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On Bishop frame of a partially null curve in Minkowski space-time $\mathbb{E}^4_1$

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Abstract. In this paper, we introduce Bishop frame of a partially null curve α in Minkowski space-time $\mathbb{E}^4_1$. We prove that such curve has three Bishop frames determined by the particular solutions of the third order non-linear differential equation. We show that the Frenet frame of α can be obtained by rotating the Bishop frame that corresponds to the particular solution $\theta(s) = \int \kappa_1(s)\, ds$ of the mentioned differential equation. As an application, we obtain the parametrization of a lightlike hypersurface and focal surface with base curve α in terms of its Bishop frame. Finally, we prove that a lightlike focal surface along a partially null helix has no critical value set.

1. Introduction

The Bishop frame (relatively parallel adapted frame, rotation minimizing frame) $\{T, N_1, N_2\}$ of a regular curve in Euclidean 3-space is defined in [1] as a positively oriented frame obtained by rotating the Frenet frame about the tangential vector field T for an angle of rotation $\theta(s) = \int \tau(s)\, ds$, where τ is the torsion of the curve. After such rotation, the vector fields N_1' and N_2' become collinear with T, which means that they make no rotations in the planes $N_1^\perp$ and $N_2^\perp$, respectively. The Bishop frame of a regular curve is not a unique and it is well defined even in the points of the curve where the first Frenet curvature vanishes. As such, it has various applications in rigid body mechanics [2], computer graphics [3], deformation of tubes [4], sweep surface modeling [5] and in differential geometry in studying different types of curves (see for example [6–9]).

New versions of the Bishop frames in Euclidean space $\mathbb{E}^3$ are introduced in [10, 11]. In the three dimensional Minkowski space $\mathbb{E}^3_1$ and four dimensional Minkowski space-time $\mathbb{E}^4_1$, the Bishop frames of spacelike, timelike, pseudo null and null Cartan curves are defined in [12–17]. The curves in $\mathbb{E}^4_1$ along which the Bishop frame is not defined yet, are partially null curves. Partially null curves are the spacelike curves that lie in a lightlike hyperplane of $\mathbb{E}^4_1$ and whose Frenet frame $\{T, N, B_1, B_2\}$ contains two null binormal vector fields B_1 and B_2. According to the Frenet frame's equations, the vector fields T', B_1' and B_2' are collinear with the principal normal vector field N at each point of the curve, so they make minimal rotations in the corresponding hyperplanes $T^\perp = span\{N, B_1, B_2\}$, $B_1^\perp = span\{T, N, B_1\}$ and $B_2^\perp = span\{T, N, B_2\}$, respectively. Hence the Frenet frame of a partially null curve α has a rotation minimizing property. In

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this paper, we show that such curve also has another frame with rotation minimizing property and we called it the *Bishop frame*. In particular, we prove that there exist three Bishop frames of α determined by the particular solutions of the corresponding the third order non-linear differential equation. Moreover, we find that the Frenet frame of α can be obtained by rotating the Bishop frame that corresponds to the particular solution $\theta(s) = \int \kappa_1(s)\,ds$ of the mentioned differential equation, where $\kappa_1(s)$ is the first Frenet curvature of α.

Lightlike hypersurfaces in Minkowski space $\mathbb{E}^n_1$ are the ruled submanifolds whose induced first fundamental form is a positive semi-definite. Such hypersurfaces are tangent to the lightcone at any regular point. Their singularities along a spacelike submanifold are classified in [19, 20]. It is known that they provide models for studying different types of horizons (event, Cauchy, Kruskal) and play an important role in the quantum theory of gravity. The critical value set of a lightlike hypersurface along a spacelike submanifold is called the lightlike focal set of the submanifold. Lightlike hypersurfaces and focal surfaces along a spacelike curve γ in $\mathbb{E}^4_1$ whose Frenet frame contains non-null vector fields, are introduced in [20]. Such submanifolds along a partially null curve α in $\mathbb{E}^4_1$ are not investigated yet. In this paper, we obtain their parametrizations in terms of Bishop frame of α obtained by rotating the Frenet frame. We define the Lorentzian distance-squared function $G : I \times \mathbb{E}^4_1 \to \mathbb{R}$ along α and prove that a lightlike focal surface along a partially null helix has no critical value set. The obtained results can be used in classifications of singularities of lightlike hypersurfaces along partially null curves and in many mathematical and physical applications related with such curves, such as inextensible flows, Bäcklund transformations, canal surfaces, etc.

2. Preliminaries

Minkowski space-time $\mathbb{E}^4_1$ is the real vector space $\mathbb{E}^4$ equipped with the standard indefinite flat metric $\langle \cdot, \cdot \rangle$ given by

$$\langle x, y \rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4,$$

for any two vectors $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ in $\mathbb{E}^4_1$. Since $\langle \cdot, \cdot \rangle$ is an indefinite metric, a vector $x \neq 0$ in $\mathbb{E}^4_1$ can be *spacelike*, *timelike*, or *null* (*lightlike*), if $\langle x, x \rangle > 0$,$\langle x, x \rangle < 0$, or $\langle x, x \rangle = 0$, respectively ([21]). In particular, the vector $x = 0$ is said to be a spacelike. The *norm* (*length*) of a vector x in $\mathbb{E}^4_1$ is given by $||x|| = \sqrt{|\langle x, x \rangle|}$.

