_2)_{\mathbf{m}} = \Lambda_\lambda\sum_{n=1}^{\Bbbk_1}\sum_{k=2}^{\widetilde{\Bbbk}_n}\left(-\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}f_n(\tau_k), \Psi_2\right)_{\mathbf{m}}; \ (f, \Psi_2)_{\mathbf{m}} = \sum_{n=1}^{\Bbbk_1}\sum_{k=2}^{\widetilde{\Bbbk}_n}\left(\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}f_n(\tau_k), \Psi_2\right)_{\mathbf{m}}; \tag{108}$$

$$-(\Phi_2, \Lambda_\lambda\Psi_2)_{\mathbf{m}} = \left(-\Phi_2, \Lambda_\lambda\sum_{n=1}^{\Bbbk_1}\left(2\sum_{r=1}^{\Bbbk_1}\sum_{k=2}^{\widetilde{\Bbbk}_r}(\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_r,\beta_r)}\mu_{nrk}) - \sum_{k=2}^{\widetilde{\Bbbk}_n}\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}f_n(\tau_k)\right)\right)_{\mathbf{m}}; \tag{109}$$

$$-(\Phi_2, f)_{\mathbf{m}} = \left(-\Phi_2, \sum_{n=1}^{\Bbbk_1}\sum_{k=2}^{\widetilde{\Bbbk}_n}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}f_n(\tau_k)\right)_{\mathbf{m}}. \tag{110}$$

Summing equalities (108), (109), (110), we get

$$((\Lambda_\lambda\Phi_2,\Psi_2)_{\mathbf{m}}+(f,\Psi_2)_{\mathbf{m}}-(\Phi_2,\Lambda_\lambda\Psi_2)_{\mathbf{m}}-(\Phi_2,f)_{\mathbf{m}}=$$
$$=2\left(\sum_{n=1}^{\Bbbk_1}\sum_{k=2}^{\widetilde{\Bbbk}_n}\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}f_n(\tau_k),\sum_{r=1}^{\Bbbk_1}\sum_{k=2}^{\widetilde{\Bbbk}_r}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_r,\beta_r)}\sum_{n=1}^{\Bbbk_1}\mu_{nrk}\right)_{\mathbf{m}}=$$
$$=2\sum_{n=1}^{\Bbbk_1}\sum_{k=2}^{\widetilde{\Bbbk}_n}\left(\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}f_n(\tau_k),\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu nk}\right)_{\mathbf{m}}. \quad (111)$$

We denote $\widetilde{\Phi}_1=\sum_{n=1}^{\Bbbk_1}\widetilde{\varphi}_{1n}$, $\widetilde{\Psi}_1=\sum_{n=1}^{\Bbbk_1}\widetilde{\psi}_{1n}$, $\widetilde{\Psi}_3=\sum_{n=1}^{\Bbbk_1}\widetilde{\psi}_{3n}$, where

$$\widetilde{\varphi}_{1n}(t)=-w_n(t,\lambda)iJ\int_{\alpha_n}^t w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s)+w_n(t,\lambda)iJx_{n1}; \quad (112)$$

$$\widetilde{\psi}_{1n}(t)=2\sum_{r=1}^{\Bbbk_1}w_r(t,\lambda)\mu_{nr1}-w_n(t,\lambda)iJ\int_{\alpha_n}^t w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s); \quad (113)$$

$$\widetilde{\psi}_{3nrk}(t)=\begin{cases}0 \text{ for } t<\tau_k,\\ -2w_r(t,\lambda)iJw_r^*(\tau_k,\overline{\lambda})\mathbf{m}(\{\tau_k\})\mu_{nrk} \text{ for } t\geqslant\tau_k, \text{ where } \tau_k\in\mathcal{S}_{\mathbf{m}}\cap(\alpha_r,\beta_r)\end{cases};\quad \widetilde{\psi}_{3n}(t)=\sum_{r=1}^{\Bbbk_1}\sum_{k=2}^{\widetilde{\Bbbk}_r}\widetilde{\psi}_{3nrk}(t).$$

