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Geodesic equations of Weyl conformal gravity theory in CSS metric

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Abstract. In our work we presented the modified field equations generated by action of Weyl conformal gravity. Assuming static cylindric symmetry, we derived the corresponding Bach tensor and from field equation we find gravity potential. We solved geodesic equations in the case of conformal gravity potential. Also we consider precession perihelion of Solar planets and S stars.

1. Introduction

The modified theories of gravity have been proposed like alternative approaches to Einstein theory of gravity [1]. In this work we consider Weyl gravity theories of gravity [4]. Weyl gravity is a straightforward extension of General Relativity (GR) where, instead of the Hilbert-Einstein action, linear in the Ricci scalar R, one considers in the gravity Lagrangian density [2, 3, 7]:

$$L = R^{\mu\nu}R_{\mu\nu} - \frac{1}{3}R^2. \tag{1}$$

In the weak field limit, a gravitational potential may be written as [2, 3]:

$$\Phi = \frac{c^2}{2}\left[-\frac{2\beta}{r} + \gamma r - kr^2\right]. \tag{2}$$

and β, γ, and k are integration constants. This solution includes as special cases the Schwarzschild solution ($\gamma = k = 0$) and the Schwarzschild-de Sitter ($\gamma = 0$) solution; the latter requiring the presence of a cosmological constant in Einstein gravity.

We considered geodesic equations for cilindrically symmetric static (CSS) metric and Weyl gravity. In Section 2 we presented basic properties of CSS metric, while in section 3 we design an action integral, also find field equations. In section 4. we find geodesic equations as proposed in [1]. In section 5 we considered expression for precession perihelion for body rotating around supermassive star or black hole.

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In next section we have done the theoretical results for precession perihelion of potential Weyl's gravity. The section 7 is devoted to concluding remarks. Section 8 or Appendix A present detail calculation of Bach tensor W^{rr} [3], while in section 9 or appendix B we repesent solution of differential equation $W^{rr}=0$. After that at finally in section 9 or appendix C are given calculation of precession perihelion.

2. General properties

We proposed metric for cilindical static symmetric (CSS) space [3], because the metric tensor with components $g^{\mu\mu}$ depend only of radius r

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\varphi^2 + r^2 dz^2, \tag{3}$$

where is: $g_{00} = -B(r)$, $g_{11} = A(r)$, $g_{22} = r^2$, $g_{33} = r^2$, $g^{\mu\mu} = \dfrac{1}{g_{\mu\mu}}$, $g = g_{00}g_{11}g_{22}g_{33} = -ABr^4$.

Cristoffel symbols are given: $\Gamma^{\alpha}_{\varepsilon\nu} = \dfrac{1}{2} g^{\alpha\sigma}\left(g_{\sigma\varepsilon,\nu} + g_{\sigma\nu,\varepsilon} - g_{\varepsilon\nu,\sigma}\right)$ and $\Gamma^{\alpha}_{\mu\alpha,\nu} = \dfrac{\partial \Gamma^{\alpha}_{\mu\alpha}}{\partial x^{\nu}}$ and $g_{\mu\nu,\alpha} = \dfrac{\partial g_{\mu\nu}}{\partial x^{\alpha}}$.
Crystoffel symbols $\Gamma^{\alpha}_{\varepsilon\nu}$ different from zero are:
$\Gamma^1_{00} = \dfrac{1}{2A}\dfrac{dB}{dr}, \Gamma^0_{10} = \Gamma^0_{01} = \dfrac{1}{2B}\dfrac{dB}{dr}, \Gamma^1_{11} = \dfrac{1}{2A}\dfrac{dA}{dr}, \Gamma^1_{22} = \Gamma^1_{33} = -\dfrac{r}{A}, \Gamma^2_{12} = \Gamma^3_{13} = \dfrac{1}{r}.$

Rimman curvator tensor $R_{\rho\nu\alpha\beta}$ and Ricci tensor $R_{\nu\beta}$ are given by the following relations [1,5]:

$$R^{\mu}_{\nu\alpha\beta} = g^{\mu\rho} R_{\rho\nu\alpha\beta} = \Gamma^{\mu}_{\nu\alpha,\beta} - \Gamma^{\mu}_{\nu\beta,\alpha} + \Gamma^{\mu}_{\varepsilon\beta}\Gamma^{\varepsilon}_{\nu\alpha} - \Gamma^{\mu}_{\epsilon\alpha}\Gamma^{\epsilon}_{\nu\beta}, \tag{4}$$

$$R_{\nu\beta} = R^{\mu}_{\nu\mu\beta}, \tag{5}$$

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\varepsilon\nu}\Gamma^{\varepsilon}_{\mu\alpha} - \Gamma^{\varepsilon}_{\mu\nu}\Gamma^{\alpha}_{\varepsilon\alpha}. \tag{6}$$

Weyl tensor $C_{\mu\nu\alpha\beta}$ is defined by expression [5,7]:

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + \frac{1}{2}\left(g_{\nu\alpha}R_{\mu\beta} + g_{\mu\beta}R_{\nu\alpha} - g_{\nu\beta}R_{\mu\alpha} - g_{\mu\alpha}R_{\nu\beta}\right) + \frac{1}{6}R\left(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}\right). \tag{7}$$