An arbitrary curve $\alpha : I \to \mathbb{E}^4_1$ can locally be *spacelike, timelike*, or *null* (*lightlike*), if all of its velocity vectors α' are spacelike, timelike, or null, respectively ([21]).

Partially null curve α in $\mathbb{E}^4_1$ is a spacelike curve whose Frenet frame $\{T, N, B_1, B_2\}$ satisfies the conditions

$$\begin{gathered} \langle T, T \rangle = \langle N, N \rangle = 1, \quad \langle B_1, B_1 \rangle = \langle B_2, B_2 \rangle = 0, \\ \langle T, N \rangle = \langle T, B_1 \rangle = \langle T, B_2 \rangle = \langle N, B_1 \rangle = \langle N, B_2 \rangle = 0, \quad \langle B_1, B_2 \rangle = 1, \end{gathered} \tag{1}$$

$$\langle T, N \times B_1 \times B_2 \rangle = det(T, N, B_1, B_2) = 1,$$

where T, N, B_1 and B_2 are the tangent, the principal normal, the first binormal and the second binormal vector field of α, respectively.

The Frenet frame's equations read ([22])

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 & 0 \\ 0 & -\kappa_2 & 0 & -\kappa_3 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}. \tag{2}$$

The curvature functions $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s) = 0$ are called the *first*, the *second* and the *third Frenet curvature* of α, respectively.

The *lightcone* LC_p in $\mathbb{E}^4_1$ with vertex at a point p is a lightlike hypersurface defined by ([21])

$$LC_p = \{X \in \mathbb{E}^4_1 \setminus \{p\} \mid \langle X - p, X - p \rangle = 0\}.$$

A hypersurface $H \subset \mathbb{E}^4_1$ is called a *lightlike*, if it is tangential to the lightcone at any regular point ([20]).

If $\gamma : I \to \mathbb{E}^4_1$ is a spacelike curve with a non-null vector fields T,N,B_1 and B_2, *pseudo-normal space* of γ is defined as ([20])

$$\gamma^{\perp} = \{V \in \mathbb{E}^4_1 \,|\, \langle V, T\rangle = 0\}.$$

The unit *pseudosphere* with respect to a future directed timelike normal vector field n^T along γ, is given by ([20])

$$N_1(\gamma)_p[n^T] = \{\xi \in \gamma^{\perp} \,|\, \langle \xi, n^T(p)\rangle = 0,\ \langle \xi, \xi\rangle = 1\},$$

where $p = \gamma(s)$. *The lightcone Gauss image* of pseudosphere $N_1(\gamma)_p[n^T]$ has parametrization of the form ([20])

$$LG(n^T)(s, \theta) = n^T(s) + \cos\theta B_1(s) + \sin\theta B_2(s).$$

The paramerization of a lightlike hypersurface along γ relative to n^T reads

$$LH_{\gamma}((s, \theta), t) = \gamma(s) + t(n^T(s) + \cos\theta B_1(s) + \sin\theta B_2(s)).$$

The Lorentzian distance-squared functions $G : I \times \mathbb{E}^4_1 \to \mathbb{R}$ on γ are defined as ([20])

$$G(p, \lambda) = G(s, \lambda) = \langle \gamma(s) - \lambda, \gamma(s) - \lambda\rangle,$$

where $p = \gamma(s)$ and $\lambda \in \mathbb{E}^4_1$. *The discriminant set of order k* of $G(p, \lambda)$ is given by ([20])

$$D^k_G = \Big\{\lambda \in \mathbb{R}^4_1 \,|\, \exists s \in I,\ G(s, \lambda) = \frac{\partial G}{\partial s}(s, \lambda) = \cdots = \frac{\partial^k G}{\partial s^k}(s, \lambda) = 0\Big\}.$$

3. Bishop frame of a partially null curve in $\mathbb{E}^4_1$

In this section, we introduce Bishop frame $\{N_0, N_1, N_2, N_3\}$ of a partially null curve α in $\mathbb{E}^4_1$ with the Frenet curvatures $\kappa_1(s) \neq 0$, $\kappa_2(s) \neq 0$ and $\kappa_3(s) = 0$. We find the relation between the Frenet and Bishop frame of α and derive the Bishop frame's equations. We show that Bishop frame and Bishop curvatures are determined by the particular solutions of the corresponding the third order non-linear differential equation. We prove that the Frenet frame of α can be obtained by rotating the Bishop frame that corresponds to the particular solution $\theta(s) = \int \kappa_1(s)ds$ about timelike plane spanned by $\{N_2, N_3\}$.