Then the equalities $\widetilde{\varphi}_{1n}(t)-\varphi_{1n}(t)=\mathfrak{X}_{[\alpha_n,\beta_n)\cap\mathcal{S}_{\mathbf{m}}}(t)\widetilde{\varphi}_{1n}(t)$, $\widetilde{\psi}_{1n}(t)-\psi_{1n}(t)=\mathfrak{X}_{[\alpha_n,\beta_n)\cap\mathcal{S}_{\mathbf{m}}}(t)\widetilde{\psi}_{1n}(t)$, $\widetilde{\psi}_{3nrk}(t)-\psi_{3nrk}(t)=\mathfrak{X}_{\{\tau_k\}\cap[\alpha_r,\beta_r)}(t)\widetilde{\psi}_{3nrk}(t)$ hold. This implies that

$$(\varphi_{1n},\psi_{1n})_{\mathbf{m}_0}=(\widetilde{\varphi}_{1n},\widetilde{\psi}_{1n})_{\mathbf{m}_0},\ (\varphi_{1n},f)_{\mathbf{m}_0}=(\widetilde{\varphi}_{1n},f)_{\mathbf{m}_0},\ (f,\psi_{1n})_{\mathbf{m}_0}=(f,\widetilde{\psi}_{1n})_{\mathbf{m}_0},\ (\varphi_{1n},\psi_{3nk})_{\mathbf{m}_0}=(\widetilde{\varphi}_{1n},\widetilde{\psi}_{3nk})_{\mathbf{m}_0}. \quad (114)$$

Using (111), (114), we continue equality (107)

$$(u_-,y+z)_{\mathbf{m}}-(y-z,u_+)_{\mathbf{m}}=[(\Lambda_\lambda\widetilde{\Phi}_1+f,\widetilde{\Psi}_1)_{\mathbf{m}_0}-(\widetilde{\Phi}_1,\Lambda_\lambda\widetilde{\Psi}_1+f)_{\mathbf{m}_0}]+[(\Lambda_\lambda\widetilde{\Phi}_1+f,\widetilde{\Psi}_3)_{\mathbf{m}_0}-(\widetilde{\Phi}_1,\Lambda_\lambda\widetilde{\Psi}_3)_{\mathbf{m}_0}]+$$
$$+2\sum_{n=1}^{\Bbbk_1}\sum_{k=2}^{\widetilde{\Bbbk}_n}\left(\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}f_n(\tau_k),\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu nk}\right)_{\mathbf{m}}. \quad (115)$$

We claim that the equality

$$(\Lambda_\lambda\widetilde{\Phi}_1+f,\widetilde{\Psi}_1)_{\mathbf{m}_0}-(\widetilde{\Phi}_1,\Lambda_\lambda\widetilde{\Psi}_1+f)_{\mathbf{m}_0}=\sum_{n=1}^{\Bbbk_1}\left(\left(\Lambda_\lambda\widetilde{\varphi}_{1n}+f_n,\widetilde{\psi}_{5n}\right)_{\mathbf{m}_0}-\left(\widetilde{\varphi}_{1n},\Lambda_\lambda\widetilde{\psi}_{5n}+f_n\right)_{\mathbf{m}_0}\right). \quad (116)$$

holds, where

$$\widetilde{\psi}_{5n}(t)=2w_n(t,\lambda)\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu n1}-w_n(t,\lambda)iJ\int_{\alpha_n}^t w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s). \quad (117)$$

Indeed, using (44), (113), and the equality $(h_n,w_r)_{\mathbf{m}_0}=0$ for $n\neq r$, we obtain

$$\left(\sum_{n=1}^{\Bbbk_1}h_n,\widetilde{\Psi}_1\right)_{\mathbf{m}_0}=\left(\sum_{n=1}^{\Bbbk_1}h_n(t),2\sum_{n=1}^{\Bbbk_1}\sum_{r=1}^{\Bbbk_1}w_r(t,\lambda)\mu_{nr1}-\sum_{n=1}^{\Bbbk_1}w_n(t,\lambda)iJ\int_{\alpha_n}^t w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s)\right)_{\mathbf{m}_0}=$$
$$=\left(\sum_{n=1}^{\Bbbk_1}h_n(t),2\sum_{n=1}^{\Bbbk_1}w_n(t,\lambda)\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu n1}-\sum_{n=1}^{\Bbbk_1}w_n(t,\lambda)iJ\int_{\alpha_n}^t w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s)\right)_{\mathbf{m}_0}=$$
$$=\left(\sum_{n=1}^{\Bbbk_1}h_n,\sum_{n=1}^{\Bbbk_1}\widetilde{\psi}_{5n}\right)_{\mathbf{m}_0}=\sum_{n=1}^{\Bbbk_1}\left(h_n,\widetilde{\psi}_{5n}\right)_{\mathbf{m}_0} \quad (118)$$

for any function $h_n(t)$ such that $h_n(t)=\mathfrak{X}_{[\alpha_n,\beta_n)}h(t)$, where $h\in\mathfrak{H}$. It follows from (112), (117), (118) that equality (116) holds.

