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Existence and uniqueness results on coupled Caputo-Hadamard fractional differential equations in a bounded domain

Karthikeyan Buvaneswari[a], Panjaiyan Karthikeyan[b], Kulandhivel Karthikeyan[c], Ozgur Ege[d,*]

[a]*Department of Mathematics, Sona College of Technology, Salem - 636 005, Tamil Nadu, India.*
[b]*Department of Mathematics, Sri Vasavi College, Erode, Tamil Nadu, India.*
[c]*Department of Mathematics, Centre for Research and Development, KPR Institute of Engineering and Technology, Coimbatore - 641407, Tamil Nadu, India.*
[d]*Department of Mathematics, Ege University, Bornova, Izmir, 35100, Turkey.*

Abstract. In this article, we study the existence and uniqueness of solutions for a boundary value problem of coupled system of Caputo-Hadamard fractional differential equations in a bounded domain. Banach contraction mapping principle and Schaefer's fixed point theorem are the main tools of our study. An example is presented at the end to support the main result.

1. Introduction

During the last three decades, fractional calculus and its applications become diversified more and has materialize as a significant tool for the comprehensive applications in mathematical modeling of nonlinear systems. The nonlocal nature of fractional order operators accounts the hereditary properties involved in various systems in terms of fractional differential operator. For further reference, see [7, 13, 23, 24] and the references cited therein. The definitions like Riemann-Liouville (1832), Grunwald-Letnikov (1867), Hadamard (1891,[11]) and Caputo (1997) are used to model problems in applied sciences and the formulations are used to model the physical systems and has given more accurate results. In 1891, Hadamard introduced the new derivative. For more details, one can refer [3, 20–22] and the references cited therein. A new approach called Caputo-Hadamard derivative [15], obtained from the Hadamard derivative and is applied to solve for physically interpretable initial condition problems. For the recent results in Caputo-Hadamard derivative, one can cite [1, 2, 4, 6, 9, 12, 14, 25–27] and the references therein.

Recently, nonlinear boundary value problems for coupled systems of hybrid differential equations of fractional order have many more applications. For more details, one can refer to [5, 16–19]. In 2008, Benchohra et al.[10] discussed the Caputo fractional derivative of order p

$$^{c}D^{p}\vartheta(t) = f_1(t, \vartheta(t)), \text{ for a. e. } t \in [0, T], \quad 0 < p \leq 1,$$

$$a_1\vartheta(0) + b_1\vartheta(T) = c_1$$

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* Corresponding author: Ozgur Ege
Email addresses: buvanasona@gmail.com (Karthikeyan Buvaneswari), karthi@srivasavi.ac.in (Panjaiyan Karthikeyan), karthi_phd2010@yahoo.co.in (Kulandhivel Karthikeyan), ozgur.ege@ege.edu.tr (Ozgur Ege)

with $f_1 : [0, T] \times R \to R$ is a given continuous function and $a_1, b_1, c_1 \in R$ such that $a_1 + b_1 \neq 0$.

In 2017, Arioua et al. [8] consider the following problem

$$ {}^cD^p_{1^+}\vartheta(t) + f_1(t, \vartheta(t)) = 0, \text{ for } 1 < t < e, \ 2 < p \leq 3, $$

with the fractional boundary conditions:

$$ \vartheta(1) = \vartheta'(1) = 0, \quad ({}^cD^{p-1}_{1^+}\vartheta)(e) = ({}^cD^{p-2}_{1^+}\vartheta)(e) = 0 $$

where ${}^cD^p$ denotes the Caputo-Hadamard fractional differential equations of order p and $f_1 : [1, e] \times R \to R$.

In 2018, Benhamida et al. [11] investigated the following Caputo-Hadamard fractional differential equations with the boundary conditions:

$$ {}^c_HD^p\vartheta(t) = f_1(t, \vartheta(t)), \text{ for a. e. } t \in [1, T], \ 0 < p \leq 1, $$

$$ a_1\vartheta(1) + b_1\vartheta(T) = c_1, $$

where ${}^c_HD^p$ denotes the Caputo-Hadamard fractional differential equations of order p with $f_1 : [1, T] \times R \to R$ and the real constants a_1, b_1 and c_1 such that $a_1 + b_1 \neq 0$.