Let us consider a partially null curve α in $\mathbb{E}^4_1$ parameterized by arc-length s with the Frenet frame $\{T, N, B_1, B_2\}$. Denote by $\{N_0, N_1, N_2, N_3\}$ a new frame along α, satisfying the conditions

$$\begin{aligned} \langle N_2, N_2\rangle = \langle N_3, N_3\rangle = 0, \quad \langle N_0, N_0\rangle = \langle N_1, N_1\rangle = \langle N_2, N_3\rangle = 1, \\ \langle N_0, N_1\rangle = \langle N_0, N_2\rangle = \langle N_0, N_3\rangle = \langle N_1, N_2\rangle = \langle N_1, N_3\rangle = 0, \end{aligned} \tag{3}$$

where $N_3 = B_2$ and

$$\langle N_0, N_1 \times N_2 \times N_3\rangle = det(N_0, N_1, N_2, N_3) = 1. \tag{4}$$

We introduce *relatively parallel vector fields* along α as follows.

Definition 3.1. *The vector fields N_0, N_1 and N_2 along a partially null curve α in $\mathbb{E}^4_1$ satisfying the conditions (3) and (4) are said to be relatively parallel, if their derivatives N_0', N_1' and N_2' are collinear with N_2 at each point the curve.*

According to Definition 3.1, the vector fields N_0', N_1' and N_2' make minimal rotations in the hyperplanes $N_0^{\perp} = span\{N_1, N_2, N_3\}$, $N_1^{\perp} = span\{N_0, N_2, N_3\}$ and $N_2^{\perp} = span\{N_0, N_1, N_2\}$, respectively.

Definition 3.2. *The Bishop frame of a partially null curve α in $\mathbb{E}^4_1$ is the frame $\{N_0, N_1, N_2, N_3\}$ containing the second binormal vector field $N_3 = B_2$ and relatively parallel vector fields N_0, N_1 and N_2 satisfying the conditions (3) and (4).*

In what follows, we define the curvature functions of α with respect to the Bishop frame and we called them the *Bishop curvatures*.

Definition 3.3. *Let α be a partially null curve in $\mathbb{E}^4_1$ with Bishop frame $\{N_0, N_1, N_2, N_3\}$. The Bishop curvatures of α are given by*

$$\sigma_1 = -\langle N_0', N_3\rangle, \quad \sigma_2 = -\langle N_1', N_3\rangle, \quad \sigma_3 = -\langle N_2', N_3\rangle. \tag{5}$$

In the next theorem, we derive the Bishop frame's equations. Analogous Bishop frame's equations for another types of curves in $\mathbb{E}^4_1$, can be found in [12–14].

Theorem 3.4. *If α is a partially null curve in $\mathbb{E}^4_1$ with the Bishop frame $\{N_0, N_1, N_2, N_3\}$, then the Bishop frame's equations read*

$$\begin{bmatrix} N_0'(s) \\ N_1'(s) \\ N_2'(s) \\ N_3'(s) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\sigma_1(s) & 0 \\ 0 & 0 & -\sigma_2(s) & 0 \\ 0 & 0 & -\sigma_3(s) & 0 \\ \sigma_1(s) & \sigma_2(s) & 0 & \sigma_3(s) \end{bmatrix} \begin{bmatrix} N_0(s) \\ N_1(s) \\ N_2(s) \\ N_3(s) \end{bmatrix}. \tag{6}$$

Proof. Assume that α is a partially null curve parameterized by arc length parameter s with the Bishop frame $\{N_0, N_1, N_2, N_3\}$. Since N_0 is a relatively parallel vector field, according to Definition 1, we have

$$N_0'(s) = c_0(s)N_2(s), \tag{7}$$

where $c_0(s)$ is some differentiable function. By using the relations (3), (5) and (7), we find

$$\langle N_0'(s), N_3(s)\rangle = c_0(s) = -\sigma_1(s).$$

Substituting this in (7), we obtain

$$N_0'(s) = -\sigma_1(s)N_2(s). \tag{8}$$

In a similar way, we get

$$\begin{aligned} N_1'(s) &= -\sigma_2(s)N_2(s), \\ N_2'(s) &= -\sigma_3(s)N_2(s), \\ N_3'(s) &= \sigma_1(s)N_0(s) + \sigma_2(s)N_1(s) + \sigma_3(s)N_3(s). \end{aligned} \tag{9}$$

Hence relations (8) and (9) imply relation (6). □

Next, we obtain the relation between Bishop frame and Frenet frame, as well as between the curvature functions with respect to these frames. Similar relations between the frames of another types of curves in $\mathbb{E}^4_1$, are obtained in [12–14].