Let us transform the formula

$$(\Lambda_\lambda\widetilde{\varphi}_{1n}+f_n,\widetilde{\psi}_{5n})_{\mathbf{m}_0}-(\widetilde{\varphi}_{1n},\Lambda_\lambda\widetilde{\psi}_{5n}+f_n)_{\mathbf{m}_0}.$$

It follows from Lemma 2.3, Remark 2.4, and equalities (112), (117) that

$$\widetilde{\varphi}_{1n}(t)=iJx_{n1}-iJ\int_{\alpha_n}^{t}dp_0(s)\widetilde{\varphi}_{1n}(s)-iJ\int_{\alpha_n}^{t}d\mathbf{n}_{0\lambda}(s)\widetilde{\varphi}_{1n}(s)-iJ\int_{\alpha_n}^{t}d\mathbf{m}_0(s)f_n(s),$$

$$\widetilde{\psi}_{5n}(t)=2\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu n1}-iJ\int_{\alpha_n}^{t}dp_0(s)\widetilde{\psi}_{5n}(s)-iJ\int_{\alpha_n}^{t}d\mathbf{n}_{0\lambda}(s)\widetilde{\psi}_{5n}(s)-iJ\int_{\alpha_n}^{t}d\mathbf{m}_0(s)f_n(s).$$

Using these equalities and (26), we get

$$\widetilde{\varphi}_{1n}(t)=iJx_{n1}-iJ\int_{\alpha_n}^{t}dp_0(s)\widetilde{\varphi}_{1n}(s)-iJ\int_{\alpha_n}^{t}d\mathbf{m}_0(s)(\Lambda_\lambda\widetilde{\varphi}_{1n})(s)-iJ\int_{\alpha_n}^{t}d\mathbf{m}_0(s)f_n(s), \tag{119}$$

$$\widetilde{\psi}_{5n}(t)=2\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu n1}-iJ\int_{\alpha_n}^{t}dp_0(s)\widetilde{\psi}_{5n}(s)-iJ\int_{\alpha_n}^{t}d\mathbf{m}_0(s)(\Lambda_\lambda\widetilde{\psi}_{5n})(s)-iJ\int_{\alpha_n}^{t}d\mathbf{m}_0(s)f_n(s). \tag{120}$$

We denote $\widetilde{F}_{1n}=\Lambda_\lambda\widetilde{\varphi}_{1n}+f_n$, $\widetilde{P}_{1n}=\Lambda_\lambda\widetilde{\psi}_{5n}+f_n$. Then

$$(\Lambda_\lambda\widetilde{\varphi}_{1n}+f_n,\widetilde{\psi}_{5n})_{\mathbf{m}_0}-(\widetilde{\varphi}_{1n},\Lambda_\lambda\widetilde{\psi}_{5n}+f_n)_{\mathbf{m}_0}=(\widetilde{F}_{1n},\widetilde{\psi}_{5n})_{\mathbf{m}_0}-(\widetilde{\varphi}_{1n},\widetilde{P}_{1n})_{\mathbf{m}_0}.$$

Using (119), (120), we obtain that equations (121), (122) hold (see below)

$$\widetilde{\varphi}_{1n}(t)=iJx_{n1}-iJ\int_{\alpha_n}^{t}dp_0(s)\widetilde{\varphi}_{1n}(s)-iJ\int_{\alpha_n}^{t}d\mathbf{m}_0(s)\widetilde{F}_{1n}(s); \tag{121}$$

$$\widetilde{\psi}_{5n}(t)=2\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu n1}-iJ\int_{\alpha_n}^{t}dp_0(s)\widetilde{\psi}_{5n}(s)-iJ\int_{\alpha_n}^{t}d\mathbf{m}_0(s)\widetilde{P}_{1n}(s). \tag{122}$$