Ricci tensor $R_{\mu\nu}$ and Ricci scalar R are expressed in the sense of CSS metric:

$$R_{00} = -\frac{1}{2A}\frac{d^2B}{dr^2} + \frac{1}{4A^2}\frac{dA}{dr}\frac{dB}{dr} + \frac{1}{4AB}\left(\frac{dB}{dr}\right)^2 - \frac{1}{rA}\frac{dB}{dr}, \tag{8}$$

$$R_{11} = \frac{1}{2B}\frac{d^2B}{dr^2} - \frac{1}{4B^2}\left(\frac{dB}{dr}\right)^2 - \frac{1}{4AB}\frac{dA}{dr}\frac{dB}{dr} - \frac{1}{rA}\frac{dA}{dr}, \tag{9}$$

$$R_{22} = \frac{1}{A} + r\frac{1}{2AB}\frac{dB}{dr} - r\frac{1}{2A^2}\frac{dA}{dr}, \quad R_{33} = R_{22}, \tag{10}$$

$$R = g^{\mu\nu}R_{\mu\nu} = \frac{1}{AB}\frac{d^2B}{dr^2} - \frac{1}{2AB^2}\left(\frac{dB}{dr}\right)^2 - \frac{1}{2BA^2}\frac{dA}{dr}\frac{dB}{dr} - \frac{2}{r^2}\left(-\frac{1}{A}\right) - \frac{2}{rA^2}\frac{dA}{dr} + \frac{2}{rAB}\frac{dB}{dr}. \tag{11}$$

As we take $AB = 1$ it has shown that relations (8), (9), (10) and (11) become:

$$R_{00} = -\frac{B}{2r}\left(\frac{d^2B}{dr^2}r + 2\frac{dB}{dr}\right), \tag{12}$$

$$R_{11} = \frac{1}{2rB}\left(\frac{d^2B}{dr^2}r + 2\frac{dB}{dr}\right), \tag{13}$$

$$R_{22} = R_{33} = r\frac{dB}{dr} + B, \tag{14}$$

$$\mathrm{R} = \frac{d^2B}{dr^2} + \frac{4}{r}\frac{dB}{dr} + \frac{2B}{r^2}. \tag{15}$$

3. Weyl conformal CSS gravity

Action is: $S = -\alpha \int d^4x \sqrt{-g} C^{\mu\nu\alpha\beta} C_{\mu\nu\alpha\beta} = -2\alpha \int d^4x \sqrt{-g} [R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2]$, [2,3,7] where lagrangian density is:

$$L = R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2. \tag{16}$$

The Euler-Lagrange equations turn out to be second order [2]

$$\sqrt{-g} W^{\mu\nu} = \frac{\partial}{\partial g_{\mu\nu}} \left(\sqrt{-g} L \right) - \frac{\partial}{\partial x^\mu} \left(\sqrt{-g} \frac{\partial L}{\partial \left(g_{\mu\nu} \right)'} \right) + \frac{\partial^2}{\partial \left(x^\mu \right)^2} \left(\sqrt{-g} \frac{\partial L}{\partial \left(g_{\mu\nu} \right)''} \right), \tag{17}$$

where $\left(g_{\mu\nu} \right)' = \frac{\partial g_{\mu\nu}}{\partial r}$, $\left(g_{\mu\nu} \right)'' = \frac{\partial^2 g_{\mu\nu}}{\partial r^2}$, and $W^{\mu\nu}$ is Bach tensor [5,6,7], which is given by the following equations expression

$$W^{\mu\nu} = C^{\mu\beta\nu\alpha}{}_{;\alpha\beta} + \frac{1}{2} C^{\mu\alpha\nu\beta} R_{\alpha\beta}, \tag{18}$$

where $C^{\mu\beta\nu\alpha}{}_{;\alpha\beta}$ is covariant derivate second order of Weyl tensor.

The Euler-Lagrange equations is $W^{\mu\nu} = 0$ [2,3]. In the static geometry it is enough to find out W^{rr}. Two other components of Bach tensor W^{zz} and W^{tt} can get from the trace of Bach tensor $W^{\mu\nu}$ and the divergence of $W^{\mu\nu}$ [2,6,7] , ie.

$$g_{\mu\nu} W^{\mu\nu} = 0 \, , W^{\mu\nu}{}_{;\mu} = 0. \tag{19}$$

After some manipulation we get from equations (19) following two equations:

$$-B(r) W^{tt} + A(r) W^{rr} + 2r^2 W^{zz} = 0, \tag{20}$$

$$\left(\frac{\partial}{\partial r} + \frac{1}{A} \frac{dA}{dr} + \frac{1}{2B} \frac{dB}{dr} + \frac{2}{r} \right) W^{rr} + \frac{1}{2A} \frac{dB}{dr} W^{tt} - \frac{2r}{A} W^{zz} = 0. \tag{21}$$

From equation (15) and Appendix A [2,3,4] we get third equation:

$$\frac{12r^4}{B} W^{rr} = -4B^2 - 4rB \left[-2\frac{dB}{dr} + r \left(\frac{d^2B}{dr^2} + r \frac{d^3B}{dr^3} \right) \right] + r^2 \left[-4 \left(\frac{dB}{dr} \right)^2 - r^2 \left(\frac{d^2B}{dr^2} \right)^2 + 2r \frac{dB}{dr} \left(2\frac{d^2B}{dr^2} + r \frac{d^3B}{dr^3} \right) \right]. \tag{22}$$

In our case field equations are

$$W^{\mu\mu} = 0, \tag{23}$$

then from Appendix B and [2,3] we get

$$B(r) = 1 - \frac{2\beta}{r} + \gamma r - kr^2. \tag{24}$$

and β, γ, and k are integration constants. This solution includes as special cases the Schwarzschild solution ($\gamma = k = 0$) and the Schwarzschild-de Sitter ($\gamma = 0$) solution; the latter requiring the presence of a cosmological constant in Einstein gravity.