Theorem 3.5. *Let α be a partially null curve in $\mathbb{E}^4_1$ parameterized by arc-length s, with the Frenet curvatures $\kappa_1(s) \neq 0$, $\kappa_2(s) \neq 0$, $\kappa_3(s) = 0$ and Bishop curvatures $\sigma_1(s)$, $\sigma_2(s)$ and $\sigma_3(s)$. Then:*

(i) the Bishop curvatures of α have the form

$$\begin{aligned} \sigma_1(s) &= \kappa_2(s)\cos\theta(s), \\ \sigma_2(s) &= \kappa_2(s)\sin\theta(s), \\ \sigma_3(s) &= \frac{\kappa_2(s)}{\theta'(s)}\Big(\frac{\kappa_1(s) - \theta'(s)}{\kappa_2(s)}\Big)', \end{aligned} \tag{10}$$

where $\theta(s) \neq$ constant satisfies the third order non-linear differential equation

$$\Big(\frac{1}{\theta'(s)}\Big(\frac{\kappa_1(s) - \theta'(s)}{\kappa_2(s)}\Big)'\Big)' = \frac{\theta'^2(s) - \kappa_1^2(s)}{2\kappa_2(s)} - \frac{\kappa_2(s)}{2\theta'^2(s)}\Big(\frac{\kappa_1(s) - \theta'(s)}{\kappa_2(s)}\Big)'^2; \tag{11}$$

(ii) the Bishop frame and the Frenet frame of α are related by

$$\begin{aligned} N_0 &= -\sin\theta T - \cos\theta N + \Big[\sin\theta\Big(\frac{\theta' - \kappa_1}{\kappa_2}\Big) - \cos\theta\Big(\frac{1}{\theta'}\Big(\frac{\kappa_1 - \theta'}{\kappa_2}\Big)'\Big)\Big]B_2, \\ N_1 &= \cos\theta T - \sin\theta N - \Big[\cos\theta\Big(\frac{\theta' - \kappa_1}{\kappa_2}\Big) + \sin\theta\Big(\frac{1}{\theta'}\Big(\frac{\kappa_1 - \theta'}{\kappa_2}\Big)'\Big)\Big]B_2, \\ N_2 &= \Big(\frac{\theta' - \kappa_1}{\kappa_2}\Big)T + \frac{1}{\theta'}\Big(\frac{\theta' - \kappa_1}{\kappa_2}\Big)'N + B_1 - \frac{1}{2}\Big[\Big(\frac{\theta' - \kappa_1}{\kappa_2}\Big)^2 + \Big(\frac{1}{\theta'}\Big(\frac{\theta' - \kappa_1}{\kappa_2}\Big)'\Big)^2\Big]B_2, \\ N_3 &= B_2. \end{aligned} \tag{12}$$

Proof. Assume that $\{N_0, N_1, N_2, N_3\}$ is the Bishop frame of a partially null curve α parameterized by arc length s. Differentiating the equation $N_3 = B_2$ with respect to s and using (2) and (6), we find

$$N_3' = B_2' = \sigma_1 N_0 + \sigma_2 N_1 + \sigma_3 N_3 = -\kappa_2 N.$$

The last relation gives

$$N = -\frac{\sigma_1}{\kappa_2}N_0 - \frac{\sigma_2}{\kappa_2}N_1 - \frac{\sigma_3}{\kappa_2}N_3. \tag{13}$$

By using the condition $\langle N, N\rangle = 1$ and relations (3) and (13), we obtain

$$\sigma_1^2 + \sigma_2^2 = \kappa_2^2. \tag{14}$$

Up to a parametrization, we may assume that

$$\sigma_1 = \kappa_2 \cos\theta, \quad \sigma_2 = \kappa_2 \sin\theta, \tag{15}$$

where θ is some differentiable function in s. Next we may consider two cases:

(A) $\theta = \theta_0 \in R$.

By using the relations $\theta = \theta_0$, (13) and (15), we obtain

$$N = -\cos\theta_0 N_0 - \sin\theta_0 N_1 - \frac{\sigma_3}{k_2}N_3. \tag{16}$$

Decompose the tangent vector field T of α as

$$T = \lambda N_0 + \mu N_1 + \nu N_2 + \omega N_3, \tag{17}$$

where λ, μ, ν and ω are some differentiable functions in s. Relations (16), (17) and the conditions $\langle T, T\rangle = 1$, $\langle T, N\rangle = \langle T, B_2\rangle = 0$ give

$$\nu = 0, \quad -\lambda\cos\theta_0 - \mu\sin\theta_0 = 0, \quad \lambda^2 + \mu^2 = 1. \tag{18}$$

Up to a parametrization, the solution of the previous system of equations reads

$$\lambda = -\sin\theta_0, \quad \mu = \cos\theta_0. \tag{19}$$

Substituting (19) and $\nu = 0$ in (17), we get

$$T = -\sin\theta_0 N_0 + \cos\theta_0 N_1 + \omega N_3. \tag{20}$$

Relations (2) and (16) yield

$$T' = \kappa_1 N = -\kappa_1\Big(\cos\theta_0 N_0 + \sin\theta_0 N_1 + \frac{\sigma_3}{\kappa_2}N_3\Big). \tag{21}$$

On the other hand, differentiating the relation (20) with respect to s and using (6), (15) and (21), we find

$$T' = \omega\kappa_2 \cos\theta_0 N_0 + \omega\kappa_2 \sin\theta_0 N_1 + (\omega' + \omega\sigma_3)N_3. \tag{22}$$

By using (21) and (22), we obtain

$$\omega = -\frac{\kappa_1}{\kappa_2}, \quad -\Big(\frac{\kappa_1}{\kappa_2}\Big)' - \frac{\kappa_1}{\kappa_2}\sigma_3 = -\frac{\kappa_1}{\kappa_2}\sigma_3. \tag{23}$$

so relation (23) implies

$$\frac{\kappa_1}{\kappa_2} = constant. \tag{24}$$

However, this is a contradiction, since the Frenet curvatures κ_1 and κ_2 of α do not satisfy relation (24) in a general case.