Therefore, we can apply Lagrange formula (10) to the functions $\widetilde{\varphi}_{1n},\widetilde{F}_{1n},\widetilde{\psi}_{5n},\widetilde{P}_{1n}$ for $c_1=\alpha_n$, $c_2=\beta_n$, $\mathbf{q}=\mathbf{m}_0$, $\mathbf{p}_1=\mathbf{p}_2=\mathbf{p}_0$. By (96), (112), it follows that $\lim_{t\to\beta_n-0}\widetilde{\varphi}_1(t)=\widetilde{\varphi}_1(\beta_n)=0$. Using (10), we get

$$(\Lambda_\lambda\widetilde{\varphi}_{1n}+f_n,\widetilde{\psi}_{5n})_{\mathbf{m}_0}-(\widetilde{\varphi}_{1n},\Lambda_\lambda\widetilde{\psi}_{5n}+f_n)_{\mathbf{m}_0}=\int_{\alpha_n}^{\beta_n}(d\mathbf{m}_0(t)\widetilde{F}_{1n}(t),\widetilde{\psi}_{5n}(t))-\int_{\alpha_n}^{\beta_n}(d\mathbf{m}_0(t)\widetilde{\varphi}_{1n}(t),\widetilde{P}_{1n}(t))=$$

$$=(iJ\widetilde{\varphi}_{1n}(\beta_n),\widetilde{\psi}_{5n}(\beta_n))-(iJ\widetilde{\varphi}_{1n}(\alpha_n),\widetilde{\psi}_{5n}(\alpha_n))=-(iJiJx_{n1},2\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu n1})=2(x_{n1},\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu n1}). \tag{123}$$

Equalities (116), (123) imply that

$$(\Lambda_\lambda\widetilde{\Phi}_1+f,\widetilde{\Psi}_1)_{\mathbf{m}_0}-(\widetilde{\Phi}_1,\Lambda_\lambda\widetilde{\Psi}_1+f)_{\mathbf{m}_0}=2\sum_{n=1}^{\Bbbk_1}\left(x_{n1},\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu n1}\right). \tag{124}$$

We claim that the equality

$$(\Lambda_\lambda\widetilde{\Phi}_1+f,\widetilde{\Psi}_3)_{\mathbf{m}_0}-(\widetilde{\Phi}_1,\Lambda_\lambda\widetilde{\Psi}_3)_{\mathbf{m}_0}=-2\sum_{n=1}^{\Bbbk_1}\sum_{k=2}^{\widetilde{\Bbbk}_n}\left(\left(\Lambda_\lambda\widetilde{\varphi}_{1n}+f_n,\widetilde{\psi}_{4nk}\right)_{\mathbf{m}_0}-\left(\widetilde{\varphi}_{1n},\Lambda_\lambda\widetilde{\psi}_{4nk}\right)_{\mathbf{m}_0}\right) \tag{125}$$

holds, where

$$\widetilde{\psi}_{4nk}(t)=\begin{cases}0 \ \text{ for } \ t<\tau_k,\\ w_n(t,\lambda)iJw_n^*(\tau_k,\overline{\lambda})\mathbf{m}(\{\tau_k\})\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu nk} \ \text{ for } \ t\geqslant\tau_k.\end{cases} \tag{126}$$

Indeed, using (44), (113), and the equality $(h_n,w_r)_{\mathbf{m}_0}=0$ for $n\neq r$, we obtain

$$\left(\sum_{n=1}^{\Bbbk_1} h_n, \widetilde{\Psi}_3\right)_{\mathbf{m}_0} = -2\left(\sum_{n=1}^{\Bbbk_1} h_n(t), \sum_{n=1}^{\Bbbk_1}\sum_{r=1}^{\Bbbk_1}\sum_{k=2}^{\widetilde{\Bbbk}_r} \mathfrak{X}_{[\tau_k,b]} w_r(t,\lambda) iJ w_r^*(\tau_k,\overline{\lambda})\mathbf{m}(\{\tau_k\})\mu_{nrk}\right)_{\mathbf{m}_0} =$$