4. Geodesic equations in Weyl gravity

We are solving relativistic equations of motion for massive particles in Weyl gravity [1] with common assumption given in the paper by Capozziello at al. [14], Manheim at al. [2], Jackson Levi Said at al. [3]: $AB = 1$.

Geodesic equations are:

$$\frac{d^2x^\mu}{dp^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{dp}\frac{dx^\beta}{dp} = 0. \tag{25}$$

These equations provide differential equations for the four space-time components: $x^\mu = (t(p), r(p), \varphi(p), z(p))$, where p is affine parameter describing the trajectory. The coordinate system may be oriented so that the orbit of the particle lies in $(r(p), \varphi(p))$ plane, and fix the $z = 0$ [1,4]. These equations become:

$$\frac{d^2t}{dp^2} + \frac{1}{B}\frac{dB}{dr}\frac{dr}{dp}\frac{dt}{dp} = 0, \tag{26}$$

$$\frac{d^2r}{dp^2} + \frac{1}{2A}\frac{dB}{dr}\left(\frac{dt}{dp}\right)^2 + \frac{1}{2A}\frac{dA}{dr}\left(\frac{dr}{dp}\right)^2 - \frac{r}{A}\left(\frac{d\varphi}{dp}\right)^2 - \frac{r}{A}\left(\frac{dz}{dp}\right)^2 = 0, \tag{27}$$

$$\frac{d^2\varphi}{dp^2} + \frac{2}{r}\frac{dr}{dp}\frac{d\varphi}{dp} = 0, \tag{28}$$

$$\frac{d^2z}{dp^2} + \frac{2}{r}\frac{dr}{dp}\frac{dz}{dp} = 0. \tag{29}$$

From the first equation we get:

$$\frac{dt}{dp} = \frac{1}{B}. \tag{30}$$

From the third equation we obtain:

$$J = r^2\frac{d\varphi}{dp} = const. = \sqrt{GML} = \sqrt{GMa(1-e^2)}, \tag{31}$$

where J is sector velocity, G is gravitation constant, M is mass of supermasive black hole, a is semimajor axis, L is semilatus rectum, e is eccentricity. From the fourth equation we obtain in general case:

$$K = r^2\frac{dz}{dp} = const. \tag{32}$$

where K is second sector velocity and in our case is zero. We assumed that the orbital of particle lies in the plane $z = 0$, i.e. coordinate z is fixed and does not depend on p. From the second equation we obtain:

$$A\left(\frac{dr}{dp}\right)^2 + \frac{J^2 + K^2}{r^2} - \frac{1}{B} = const. = E, \tag{33}$$

and using the second equation we finally have:

$$\left(\frac{dr}{d\varphi}\right)^2 + \frac{r^2}{A}\left(\frac{J^2 + K^2}{J^2} - \frac{Er^2}{J^2}\right) = \frac{c^2r^4}{ABJ^2}, \tag{34}$$

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{c^2}{A} + \frac{c^4}{ABE} - \frac{c^2(J^2 + K^2)}{Er^2A}, \tag{35}$$

$$\left(\frac{dr}{dt}\right)^2 = \frac{EB^2}{A} + \frac{Bc^2}{A} - \frac{(J^2 + K^2)B^2}{r^2 A}, \tag{36}$$

where E, J and K are constants of the motion and c is velocity of light [1,4].

Also,

$$ds^2 = c^2 d\tau^2 = Edp^2. \tag{37}$$

where τ is proper time. Angle $\varphi(r)$ is given by expression:

$$\varphi(r) = \varphi(r_-) + \int_{r_-}^{r} \frac{\sqrt{A}dr}{r^2\sqrt{\frac{E}{J^2} + \frac{c^2}{BJ^2} - \frac{J^2+K^2}{r^2 J^2}}} \tag{38}$$

and

$$r_\pm = (1 \pm e)\, a \ \wedge \ L = \left(1 - e^2\right) a, \tag{39}$$

where $r_\pm$ is maximum (aphelia) and minimum (perihelia) values of r, a - semimajor axis, L - semilatus rectum, e - eccentricity.

The angle of orbital precession per revolution is [1,4]:

$$\Delta\varphi = 2|\varphi(r_+) - \varphi(r_-)| - 2\pi. \tag{40}$$

In the case of Weyl gravity, taking into account the following equations:

$$B = 1 + \frac{2\Phi}{c^2}, \tag{41}$$

which is $g_{00} = 1 + \frac{2\Phi}{c^2}$ in Newtonian limit in weak field, and Φ is gravitation potential.

$$B\,(r) = A^{-1}\,(r) = 1 - \frac{2\beta}{r} + \gamma r - kr^2, \tag{42}$$

$$\Phi = \frac{c^2}{2}\left[-\frac{2\beta}{r} + \gamma r - kr^2\right]. \tag{43}$$

We also obtained angular velocity ω in Weyl gravity:

$$\omega = \frac{d\varphi}{dt} = \frac{JB}{r^2} = \frac{J}{r^2}\left[1 - \frac{2\beta}{r} + \gamma r - kr^2\right], \tag{44}$$

and:

$$\frac{dr}{dt} = B\sqrt{c^2 + B\left(E - \frac{(J^2+K^2)}{r^2}\right)}. \tag{45}$$