Motivated by the above mentioned works, we consider the system of hybrid nonlinear Caputo-Hadamard fractional differential equations:

$$ \begin{aligned} &{}^c_HD^{\gamma_1}[z(t)] = \theta_1(t, z(t), \vartheta(t)), \ t \in [1, T], \ 0 < \gamma_1 \leq 1, \\ &{}^c_HD^{\delta_1}[\vartheta(t)] = \theta_2(t, z(t), \vartheta(t)), \ t \in [1, T], \ 0 < \delta_1 \leq 1, \end{aligned} \tag{1} $$

supplemented with

$$ a_1z(1) + b_1z(T) = c_1, \quad a_2\vartheta(1) + b_2\vartheta(T) = c_2 \tag{2} $$

where ${}^c_HD^{\gamma_1}, {}^c_HD^{\delta_1}$ denote the Caputo-Hadamard fractional derivatives of orders γ_1 and δ_1, respectively, the given continuous functions $\theta_i : [1, T] \times R \times R \to R, \ i = 1, 2$ with a_i, b_i and $c_i \in R, \ i = 1, 2$.

Now, we extend the problem considered in [11] to a boundary value problem of coupled hybrid Caputo-Hadamard fractional differential equations. For the existence part of the solution, we use Schaefer's fixed point theorem and the uniqueness, we apply Banach contraction mapping principle.

Remark 1.1. *Problems [10] defined on* (1) *and* (2) *are applied for an initial value problem when (a_i=1 and b_i=0), boundary value problem when (a_i=0 and b_i=1) and have antiperiodic solutions ($a_i = 1$ and $b_i = 1, c_i = 0, i = 1, 2$).*

Section 2 states the preliminary concepts and the discussion of auxiliary lemma related to the problem at hand. Section 3 dealt with the main proof the existence results of problem (1) and (2) while an illustrative example for the obtained result is discussed in Section 4.

2. Preliminaries

Definition 2.1. ([20]) *If $h_1 : [1, +\infty) \to R$, a continuous function then the Hadamard fractional integral of order q_1 is defined by*

$$ {}_HI^{q_1}h_1(t) = \frac{1}{\Gamma(q_1)}\int_1^t \Big(log\frac{t}{s}\Big)^{q_1-1}\frac{h_1(s)}{s}ds, \ q_1 > 0, \ t > 1 $$

provided the integral exists.

Definition 2.2. ([20]) *For the function $h_1 : [1, +\infty] \to R$, the Hadamard fractional derivative of order γ_1 is defined as*

$$ \begin{aligned} ({}_HD^{q_1}h_1)(t) &= \frac{1}{\Gamma(n-q_1)}\Big(\frac{d}{dt}\Big)^n\int_1^t \Big(ln\frac{t}{s}\Big)^{n-q_1-1}\frac{h_1(s)}{s}ds, \quad n-1 < q_1 < n, \\ &= \delta^n({}_HI^{n-q_1}h_1)(t), \end{aligned} $$

where $n = [q_1] + 1$ $[q_1]$ is the integer part of the real number.

Definition 2.3. ([15]) *The Caputo-Hadamard fractional derivative of order q_1 where $q_1 \geq 0$, $n-1 < q_1 < n$, with n=$[q_1]+1$ and $h_1 \in AC^n_\delta[1,\infty)$*

$$({}^c_H D^{q_1} h_1)(t) = \frac{1}{\Gamma(n-q_1)} \int_1^t (log\frac{t}{s})^{n-q_1-1} \delta^n h_1(s) \frac{ds}{s} =_H I^{n-q_1}(\delta^n h_1)(t).$$

Lemma 2.4. ([15]) *Let $h_1 \in AC^n_\delta[1,+\infty)$ and $q_1 > 0$. Then*

$$_H I^{q_1}({}^c_H D^{q_1} h_1)(t) = h_1(t) - \sum_{i=0}^{n-1} \frac{\delta^i h_1(1)}{i!}(logt)^i.$$