$$= -2\left(\sum_{n=1}^{\Bbbk_1} h_n(t), \sum_{r=1}^{\Bbbk_1} \mathfrak{X}_{[\tau_k,b]} w_r(t,\lambda) iJ \sum_{k=2}^{\widetilde{\Bbbk}_r} w_r^*(\tau_k,\overline{\lambda})\mathbf{m}(\{\tau_k\})\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu rk}\right)_{\mathbf{m}_0} =$$

$$= -2\left(\sum_{n=1}^{\Bbbk_1} h_n(t), \sum_{n=1}^{\Bbbk_1}\sum_{k=2}^{\widetilde{\Bbbk}_n} \mathfrak{X}_{[\tau_k,b]} w_n(t,\lambda) iJ w_n^*(\tau_k,\overline{\lambda})\mathbf{m}(\{\tau_k\})\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu nk}\right)_{\mathbf{m}_0} =$$

$$= -2\sum_{n=1}^{\Bbbk_1}\left(h_n(t), \sum_{k=2}^{\widetilde{\Bbbk}_n} \mathfrak{X}_{[\tau_k,b]} w_n(t,\lambda) iJ w_n^*(\tau_k,\overline{\lambda})\mathbf{m}(\{\tau_k\})\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu nk}\right)_{\mathbf{m}_0} = -2\sum_{n=1}^{\Bbbk_1}\sum_{k=2}^{\widetilde{\Bbbk}_n}\left(h_n, \widetilde{\psi}_{4nk}\right)_{\mathbf{m}_0}. \quad (127)$$

for any function $h_n(t)$ such that $h_n(t) = \mathfrak{X}_{[\alpha_n,\beta_n)} h(t)$, where $h \in \mathfrak{H}$. It follows from (112), (126), (127) that equality (125) holds.

Now let us transform the formula

$$(\Lambda_\lambda \widetilde{\varphi}_{1n} + f_n, \widetilde{\psi}_{4nk})_{\mathbf{m}_0} - (\widetilde{\varphi}_{1n}, \Lambda_\lambda \widetilde{\psi}_{4nk})_{\mathbf{m}_0}.$$

It follows from Lemma 2.3, equalities (126), (45), (13) that the equality

$$\widetilde{\psi}_{4nk}(t) = iJ\mathbf{m}(\{\tau_k\})\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu nk} - iJ\int_{\tau_k}^t dp_0(s)\widetilde{\psi}_{4nk}(s) - iJ\int_{\tau_k}^t d\mathbf{n}_{0\lambda}(s)\widetilde{\psi}_{4nk}(s)$$

holds. Using this equality and (26), we get

$$\widetilde{\psi}_{4nrk}(t) = iJ\mathbf{m}(\{\tau_k\})\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu nk} - iJ\int_{\tau_k}^t dp_0(s)\widetilde{\psi}_{4nk}(s) - iJ\int_{\tau_k}^t d\mathbf{m}_0(s)(\Lambda_\lambda\widetilde{\psi}_{4nk})(s).$$

We denote $\widetilde{P}_{4nk} = \Lambda_\lambda \widetilde{\psi}_{4nk}$. Then the pair $\{\widetilde{\psi}_{4nk}, \widetilde{P}_{4nk}\}$ satisfies the equation

$$\widetilde{\psi}_{4nk}(t) = iJ\mathbf{m}(\{\tau_k\})\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu nk} - iJ\int_{\tau_k}^t dp_0(s)\widetilde{\psi}_{4nk}(s) - iJ\int_{\tau_k}^t d\mathbf{m}_0(s)\widetilde{P}_{4nk}(s).$$

We can apply Lagrange formula (10) to the functions $\widetilde{\varphi}_{1n}$, $\widetilde{F}_{1n} = \Lambda_\lambda\widetilde{\varphi}_{1n} + f_n$, $\widetilde{\psi}_{4nk}$, $\widetilde{P}_{4nk}$ for $c_1 = \tau_k$, $c_2 = \beta_r$, $\mathbf{q} = \mathbf{m}_0$, $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}_0$. We first note that equalities (112), (96) imply

$$\widetilde{\varphi}_{1n}(\tau_k) = -w_n(\tau_k,\lambda) iJ\int_{\alpha_n}^{\tau_k} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s) +$$

$$+ w_n(\tau_k,\lambda) iJ\int_{\alpha_n}^{\beta_n} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s) = w_n(\tau_k,\lambda) iJ\int_{\tau_k}^{\beta_n} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s).$$