5. Orbital precession in the potential of Weyl gravity

As proposed in the reference [4] and described in Appendix C following expression is used to calculate precession:

$$\Delta\varphi = \varphi\,(\pi) - \varphi\,(0) = \frac{1}{\sqrt{C}}\int_0^{\pi} \sqrt{A(\Psi)}d\Psi, \tag{46}$$

where Ψ is parameter which substitute of radius r to simplify the integral of precession and C is a constant.

$$C\left(L^{-1}\right) \approx 1 - \frac{(u_+ - u_-)(u_+ + u_-)A''(L^{-1})}{2\left[A(u_+) - A(u_-)\right]} \approx 1 - \frac{A''(L^{-1})}{LA'(L^{-1})}, \tag{47}$$

and

$$u = \frac{1}{r} = \frac{1 + ecos\Psi}{L}, \tag{48}$$

5.1. The case of Schwarzschild metric

In this case $B(r)$ can be simplified, taking $\gamma = 0$ and $k = 0$:

$$B(r) = A^{-1}(r) = 1 - \frac{2\beta}{r} = 1 - 2\beta u. \tag{49}$$

By taking series expansion, concluding in second order in u:

$$\sqrt{A(r)} = 1 + \beta u + 2\beta^2 u^2. \tag{50}$$

We also simplified expansion involving C:

$$\frac{1}{\sqrt{C}} = 1 + \frac{2\beta}{L}. \tag{51}$$

Precession is then:

$$\Delta\varphi = \left[1 + \frac{2\beta}{L}\right]\int_0^{\pi}\left(1 + \beta u + 2\beta^2 u^2\right)d\Psi, \tag{52}$$

$$\Delta\varphi = \left[1 + \frac{2\beta}{L}\right]\int_0^{\pi}\left(1 + \beta\left[\frac{1 + ecos\Psi}{L}\right] + 2\beta^2\left[\frac{1 + ecos\Psi}{L}\right]^2\right)d\Psi, \tag{53}$$

$$\Delta\phi + 2\pi = 2\Delta\varphi + 2\pi = 2\pi + \frac{6\pi\beta}{L} + \pi\frac{4\beta^2}{L^2}\left[\frac{4 + e^2}{2}\right] = 2\pi + \frac{3\pi r_s}{L} + \pi\frac{r_s^2}{L^2}\left[\frac{4 + e^2}{2}\right], \tag{54}$$

$$\Delta\phi \approx \frac{3\pi r_s}{L}, \tag{55}$$

where $r_s = \dfrac{2GM}{c^2}$ is Schwarzschild radius.

5.2. The case of Weyl potential

Using now completing formula (42), and then series expansion we have:

$$B(r) = A^{-1}(r) = 1 - \frac{2\beta}{r} + \gamma r - kr^2, \tag{56}$$

$$\sqrt{A(r)} = 1 + \beta u + 2\beta^2 u^2 - \frac{\gamma}{2u} + \frac{k}{2u^2}. \tag{57}$$

And similary for C:

$$\frac{1}{\sqrt{C}} = 1 + \frac{2\beta}{L}. \tag{58}$$

Precession is given by integral expression:

$$\Delta\varphi = \left[1 + \frac{2\beta}{L}\right] \int_0^{\pi} \left(1 + \beta u + 2\beta^2 u^2 - \frac{\gamma}{2u} + \frac{k}{2u^2}\right) d\Psi, \tag{59}$$

$$\Delta\varphi = \left[1 + \frac{2\beta}{L}\right] \int_0^{\pi} \left(1 + \beta\left[\frac{1 + ecos\Psi}{L}\right] + 2\beta^2\left[\frac{1 + ecos\Psi}{L}\right]^2 - \frac{\gamma L}{2\,(1 + ecos\Psi)} + \frac{kL^2}{2(1 + ecos\Psi)^2}\right) d\Psi. \tag{60}$$

From [9] we get the solution of following integrals:

$$\int_0^{\pi} \frac{1}{(1 + ecos\Psi)} d\Psi = \frac{\pi}{\sqrt{1 - e^2}}, \quad \int_0^{\pi} \frac{1}{(1 + ecos\Psi)^2} d\Psi = \frac{\pi}{\sqrt{(1 - e^2)^3}}. \tag{61}$$

Finaly integral expresion reduces to:

$$\Delta\phi = \frac{3\pi r_s}{L} - \frac{\pi\gamma L}{\sqrt{1 - e^2}} + \frac{\pi k L^2}{\sqrt{(1 - e^2)^3}}, \tag{62}$$

and we take following expresion for integral constants γ and k, which can be compared with Einstain precession in equation (55),

$$\frac{\gamma}{2} = \frac{r_s}{L^2}\frac{r_s^m}{L^m}, \quad \frac{k}{2} = \frac{r_s}{L^3}\frac{r_s^m}{L^m}, \tag{63}$$

where m is scaling parameter.

$$\Delta\phi = \frac{3\pi r_s}{L}\left[1 - \frac{2r_s^m}{3L^m\sqrt{1 - e^2}} + \frac{2r_s^m}{3L^m\sqrt{(1 - e^2)^3}}\right]. \tag{64}$$

6. Results

In this section we compare our calculations with some astronomical observations for Solar planets and S-stars. In tables 1 and 2 first column present a semimajor axis, second is e eccentricity, thrid is observation period of revolution (T), fourt is orbital precession ($\Delta\phi$) in general theory of relativity, fifty column is orbital precession ($\Delta\phi$) given by astronomical observations, sixty column is orbital precession ($\Delta\phi$) calculate by equation (64) and m is scaling parameter. In table 1 we get results for solar planets and the results are for S-stars (S2, S38 and S55) in second table. The observed orbital elements from Table 2 are taken from references [11, 12, 13, 15, 16] while the observed orbital elements in Table 1 are taken from Table 8.1 of [1,10]. $\Delta\phi$ in Table 1 is in unit arcseconds for one century, while in second table is in degrees for one revolution.