(B) $\theta \neq constant$.

Substituting (15) in (13), we obtain

$$N = -\cos\theta N_0 - \sin\theta N_1 - \frac{\sigma_3}{k_2}N_3. \tag{25}$$

By using (2) and (25), we find

$$T' = \kappa_1 N = -\kappa_1(\cos\theta N_0 + \sin\theta N_1 + \frac{\sigma_3}{k_2}N_3). \tag{26}$$

Differentiating the relation (25) with respect to s yields

$$N' = (\theta'\sin\theta - \sigma_3\cos\theta)N_0 - (\theta'\cos\theta + \sigma_3\sin\theta)N_1 + \kappa_2 N_2 - \Big(\Big(\frac{\sigma_3}{\kappa_2}\Big)' + \frac{\sigma_3^2}{\kappa_2}\Big)N_3. \tag{27}$$

According to (2), we have

$$N' = -\kappa_1 T + \kappa_2 B_1. \tag{28}$$

Relations (1), (3), (27) and (28) give

$$\langle N', N'\rangle = \kappa_1^2 = \theta'^2 - \sigma_3^2 - 2\kappa_2\Big(\frac{\sigma_3}{\kappa_2}\Big)'.$$

Consequently,

$$\Big(\frac{\sigma_3}{\kappa_2}\Big)' = \frac{1}{2\kappa_2}(\theta'^2 - \kappa_1^2 - \sigma_3^2). \tag{29}$$

By using the conditions $\langle T, T\rangle = 1$, $\langle T, N\rangle = 0$, $\langle T, B_2\rangle = 0$ and relation (25), we find

$$T = -\sin\theta N_0 + \cos\theta N_1 + dN_3, \tag{30}$$

where d is some differentiable function in s. Differentiating relation (30) with respect to s and using (15), we get

$$T' = (-\theta'\cos\theta + d\kappa_2\cos\theta)N_0 + (-\theta'\sin\theta + d\kappa_2\sin\theta)N_1 + (d' + d\sigma_3)N_3. \tag{31}$$

Next from relations (26) and (31) we obtain

$$d = \frac{\theta' - k_1}{k_2}, \qquad d' + d\sigma_3 = -\frac{\kappa_1}{\kappa_2}\sigma_3. \tag{32}$$

Hence the third Bishop curvature of α can be expressed as

$$\sigma_3 = \frac{\kappa_2}{\theta'}\Big(\frac{\kappa_1 - \theta'}{\kappa_2}\Big)'. \tag{33}$$

Therefore, relations (15) and (33) imply that relation (10) holds. In particular, substituting (33) in (29), we obtain the third order non-linear differential equation (11), which proves statement (i).

Multiplying relation (30) with $-\sin\theta$, relation (25) with $-\cos\theta$, adding the obtained equations and using (32) and (33), we find

$$N_0 = -\sin\theta T - \cos\theta N + \Big[\sin\theta\Big(\frac{\theta' - \kappa_1}{\kappa_2}\Big) - \cos\theta\Big(\frac{1}{\theta'}\Big(\frac{\kappa_1 - \theta'}{\kappa_2}\Big)'\Big)\Big]B_2. \tag{34}$$

Analogously, multiplying relation (30) with $\cos\theta$, relation (25) with $-\sin\theta$, adding the obtained equations and using (32) and (33), we get

$$N_1 = \cos\theta T - \sin\theta N - \Big[\cos\theta\Big(\frac{\theta' - \kappa_1}{\kappa_2}\Big) + \sin\theta\Big(\frac{1}{\theta'}\Big(\frac{\kappa_1 - \theta'}{\kappa_2}\Big)'\Big)\Big]B_2. \tag{35}$$

Substituting (34) and (35) in (27) and using (28) yields

$$N_2 = \Big(\frac{\theta' - \kappa_1}{\kappa_2}\Big)T + \frac{1}{\theta'}\Big(\frac{\theta' - \kappa_1}{\kappa_2}\Big)'N + B_1 - \frac{1}{2}\Big[\Big(\frac{\theta' - \kappa_1}{\kappa_2}\Big)^2 + \Big(\frac{1}{\theta'}\Big(\frac{\theta' - \kappa_1}{\kappa_2}\Big)'\Big)^2\Big]B_2.$$

The last relation together with (34) and (35) gives relation (12), which proves statement (ii). □

According to Theorem 3.5, every particular solution of the third order non-linear differential equation (11) provides the corresponding Bishop frame and Bishop curvatures. It can be easily seen that $\theta(s) = \int \kappa_1(s)\,ds$ is one particular solution. In general case, it is not always possible to get all three particular solutions explicitly. In the next theorem, we prove that the Frenet frame of α can be obtained by rotating the Bishop frame that corresponds to the particular solution $\theta(s) = \int k_1(s)\,ds$.