Moreover, it follows from (45), (126) that $\widetilde{\psi}_{4nk}(\tau_k) = iJ\mathbf{m}(\{\tau_k\})\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu nk}$. Now using (10) and the equality $\lim_{t\to\beta_n-0}\widetilde{\varphi}_1(t) = \widetilde{\varphi}_1(\beta_n) = 0$, we get

$$(\Lambda_\lambda\widetilde{\varphi}_{1n} + f_n, \widetilde{\psi}_{4nk})_{\mathbf{m}_0} - (\widetilde{\varphi}_{1n}, \Lambda_\lambda\widetilde{\psi}_{4nk})_{\mathbf{m}_0} = (\widetilde{F}_{1n}, \widetilde{\psi}_{4nk})_{\mathbf{m}_0} - (\widetilde{\varphi}_{1n}, \widetilde{P}_{4nk})_{\mathbf{m}_0} =$$

$$= \int_{\tau_k}^{\beta_n}(d\mathbf{m}_0(t)\widetilde{F}_{1n}(t), \widetilde{\psi}_{4nk}(t)) - \int_{\tau_k}^{\beta_n}(d\mathbf{m}_0(t)\widetilde{\varphi}_{1n}(t), \widetilde{P}_{4nk}(t)) =$$

$$= (iJ\widetilde{\varphi}_{1n}(\beta_n), \widetilde{\psi}_{4nk}(\beta_n)) - (iJ\widetilde{\varphi}_{1n}(\tau_k), \widetilde{\psi}_{4nk}(\tau_k)) = \left(-iJw_n(\tau_k,\lambda)iJ\int_{\tau_k}^{\beta_n} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s), iJ\mathbf{m}(\{\tau_k\})\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu nk}\right) =$$

$$= \left(-w_n(\tau_k,\lambda)iJ\int_{\tau_k}^{\beta_n} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s), \mathbf{m}(\{\tau_k\})\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu nk}\right). \quad (128)$$

Using (96), (97), (99), (124), (125), (128), and the equality $\mathbf{m}(\{\tau_k\}) = \mathbf{m}^*(\{\tau_k\})$, we continue equality (115)

$$(u_-, y+z)_{\mathbf{m}} - (y-z, u_+)_{\mathbf{m}} = 2\sum_{n=1}^{\Bbbk_1}\left(x_{n1}, \sum_{\nu=1}^{\Bbbk_1}\mu_{\nu n1}\right) + 2\sum_{n=1}^{\Bbbk_1}\sum_{k=2}^{\widetilde{\Bbbk}_n}\left(\Lambda_\lambda^{-1}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}f_n(\tau_k), \mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu nk}\right)_{\mathbf{m}} +$$

$$+ 2\sum_{n=1}^{\Bbbk_1}\sum_{k=2}^{\widetilde{\Bbbk}_n}\left(w_n(\tau_k,\lambda)iJ\int_{\tau_k}^{\beta_n} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s), \mathbf{m}(\{\tau_k\}\sum_{\nu=1}^{\Bbbk_1}\mu_{\nu nk}\right) = 2(\widetilde{x}, M(\lambda)\widetilde{x}). \quad (129)$$

On the other hand, using (106), we get

$$(f, y+z)_{\mathbf{m}} - (y-z, f)_{\mathbf{m}} = (u_- - \Lambda_\lambda(y-z), y+z)_{\mathbf{m}} - (y-z, u_+ - \Lambda_\lambda(y+z))_{\mathbf{m}} =$$
$$= (u_-, y+z)_{\mathbf{m}} - (y-z, u_+)_{\mathbf{m}} - (\Lambda_\lambda y, y)_{\mathbf{m}} + (\Lambda_\lambda z, y)_{\mathbf{m}} - (\Lambda_\lambda y, z)_{\mathbf{m}} +$$
$$+ (\Lambda_\lambda z, z)_{\mathbf{m}} + (y, \Lambda_\lambda y)_{\mathbf{m}} - (z, \Lambda_\lambda y)_{\mathbf{m}} + (y, \Lambda_\lambda z)_{\mathbf{m}} - (z, \Lambda_\lambda z)_{\mathbf{m}}. \quad (130)$$