Solar Planet	a($10^6 km$)	e	T (days)	GR $\Delta\phi$	Observ $\Delta\phi$	Weyl relativity $\Delta\phi$	m
Mercur	57.91	0.2056	87.9692	43.03	43.1 ± 0.5	42.912	1
Venus	108.21	0.0068	224.7091	8.6	8.4 ± 4.8	8.4	1
Earth	149.60	0.0167	365.256	3.8	5.0 ± 1.2	5.02	-0.42

Table 1: Period of revolution (T) in (yr.) and orbital precession ($\Delta\phi$) in (seconds for one century) for Solar planets (Mercur, Venera and Earth), estimated for the Weyl gravity.

Probably dark matter is reason for $m = -0.42$ for Earth, while all other cases suggest that $m = 1$ is good match.

The GRAVITY Collaboration was detected orbital precession of the S2 star around the Galactic Center [15] and found that it is similary to GR prediction which for S2 star is $\Delta\varphi = 0^{\circ}.201$ per orbital period. Also, according to data analysis in the framework of Yukawa gravity model in the paper [16], the orbital precessions of the S38 and S55 stars corespond to GR predictions for these stars, which are $0^{\circ}.119$ and $0^{\circ}.106$ per orbital period, respectively. Also the scaling parameter m is 1.

S stars	a(AU)	e	T (years)	GR $\Delta\phi$	Observ$\Delta\phi$	Weyl relativity $\Delta\phi$	m
S2	1044.2	0.8839	16.00	0.20078	0.201	0.20116	1
S38	1178.1	0.8201	19.2	0.11888	0.119	0.11950	1
S55	896.9	0.7209	12.80	0.10743	0.106	0.10745	1

Table 2: Period of revolution (T) in (yr.) and orbital precession $\Delta\phi$ in (degree for orbital period) for S stars (S2, S38 and S55), estimated for the Weyl gravity. The observed orbital elements are taken from [11,12,13,15,16]

From the Table 2. we can see the orbital precession of Weyl gravity for S-stars (S2, S38 and S55) are in good agreement with astronomical observations.

7. Conclusions

In this work we presented the modified field equations and geodesic equations in the case of a Weyl gravity. We assume cilindrical static symmetry because the metric tensor $g^{\mu\mu}$ depend only of radius r. After that we find the equations of field and geodesic equations for Weyl gravity. Solving geodesic equations of motion we find orbital precession ($\Delta\phi$) in limit of weak field. Then our results represent in Table 1 and Table 2 are compared with astronomical observations.

As we can see from tables our results are agree with observed values, while in the case of Earth the Weyl gravitation get good agreement with observation. Weyl gravitation include dark matter and dark energy which Einstain's prediction are given by term of cosmology constant. Weyl gravity consists three term.

From equation (43) we can calculate gravitation force, is given by following relation:

$$\overrightarrow{F} = -\overrightarrow{\nabla}\Phi = -\frac{\partial \Phi}{\partial r}\overrightarrow{e}_{\overrightarrow{r}} = -\frac{c^2\beta}{r^2}\overrightarrow{e}_{\overrightarrow{r}} - \frac{c^2\gamma}{2}\overrightarrow{e}_{\overrightarrow{r}} + c^2kr\overrightarrow{e}_{\overrightarrow{r}}. \tag{65}$$

The first factor is Newton force.

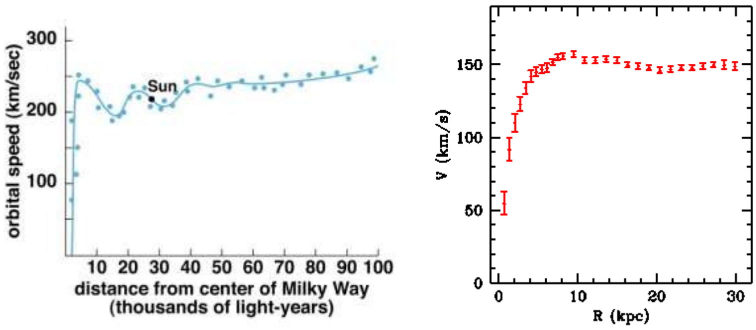
Second force is constant and present galaxy rotation curves as a factor of expansion cosmos. What is galaxy rotation curves? Spiral Galaxies rotate such that the rotation speed: Rises steadily from the center to the inner disk and becomes roughly constant (flat) with radius in the outer parts as far as can be measured in the disk. Evidence of dark matter has been confirmed through the study of rotation curves. To make a rotation curve one calculates the rotational velocity of e.g. stars along the length of a galaxy by measuring their Doppler shifts, and then plots this quantity versus their respective distance away from the center (for example Figure 1)

Vera Florence Cooper Rubin (July 23, 1928 December 25, 2016) was an American astronomer who pioneered work on galaxy rotation rates. She uncovered the discrepancy between the predicted and observed angular motion of galaxies by studying galactic rotation curves. Identifying the galaxy rotation problem, her work provided evidence for the existence of dark matter. These results were confirmed over subsequent decades.

Thrid force is given by Einstain along constant c^2k and also it is likewise force of Hook and constant c^2k is similary as Hook constant. In this case we have only expansion of cosmos because force is greater than zero.