Theorem 3.6. *The Frenet frame of a partially null curve α in $\mathbb{E}^4_1$ with Frenet curvatures $\kappa_1(s) \neq 0$, $\kappa_2(s) \neq 0$, $\kappa_3(s) = 0$ can be obtained by rotating the Bishop frame that corresponds to the particular solution $\theta(s) = \int k_1(s)\,ds$ of differential equation (11) about timelike plane spanned by $\{N_2, N_3\}$ for the hyperbolic angle $\omega(s) = -\theta(s) - 90°$.*

Proof. Assume that α is a partially null curve in $\mathbb{E}^4_1$ with Frenet curvatures $\kappa_1(s) \neq 0$, $\kappa_2(s) \neq 0$ and $\kappa_3(s) = 0$. Substituting the particular solution $\theta(s) = \int k_1(s)\,ds$ of differential equation (11) in relation (12), we obtain that Bishop frame of α has the form

$$\begin{aligned} N_0 &= -\sin\theta T - \cos\theta N,\\ N_1 &= \cos\theta T - \sin\theta N,\\ N_2 &= B_1,\\ N_3 &= B_2. \end{aligned} \tag{36}$$

It can be easily seen that relation (36) implies

$$\begin{aligned} T &= -\sin\theta N_0 + \cos\theta N_1,\\ N &= -\cos\theta N_0 - \sin\theta N_1,\\ B_1 &= N_2,\\ B_2 &= N_3. \end{aligned} \tag{37}$$

Hence the Frenet frame of α is obtained by rotating the Bishop frame about timelike plane spanned by $\{N_2, N_3\}$ for the hyperbolic angle $\omega(s) = -\theta(s) - 90°$. □

Example 3.7. *Let us consider partially null helix α in $\mathbb{E}^4_1$ with parameter equation*

$$\alpha(s) = (\sin(s) + \cos(s) - 2s, \sin(s) + \cos(s) - 2s, \cos(s), \sin(s)).$$

The Frenet curvatures of α have the form

$$\kappa_1(s) = 1, \quad \kappa_2(s) = -2, \quad \kappa_3(s) = 0. \tag{38}$$

A straightforward calculation shows that the Frenet frame of α reads

$$\begin{aligned} T(s) &= (\cos(s) - \sin(s) - 2, \cos(s) - \sin(s) - 2, -\sin(s), \cos(s)), \\ N(s) &= (-\sin(s) - \cos(s), -\sin(s) - \cos(s), -\cos(s), -\sin(s)), \\ B_1(s) &= (1, 1, 0, 0), \\ B_2(s) &= (2\cos(s) - 2\sin(s) - \frac{7}{2}, 2\cos(s) - 2\sin(s) - \frac{5}{2}, -2\sin(s) - 1, 2\cos(s) - 1). \end{aligned}$$

Substituting (38) in (11), we obtain the third order non-linear differential equation

$$-2\theta'\theta''' + 3\theta''^2 - \theta'^2(\theta'^2 - 1) = 0. \tag{39}$$

Putting $\theta'(s) = t(s)$, where $t(s)$ is some differentiable function, the previous differential equation reduces to differential equation

$$-2tt'' + 3t'^2 - t^2(t^2 - 1) = 0.$$

Putting $t'(s) = p(s)$, where $p(s)$ is some differentiable function, the last differential equation reduces to the first order Bernoulli differential equation, and then to linear differential equation. A straightforward calculation shows that differential equation (39) has three particular solutions $\theta(s) = s$, $\theta(s) = -s$ and

$$\theta(s) = 2\arctan\Big(\sqrt{5}\tan(\frac{s}{2}) - 2\Big). \tag{40}$$

By using the relations $\theta(s) = -s$, (10) and (38), we find that Bishop curvatures of α have the form

$$\sigma_1(s) = -2\cos(s), \quad \sigma_2(s) = 2\sin(s), \quad \sigma_3(s) = 0.$$

Relation (12) implies that the Bishop frame which corresponds to the particular solution $\theta(s) = -s$, reads

$$\begin{aligned} N_0(s) &= \sin(s)T(s) - \cos(s)N(s) - \sin(s)B_2(s), \\ N_1(s) &= \cos(s)T(s) + \sin(s)N(s) - \cos(s)B_2(s), \\ N_2(s) &= T(s) + B_1(s) - \frac{1}{2}B_2(s), \\ N_3(s) &= B_2(s). \end{aligned} \tag{41}$$

Substituting the second particular solution $\theta(s) = s$ in (10) and using (38), we get

$$\sigma_1(s) = -2\cos s, \quad \sigma_2(s) = -2\sin s, \quad \sigma_3(s) = 0.$$

In this case, the Bishop frame of α has the form

$$\begin{aligned} N_0(s) &= -\sin(s)T(s) - \cos(s)N(s), \\ N_1(s) &= \cos(s)T(s) - \sin(s)N(s), \\ N_2(s) &= B_1(s), \\ N_3(s) &= B_2(s). \end{aligned} \tag{42}$$