Combining (129) and (130), we obtain

$$(f, y+z)_{\mathbf{m}} - (y-z, f)_{\mathbf{m}} = 2(\widetilde{x}, M(\lambda)\widetilde{x}) - (\Lambda_\lambda y, y)_{\mathbf{m}} + (\Lambda_\lambda z, y)_{\mathbf{m}} -$$
$$- (\Lambda_\lambda y, z)_{\mathbf{m}} + (\Lambda_\lambda z, z)_{\mathbf{m}} + (y, \Lambda_\lambda y)_{\mathbf{m}} - (z, \Lambda_\lambda y)_{\mathbf{m}} + (y, \Lambda_\lambda z)_{\mathbf{m}} - (z, \Lambda_\lambda z)_{\mathbf{m}}.$$

Therefore, $\mathrm{Im}(f, y)_{\mathbf{m}} = \mathrm{Im}(\widetilde{x}, M(\lambda)\widetilde{x}) - \mathrm{Im}[(\Lambda_\lambda y, y)_{\mathbf{m}} - (\Lambda_\lambda z, z)_{\mathbf{m}}]$. Since $y = R_\lambda f$, we get

$$\mathrm{Im}(M(\lambda)\widetilde{x}, \widetilde{x}) - \mathrm{Im}(\Lambda_\lambda z, z)_{\mathbf{m}} = \mathrm{Im}(R(\lambda)f, f)_{\mathbf{m}} - \mathrm{Im}(\Lambda_\lambda R(\lambda)f, R(\lambda)f)_{\mathbf{m}}.$$

It follows from (29), (25), (23) that

$$\mathrm{Im}(M(\lambda)\widetilde{x}, \widetilde{x}) - \int_{[a,b]\setminus\mathcal{S}_0}\mathrm{Im}(d\mathbf{n}_\lambda z, z) = \mathrm{Im}(R(\lambda)f, f)_{\mathbf{m}} - \int_{[a,b]\setminus\mathcal{S}_0}\mathrm{Im}(d\mathbf{n}_\lambda R(\lambda)f, R(\lambda)f).$$

Using (6), we obtain equality (84). The theorem is proved. □

References

[1] N. I. Akhiezer, I. M. Glazman, Theory of Linear Operators in Hilbert Space. New York: Dover Publications Inc., 2013. [Russian edition: Vishcha Shkola, Kharkiv, 1978.]

[2] A. G. Baskakov, Analysis of Linear Differential Equations by Methods of the Spectral Theory of Difference Operators and Linear Relations, Uspekhi Mat. Nauk 68 (2013), No. 1, 77–128; Engl. transl.: Russian Mathematical Surveys 68 (2013), No. 1, 69–116.

[3] J.Behrndt, S.Hassi, H.Snoo, R.Wietsma, Square-Integrable Solutions and Weil functions for Singular Canonical Systems, Math. Nachr. 284 (2011), No. 11–12, 1334–1384.

[4] J.Behrndt, S.Hassi, H.Snoo, Boundary Value Problems, Weil Functions, and Differential Operators, Monographs in Mathematics, Vol. 108, Birkhauser, 2020.

[5] Yu. M. Berezanski, Expansions in Eigenfunctions of Selfadjoint Operators, Naukova Dumka, Kiev, 1965; Engl. transl.: Amer. Math. Soc., Providence, RI, 1968.

[6] V. M. Bruk, On a Number of Linearly Independent Square-Integrable Solutions of Systems of Differential Equations, Functional Analysis 5 (1975), Uljanovsk, 25–33.

[7] V. M. Bruk, On Linear Relations in a Space of Vector Functions, Mat. Zametki 24 (1978), No. 4, 499–511; Engl. transl.: Mathematical Notes 24 (1978), No. 4, 767–773.

[8] V. M. Bruk, On Linear Relations Generated by a Differential Expression and Nevanlinna Operator Function, Journal of Math. Physics, Analysis, Geometry 7 (2011), No. 2, 115–140.

[9] V. M. Bruk, On Linear Relations Generated by an Integral Equation with a Nevanlinna Measure, Izv. VUZ. Mathem. (2012), No. 10, 3–19; Engl. transl.: Rus. Mathem. 56 (2012), No. 10, 1–14.