In the context of cosmology the constant c^2k is a homogeneous energy density that causes the expansion of the universe to accelerate. Originally proposed early in the development of general relativity in order to allow a static universe solution it was subsequently abandoned when the universe was found to be

FIGURE 1. Rotation speed of spiral galaxies versus distance r from the center of galaxy. Right picture is the rotation curve for the galaxy NGC3198 from Begeman 1989. Left picture is the rotation curve for the Milky Way galaxy and show that the rotation curve is nearly flat with increasing radius. Evidently there are huge amounts of unseen "dark" matter in the outer parts of the galaxy that add gravitational field beyond that just from the center, causing the stars and gas to orbit faster. (Figure from The Essential Cosmic Perspective, by Bennett et al.)



expanding. Now the constant c^2k is invoked to explain the observed acceleration of the expansion of the universe. The constant c^2k is the simplest realization of dark energy, which is the more generic name given to the unknown cause of the acceleration of the universe.

Also, our calculations showed a good agreement with the corresponding astronomical observations of several S-stars. We hope that using this method with geodesics, we can evaluate parameters of alternative models for a gravitational potential at the Galactic Center with higher accuracy.

8. Appendix A. Calculation of W^{rr} Weyl conformal CSS gravity [2,3]

For any action $S = -\alpha \int d^4x \sqrt{-g}L$, where lagrangian density is L and any static line element

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + C(r)d\varphi^2 + D(r)dz^2, \tag{66}$$

where in our case $C(r) = D(r) = r^2$, and langragian density is

$$L = R^{\mu\nu}R_{\mu\nu} - \frac{1}{3}R^2. \tag{67}$$

We can calculate on the following way W^{rr} component of Bach tensor:

$$\sqrt{-g}W^{rr} = \frac{\partial S}{\partial A} = \frac{\partial}{\partial A}\left(\sqrt{-g}L\right) - \frac{\partial}{\partial r}\left(\sqrt{-g}\frac{\partial L}{\partial A'}\right) + \frac{\partial}{\partial r^2}\left(\sqrt{-g}\frac{\partial L}{\partial A''}\right). \tag{68}$$

We define

$$L = L_1 + L_2 + L_3 + L_4, \tag{69}$$

where

$$L_1 = R^{00}R_{00} = g_{00}^2 R_{00}^2, \quad L_2 = R^{11}R_{11} = g_{11}^2 R_{11}^2, \quad L_3 = R^{22}R_{22} = g_{22}^2 R_{22}^2, \quad L_4 = -\frac{1}{3}R^2. \tag{70}$$

Now we calculate

$$I = -\sqrt{-g}\frac{\partial L}{\partial (A)'} = I_1 + I_2 + I_3 + I_4, \tag{71}$$

where

$$I_1 = -\sqrt{-g}\frac{\partial L_1}{\partial (A)'} = -2\sqrt{-g}R_{00}\frac{\partial R_{00}}{\partial A'}\left(g^{00}\right)^2 = \frac{BB'r}{4}\left(B''r + 2B'\right), \tag{72}$$

$$I_2 = -\sqrt{-g}\frac{\partial L_2}{\partial (A)'} = -2\sqrt{-g}R_{11}\frac{\partial R_{11}}{\partial A'}\left(g^{11}\right)^2 = \left(\frac{BB'r}{4} + B^2\right)\left(B''r + 2B'\right), \tag{73}$$

$$I_3 = -\sqrt{-g}\frac{\partial L_3}{\partial (A)'} = -4\sqrt{-g}R_{22}\frac{\partial R_{22}}{\partial A'}\left(g^{22}\right)^2 = \frac{2B^2}{r}\left(B'r + B\right), \tag{74}$$

$$I_4 = \sqrt{-g}\frac{\partial L_4}{\partial (A)'} = \frac{2}{3}\sqrt{-g}R\frac{\partial R}{\partial A'} = \left(-\frac{BB'r^2}{3} - \frac{4B^2r}{3}\right)\left(\frac{B''r^2 + 4rB' + 2B}{r^2}\right). \tag{75}$$

Then we calculate the partial derivate of I by coordinate r:

$$\frac{\partial I}{\partial r} = \frac{\partial I_1}{\partial r} + \frac{\partial I_2}{\partial r} + \frac{\partial I_3}{\partial r} + \frac{\partial I_4}{\partial r}, \tag{76}$$

where the components of partial derivate of I are $\frac{\partial I_1}{\partial r}, \frac{\partial I_2}{\partial r}, \frac{\partial I_3}{\partial r}, \frac{\partial I_4}{\partial r}$ and given by next four relations:

$$\frac{\partial I_1}{\partial r} = \left[\frac{B'^2r}{4} + \frac{BB''r}{4} + \frac{BB'}{4}\right]\left(B''r + 2B'\right) + \frac{BB'r}{4}\left(B'''r + 3B''\right), \tag{77}$$

$$\frac{\partial I_2}{\partial r} = \left[\frac{B'^2r}{4} + \frac{BB''r}{4} + \frac{9BB'}{4}\right]\left(B''r + 2B'\right) + \left(\frac{BB'r}{4} + B^2\right)\left(B'''r + 3B''\right), \tag{78}$$

$$\frac{\partial I_3}{\partial r} = \left[\frac{2B^2}{r}\right]\left(B''r + 2B'\right) + \left(rB' + B\right)\left(\frac{4BB'}{r} - \frac{2B^2}{r^2}\right), \tag{79}$$

$$\frac{\partial I_4}{\partial r} = \left[-\frac{BB''r^2}{3} - \frac{B'^2r^2}{3} - \frac{10}{3}BB'r - \frac{4B^2}{3}\right]\left(\frac{B''r^2 + 4rB' + 2B}{r^2}\right) + \left(-\frac{BB'r^2}{3} - \frac{4rB^2}{3}\right)\left(B''' + \frac{4B''}{r} - \frac{2B'}{r^2} - \frac{4B}{r^3}\right). \tag{80}$$