Hence it is obtained by rotating the Frenet frame about timelike plane spanned by $\{B_1, B_2\}$. Finally, substituting the third particular solution (40) in (10) and using (38), we get

$$\sigma_1(s) = -2\cos(2\arctan(\sqrt{5}\tan(\frac{s}{2}) - 2)), \quad \sigma_2(s) = -2\sin(2\arctan(\sqrt{5}\tan(\frac{s}{2}) - 2)), \quad \sigma_3(s) = \frac{2\cos s}{2\sin s - \sqrt{5}}.$$

In this case, the Bishop frame of α has the form

$$\begin{aligned} N_0 &= -\sin\theta T - \cos\theta N + \Big[\sin\theta\Big(\frac{1-\theta'}{2}\Big) - \cos\theta\Big(\frac{1}{\theta'}\Big(\frac{\theta'-1}{2}\Big)'\Big)\Big]B_2, \\ N_1 &= \cos\theta T - \sin\theta N - \Big[\cos\theta\Big(\frac{1-\theta'}{2}\Big) + \sin\theta\Big(\frac{1}{\theta'}\Big(\frac{\theta'-1}{2}\Big)'\Big)\Big]B_2, \\ N_2 &= \Big(1-\frac{\theta'}{2}\Big)T + \frac{1}{\theta'}\Big(\frac{1-\theta'}{2}\Big)'N + B_1 - \frac{1}{2}\Big[\Big(\frac{1-\theta'}{2}\Big)^2 + \Big(\frac{1}{\theta'}\Big(\frac{1-\theta'}{2}\Big)'\Big)^2\Big]B_2, \\ N_3 &= B_2 \end{aligned} \tag{43}$$

where $\theta = \theta(s)$ is given by (40). It can be verified that the Bishop frames given by (41), (42) and (43) satisfy relation (6).

4. Lightlike hypersurfaces and focal surfaces along a partially null curve

Lightlike hypersurfaces and focal surfaces along a spacelike curve γ with a non-null Frenet frame's vector fields in $\mathbb{E}^4_1$ are defined in [20]. It is proved in [20] that the discriminant set of order 1 and 2 of the Lorentzian distance-squared functions along γ represents a lightlike hypersurface LH_γ and a lightlike focal surface LF_γ, respectively. Such submanifolds according to a partially null curve in the same space are not investigated yet. In this section, we obtain their parametrizations in terms of the Bishop frame given by relation (36). We also prove that a lightlike focal surface along a partially null helix has no critical value set.

According to [20], the image of a lightlike hypersurface along a spacelike curve γ is independent of the choice of a unit future-directed timelike normal vector field along the curve. The same property holds for the image of a lightlike hypersurface along a partially null curve α. In relation to that, let us choose

$$n^T = \frac{\sqrt{2}}{2}(B_1 - B_2), \tag{44}$$

to be a unit future-directed timelike normal vector field along α, where B_1 and B_2 are the first and the second binormal vector fields. Denote by

$$\alpha^\perp = \{V \in \mathbb{E}^4_1 \,|\, \langle \alpha', V\rangle = 0\}$$

the 3-dimensional pseudo-normal space of α. The 2-dimensional unit pseudosphere with respect to n^T is given by

$$N_1(\alpha)_p[n^T] = \{\xi \in \alpha^\perp \,|\, \langle \xi, n^T(p)\rangle = 0, \quad \langle \xi, \xi\rangle = 1\}.$$

Hence a lightcone Gauss image of pseudosphere $N_1(\alpha)_p[n^T]$ has parametrization of the form

$$LG[n^T](s,\phi) = n^T(s) + \cos\phi N(s) + \frac{\sqrt{2}}{2}\sin\phi(B_1(s) + B_2(s)). \tag{45}$$

A lightlike hypersurface $LH_\alpha(n^T) : N_1(\alpha)_p[n^T] \times \mathbb{R} \to \mathbb{E}^4_1$ along a spacelike curve $\alpha(s)$ relative to n^T has paramerization of the form ([20])

$$LH_\alpha((s,\phi),t) = \alpha(s) + tLG[n^T](s,\phi). \tag{46}$$

In the next theorem, we obtain the parametrization of a lightlike hypersurface along a partially null curve.

Theorem 4.1. *Let α be a partially null curve in $\mathbb{E}^4_1$ with Bishop frame given by (36). Then a lightlike hypersurface along α is parameterized by*

$$LH_\alpha((s,\phi),t) = \alpha(s) + t\Big[-\cos\phi\cos\theta N_0(s) - \cos\phi\sin\theta N_1(s) + \frac{\sqrt{2}}{2}(1+\sin\phi)N_2(s) + \frac{\sqrt{2}}{2}(\sin\phi - 1)N_3(s)\Big], \tag{47}$$

where $\theta(s) = \int \kappa_1(s)\,ds$.