[10] V. M. Bruk, On the Characteristic Operator of an Integral Equation with a Nevanlinna Measure in the Infinite-Dimensional Case, Journal of Math. Physics, Analysis, Geometry 10 (2014), No. 2, 163–188.

[11] V. M. Bruk, Boundary Value Problems for Integral Equations with Operator Measures, Probl. Anal. Issues Anal. 6(24) (2017), No. 1, 19–40.

[12] V. M. Bruk, On Self-adjoint Extensions of Operators Generated by Integral Equations, Taurida Journal of Computer Science Theory and Mathematics (2017), No. 1(34), 17–31.

[13] V. M. Bruk, Generalized Resolvents of Operators Generated by Integral Equations, Probl. Anal. Issues Anal. 7(25) (2018), No. 2, 20–38.

[14] V. M. Bruk, Dissipative Extensions of Linear Relations Generated by Integral Equations with Operator Measures, Journal of Math. Physics, Analysis, Geometry 16 (2020), No.4, 281–401.
[15] V. M. Bruk, Invertible Linear Relations Generated by Integral Equations with Operator Measures, Filomat, 35 (2021), No. 5, 1589–1607.
[16] V. M. Bruk, Generalized Resolvents of Linear Relations Generated by Integral Equations with Operator Measures, Filomat, 36 (2022), No. 14, 4793–4810.
[17] V. M. Bruk, On Characteristic Functions of Generalized Resolvents Generated by Integral Equations with Operator Measures, Filomat, 37 (2023), No. 23, 7699–7718.
[18] A. Dijksma, H. S. V. de Snoo, Self-adjoint Extensions of Symmetric Subspaces, Pac. J. Math., 54 (1974), No.1, 71–100.
[19] V. I. Gorbachuk, M. L. Gorbachuk, Boundary Value Problems for Differential-Operator Equations, Naukova Dumka, Kiev, 1984; Engl. transl.: Kluver Acad. Publ., Dordrecht-Boston-London, 1991.
[20] T.Kato, Perturbation Theory for Linear Operators, Springer-Verlag, Berlin, Heidelberg, New York, 1966.
[21] V. I. Khrabustovskyi, On Characteristic Matrix of Weil-Titchmarsh Type for Differential-Operator Equations which Contains the Spectral Parameter in Linearly or Nevanlinna's Manner, Mat. Fiz., Geom. 10 (2003), No. 2, 205–227.
[22] V. I. Khrabustovskyi, On the Characteristic Operators and Projections and on the Solutions of Weyl Type of Disspative and Accumulative Operator Systems. I. General Case. 2. Abstract Theory. 3. Separated Boundary Conditions. Journal of Math. Physics, Analysis, Geometry 2 (2006), No.2, 149–175; No.3, 299–317; No4, 449-473.
[23] V. Khrabustovskyi, Analogs of Generalized Resolvents for Relations Generated by a Pair of Differential Operator Expressions One of which Depends on Spectral Parameter in Nonlinear Manner, Journal of Math. Physics, Analysis, Geometry 9 (2013), No.4, 496–535.
[24] J.-L. Lions, E. Magenes, Problemes aux Limites non Homogenes et Applications, Dunod, Paris, 1968.
[25] M.M. Malamud, S.M. Malamud, On the Spectral Theory of Operator Measures, Funk. Anal., 36 (2002), No.2, 83–89; Engl. transl.: Funct. Anal. and Appl., 36 (2002), No.2, 154–158.
[26] B. C. Orcutt, Canonical Differential Equations, Dissertation, University of Virginia, 1969.
[27] F. S. Rofe-Beketov, Square-Integrable Solutions, Self-adjoint Extensions and Spectrum of Differential Systems, Differential Equations, Proc. Int. Conf., Uppsala, 1977, 169–178.
[28] A. V. Straus, Generalized Resolvents of Symmetric Operators, Izv. Akad. Nauk SSSR, Ser, Mat., 18 (1954), No.1, 51–86.
[29] A. V. Straus, On Generalized Resolvents and Spectral Functions of Differential Operators of Even Order, Izv. Akad. Nauk SSSR, Ser. Mat., 21 (1957), No.6, 785–808.