After that we find out the

$$II = \frac{\partial}{\partial A}\left(\sqrt{-g}L\right), \tag{81}$$

which are divided in four components II_1, II_2, II_3 and II_4,

$$II = II_1 + II_2 + II_3 + II_4. \tag{82}$$

In the following four equations are given by the expresions of II_1, II_2, II_3 and II_4:

$$II_1 = \frac{\partial}{\partial A}\left(\sqrt{-g}L_1\right) = \frac{\partial}{\partial A}\left(\sqrt{AB}r^2\left(\frac{R_{00}}{g_{00}}\right)^2\right) = \frac{B}{8}\left(B''r + 2B'\right)^2 - r\left(B''r + 2B'\right)\left(\frac{B''B}{2} + \frac{B'^2}{4} + \frac{B'B}{r}\right), \tag{83}$$

$$II_2 = \frac{\partial}{\partial A}\left(\sqrt{-g}L_2\right) = \frac{\partial}{\partial A}\left(\sqrt{AB}r^2\left(\frac{R_{11}}{g_{11}}\right)^2\right) = -\frac{3B}{8}\left(B^{''}r + 2B'\right)^2 + r\left(B^{''}r + 2B'\right)\left(-\frac{B^{'2}}{4} - \frac{B'B}{r}\right), \tag{84}$$

$$II_3 = \frac{\partial}{\partial A}\left(\sqrt{-g}L_3\right) = \frac{\partial}{\partial A}\left(\sqrt{AB}r^2\left(\frac{R_{22}}{g_{22}}\right)^2\right) = \frac{B}{r^2}\left(B^{'}r + B\right)^2 + \frac{4}{r^2}\left(B^{'}r + B\right)\left(-\frac{3rB^{'}B}{2} - B^2\right), \tag{85}$$

$$II_4 = \frac{\partial}{\partial A}\left(\sqrt{-g}L_4\right) = \frac{\partial}{\partial A}\left(-\frac{1}{3}\sqrt{AB}r^2R^2\right) \tag{86}$$

$$= -\frac{B}{6r^2}\left(B^{''}r^2 + 4rB^{'} + 2B\right)^2 - \frac{2}{3}\left(B^{''}r^2 + 4rB^{'} + 2B\right)\left(-B^{''}B - \frac{6BB^{'}}{r} - \frac{2B^2}{r^2} - \frac{B'^2}{2}\right). \tag{87}$$

So we can get the finally expresion for Bach tensor W^{rr} is:

$$r^2W^{rr} = \frac{\partial I}{\partial r} + II, \tag{88}$$

where

$$B^{'} = \frac{dB}{dr}, \quad B^{''} = \frac{d^2B}{dr^2}, \quad B^{'''} = \frac{d^3B}{dr^3}, \tag{89}$$

and finally we get equation (20):

$$\frac{12r^4}{B}W^{rr} = -4B^2 - 4rB\left[-2\frac{dB}{dr} + r\left(\frac{d^2B}{dr^2} + r\frac{d^3B}{dr^3}\right)\right] + r^2\left[-4\left(\frac{dB}{dr}\right)^2 - r^2\left(\frac{d^2B}{dr^2}\right)^2 + 2r\frac{dB}{dr}\left(2\frac{d^2B}{dr^2} + r\frac{d^3B}{dr^3}\right)\right]. \tag{90}$$

9. Appendix B. Calculation of differential equation $W^{rr} = 0$ Weyl conformal CSS gravity [2,3]

This appendix represent the way of calculation of equation $W^{rr} = 0$. First we take substitution for $B(r)$

$$B(r) = r^2 l(r). \tag{91}$$

Put that in W^{rr} we get following expression:

$$\frac{12r^4}{B}W^{rr} = r^6\left[8\left(\frac{dl(r)}{dr}\right)^2 - r^2\left(\frac{d^2l(r)}{dr^2}\right)^2 + 2r\frac{dl(r)}{dr}\left[4\frac{d^2l(r)}{dr^2} + r\frac{d^3l(r)}{dr^3}\right]\right]. \tag{92}$$

After that we take second substitution:

$$\frac{dl(r)}{dr} = y(r), \tag{93}$$

and put it in equation (91), we get following expression for W^{rr}

$$\frac{12r^4}{B}W^{rr} = r^6\left[8y(r)^2 - r^2\left(\frac{dy(r)}{dr}\right)^2 + 2ry(r)\left[4\frac{dy(r)}{dr} + r\frac{d^2y(r)}{dr^2}\right]\right]. \tag{94}$$

Then we take thrid substitution

$$y(r) = \frac{h(r)}{r^3}, \tag{95}$$

and put in the equation (93), we get new expression for W^{rr}.

$$\frac{12r^4}{B}W^{rr} = h(r)^2 - r^2\left(\frac{dh(r)}{dr}\right)^2 + 2rh(r)\left[\frac{dh(r)}{dr} + r\frac{d^2h(r)}{dr^2}\right]. \tag{96}$$