Proof. By using the relations (44), (45) and (46), it follows that a lightlike hypersurface along partially null curve α relative to n^T, has parametrization of the form

$$LH_\alpha((s,\phi),t) = \alpha(s) + t\Big[\frac{\sqrt{2}}{2}(B_1(s) - B_2(s)) + \cos\phi N(s) + \frac{\sqrt{2}}{2}\sin\phi(B_1(s) + B_2(s))\Big]. \tag{48}$$

Substituting (37) in (48), we get (47). □

Next we obtain the parametrization of a lightlike focal surface LF_α along α which represents a critical value set of the lightlike hypersurface LH_α. It is shown in [20] that the discriminant set of order 2 of the Lorentzian distance-squared function along a spacelike curve γ represents lightlike focal surface LF_γ along γ. In relation to that, let us define the Lorentzian distance-squared functions $G : I \times \mathbb{E}^4_1 \to \mathbb{R}$ along partially null curve α as

$$G(p,\lambda) = G(s,\lambda) = \langle \alpha(s) - \lambda, \alpha(s) - \lambda \rangle,$$

where $\lambda \in \mathbb{E}^4_1$ and $p = \alpha(s)$. If $\lambda = \lambda_0$ is fixed, let us put $G(p,\lambda_0) = g(p)$. By taking the first and the second derivative of $g(p)$ with respect to s, we obtain

$$\begin{aligned} g'(p) &= 2\langle \alpha'(s), \alpha(s) - \lambda_0 \rangle, \\ g''(p) &= 2(\langle \alpha''(s), \alpha(s) - \lambda_0 \rangle + 1). \end{aligned} \tag{49}$$

The parametrization of a lightlike focal surface along a spacelike curve in $\mathbb{E}^4_1$ is obtained in [20]. In the next theorem, we obtain the parametrization of a lightlike focal surface along a partially null curve.

Theorem 4.2. *Let α be a partially null curve in $\mathbb{E}^4_1$ with Bishop frame given by (36). Then a lightlike focal surface along α has parametrization of the form*

$$\begin{aligned} LF_\alpha(s,\phi) = \alpha(s) &+ \frac{\sqrt{2}}{2\theta'\cos\phi}[-\sqrt{2}\cos\phi\cos\theta N_0(s) - \sqrt{2}\cos\phi\sin\theta N_1(s) + (1+\sin\phi)N_2(s) \\ &+ (\sin\phi - 1)N_3(s)], \end{aligned} \tag{50}$$

where $\cos\phi \neq 0$ *and* $\theta'(s) = \kappa_1(s) \neq 0$.

Proof. Let λ_0 be a fixed point on a lightlike hypersurface along α with parametrization (48). Then we have

$$\lambda_0 = \alpha(s) + t\Big[\frac{B_1(s) - B_2(s)}{\sqrt{2}} + \cos\phi N(s) + \sin\phi\Big(\frac{B_1(s) + B_2(s)}{\sqrt{2}}\Big)\Big].$$

The previous relation gives

$$\alpha(s) - \lambda_0 = -t\Big[\frac{B_1(s) - B_2(s)}{\sqrt{2}} + \cos\phi N(s) + \sin\phi\Big(\frac{B_1(s) + B_2(s)}{\sqrt{2}}\Big)\Big]. \tag{51}$$

Substituting (51) and $\alpha''(s) = \kappa_1(s)N(s) = \theta'(s)N(s)$ in (49) and using (1), we obtain

$$g''(p) = 2(1 - t\theta'(s)\cos\phi).$$

Hence $g''(p) = 0$ if and only if

$$t = \frac{1}{\theta'(s)\cos\phi}, \tag{52}$$

where $\theta'(s) = \kappa_1(s) \neq 0$ and $\cos\phi \neq 0$. Substituting (52) in (48) we obtain

$$LF_\alpha(s,\phi) = \alpha(s) + \frac{\sqrt{2}}{2\theta'(s)\cos\phi}[B_1(s) - B_2(s) + \sqrt{2}\cos\phi N(s) + \sin\phi(B_1(s) + B_2(s))]. \tag{53}$$

In particular, substituting (37) in (53), we get (50). □

It is proved in [20] that discriminant set of order 3 of the corresponding Lorentzian distance-squared function is the critical value set of a lightlike focal surface along a spacelike curve in $\mathbb{E}^4_1$. In the last theorem, we prove that a lightlike focal surface $LF_\alpha(s,\phi)$ along a partially null helix has no critical value set, which represents the discriminant set of order 3 of $G(p,\lambda)$.

Theorem 4.3. *Let α be a partially null helix in $\mathbb{E}^4_1$ with the Bishop frame given by (36). Then a lightlike focal surface along α has no critical value set.*

Proof. Assume that α is a partially null helix. Differentiating the relation (49) with respect to s, we find

$$g'''(p) = \frac{\sqrt{2}\kappa_2(s)(1-\sin\phi)}{\cos\phi},$$

where $\cos\phi \neq 0$. Then $g'''(p) = 0$ if and only if $\sin\phi = 1$. This implies $\cos\phi = 0$, which is a contradiction. □

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