Then we take fourth substitution

$$r = e^t, \tag{97}$$

and put it in equation (95) we get following expression for W^{rr}

$$\frac{12r^4}{B}W^{rr} = -h(t)^2 - \left(\frac{dh(t)}{dt}\right)^2 + 2h(t)\frac{d^2h(t)}{dt^2}. \tag{98}$$

And finally we take last substitution

$$h(t) = v^2(t), \tag{99}$$

to get finally expression for W^{rr}:

$$\frac{12r^4}{B}W^{rr} = v(t)^3\left[4\frac{d^2v(t)}{dt^2} - v(t)\right]. \tag{100}$$

Solving the equation $W^{rr} = 0$ we finally get $B(r)$

$$4\frac{d^2v(t)}{dt^2} - v(t) = 0, \tag{101}$$

$$v^2 = c_1 + c_2 r + \frac{c_3}{r}, \tag{102}$$

$$B(r) = a + br + cr^2 + \frac{d}{r}, \tag{103}$$

where a, b, c and d the constants of integration.

10. Appendix C. Calculation of precession [1,4]

We start from equation (42)

$$B(r) = A^{-1}(r) = 1 - \frac{2\beta}{r} + \gamma r - kr^2, \tag{104}$$

We are consider the case $K = 0$ and we get following equation from equation (34):

$$\frac{A}{r^4}\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{r^2} - \frac{c^2}{J^2B} = \frac{E}{J^2}. \tag{105}$$

In the case: $AB = 1$,

$$\varphi(r_+) - \varphi(r_-) = \int_{r_-}^{r_+} \frac{\sqrt{B^{-1}}dr}{r^2\sqrt{\frac{E}{J^2} + \frac{c^2}{BJ^2} - \frac{1}{r^2}}}, \tag{106}$$

where

$$r_\pm = (1 \pm e)a, \quad L = a\left(1 - e^2\right), \tag{107}$$

$r_\pm$ is maximum (aphelia) and minimum (perihelia) values of r, a - semimajor axis, L - semilatus rectum, e - eccentricity.

At perihelia and aphelia, r reaches its minimum and maximum values r_- and r_+, and at both points $dr/d\varphi$ vanishes so gives

$$\frac{dr}{d\varphi}(r_\pm) = 0 \Rightarrow \frac{E}{J^2} - \frac{1}{r_\pm^2} + \frac{c^2}{J^2 B(r_\pm)} = 0. \tag{108}$$

From these two equations we can derive two constants of motion K and J:

$$J^2 = c^2 \frac{\frac{1}{B(r_+)} - \frac{1}{B(r_-)}}{\frac{1}{r_+^2} - \frac{1}{r_-^2}}, \tag{109}$$

$$-E = c^2 \frac{\frac{r_+^2}{B(r_+)} - \frac{r_-^2}{B(r_-)}}{r_+^2 - r_-^2}. \tag{110}$$

The angle swept out by the position vector as r increases from r_-, using equations (108) and (109), is given by

$$\varphi(r) - \varphi(r_-) = \int_{r_-}^{r} A^{\frac{1}{2}} \left(\frac{r_-^2\left(B^{-1} - B_-^{-1}\right) - r_+^2\left(B^{-1} - B_+^{-1}\right)}{r_+^2 r_-^2\left(B_+^{-1} - B_-^{-1}\right)} - \frac{1}{r^2} \right)^{\frac{1}{2}} \frac{dr}{r^2}, \tag{111}$$

where $B_+ = B(r_+)$, $B_- = B(r_-)$. The total change φ per revolution is $2(\varphi(r_+) - \varphi(r_-))$. This would equal 2π if the orbit is closed ellipsa, so in general the orbit precesses in each revolution by an angle which is given by equation (40).

We make the argument of first square root in (110) a quadratic function of $1/r$. Futhermore, it vanishes at r_- and r_+. It can be shown that expression can write in the following shape:

$$\frac{r_-^2\left(B^{-1} - B_-^{-1}\right) - r_+^2\left(B^{-1} - B_+^{-1}\right)}{r_+^2 r_-^2\left(B_+^{-1} - B_-^{-1}\right)} - \frac{1}{r^2} = C\left(\frac{1}{r_-} - \frac{1}{r}\right)\left(\frac{1}{r} - \frac{1}{r_+}\right), \tag{112}$$

where C is a constant. Then letting $u = \frac{1}{r}$ and differentiating twice with respect to u gives

$$C \approx 1 - \frac{(u_+ - u_-)(u_+ + u_-) A''(L^{-1})}{2\left[A(u_+) - A(u_-)\right]} \approx 1 - \frac{A''(L^{-1})}{LA'(L^{-1})}, \tag{113}$$

where

$$L = \frac{2}{(u_+ + u_-)}, \quad \frac{L}{e} = \frac{2}{(u_- - u_+)}. \tag{114}$$

If we put next parametrs u_-, u_+ and C in equation (110), we get next expression:

$$\varphi(u_+) - \varphi(u_-) = -\frac{1}{\sqrt{C}} \int_{u_-}^{u_+} \frac{\sqrt{A(u)}}{\sqrt{(u_- - u)(u - u_+)}} du. \tag{115}$$

If we set following substitution in equation (114)

$$u = \frac{u_+ + u_-}{2} + \frac{u_- - u_+}{2} cos\Psi, \tag{116}$$

we get for precession finally expresion:

$$\Delta\varphi = \varphi(\pi) - \varphi(0) = \frac{1}{\sqrt{C}} \int_0^{\pi} \sqrt{A(\Psi)} d\Psi. \tag{117}$$

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