Lemma 2.5. *Suppose $h_1 : [1,+\infty) \to R$ is a continuous function and a solution z is defined by*

$$z(t) = \frac{1}{\Gamma(\gamma_1)} \int_1^t (log\frac{t}{s})^{\gamma_1-1} h_1(s) \frac{d}{ds} - \frac{b_1}{\Gamma(\gamma_1)(a_1+b_1)} \int_1^T (log\frac{T}{s})^{\gamma_1-1} h_1(s) \frac{d}{ds} + \frac{c_1}{a_1+b_1} \tag{3}$$

if and only if

$${}^c_H D^{\gamma_1} z(t) = h_1(t), \qquad 0 < \gamma_1 < 1 \tag{4}$$

and

$$a_1 z(1) + b_1 z(T) = c_1. \tag{5}$$

Proof. Assume z satisfies (4). Then Lemma 2.4 implies

$$z(t) =_H I^{\gamma_1} h_1(t) + d_1 \tag{6}$$

when we apply the boundary condition (5), we get

$$\begin{aligned} z(1) &= d_1 \\ z(T) &=_H I^{\gamma_1} h_1(T) + d_1 \\ a_1 z(1) + b_1 z(T) &= c_1 \\ a_1 d_1 + b_1[{}_H I^{\gamma_1} h_1(T) + z(1)] &= c_1 \\ a_1 z(1) + b_{1H} I^{\gamma_1} h_1(T) + b_1 z(1) &= c_1 \\ (a_1+b_1) z(1) + b_{1H} I^{\gamma_1} h_1(T) &= c_1 \\ z(1) &= \frac{c_1 - b_{1H} I^{\gamma_1} h_1(T)}{(a_1+b_1)} \end{aligned}$$

which leads to the solution (3) that

$$z(t) =_H I^{\gamma_1} h_1(t) - \frac{b_1}{(a_1+b_1)} =_H I^{\gamma_1} h_1(T) + \frac{c_1}{a_1+b_1}.$$

Conversely, equations (4)-(5) hold for z. □

3. Main results

Let us now consider a Banach space $\mathfrak{W} = \{\tilde{z}(t) \mid \tilde{z}(t) \in C([1,T])\}$ from [1,T] $\times R \to R$ endowed with the norm $\|\tilde{z}\|_\infty = sup\{|\tilde{z}(t)| : 1 \leq t \leq T\}$. Let the absolutely continuous function is defined as

$$AC^m_\delta([e_1,e_2] \times R, R) = \{h_1 : [e_1,e_2] \times R \to R : \delta^{n-1} h_1(t) \in AC([e_1,e_2] \times R, R)\},$$

where $\delta = t\frac{d}{dt}$. Then the product space $(\mathfrak{W} \times \mathfrak{W}, \|(\tilde{z},\tilde{\vartheta})\|)$ endowed with the norm $\left\|(\tilde{z},\tilde{\vartheta})\right\| = \|\tilde{z}\| + \left\|\tilde{\vartheta}\right\|$, $(\tilde{z},\tilde{\vartheta}) \in \mathfrak{W} \times \mathfrak{W}$ is a Banach space. Let us now consider the Banach space $\mathfrak{S}$ of all continuous functions $\tilde{\xi} : [1,T] \to R$ endowed with the norm $\|\tilde{\xi}\|_\infty = sup\{|\tilde{\xi}(\hat{\varkappa})| : 1 \leq \hat{\varkappa} \leq T\}$. Then the product space $(\mathfrak{S} \times \mathfrak{S})$ endowed with the norm $\left\|(\tilde{\xi},\tilde{\vartheta})\right\| = \left\|\tilde{\xi}\right\| + \left\|\tilde{\vartheta}\right\|$, $(\tilde{\xi},\tilde{\vartheta}) \in \mathfrak{S} \times \mathfrak{S}$ is also a Banach space.

(A1) Let $\theta_1, \theta_2 : [1, T] \times R \times R \to R$ and there exists constants m_i, n_i such that, for all $t \in [1, T]$ and $x_i, y_i \in R, i = 1, 2,$

$$\left|\theta_1(t, x_1, x_2) - \theta_1(t, y_1, y_2)\right| \leq m_1|x_1 - y_1| + m_2|x_2 - y_2|,$$
$$\left|\theta_2(t, x_1, x_2) - \theta_2(t, y_1, y_2)\right| \leq n_1|x_1 - y_1| + n_2|x_2 - y_2|.$$

(A2) $\sup_{t\in[1,T]} \theta_1(t, 0, 0) = \mathcal{N}_1 < \infty$ and $\sup_{t\in[1,T]} \theta_2(t, 0, 0) = \mathcal{N}_2 < \infty$.

(A3) There exists $M_1 > 0, M_2 > 0$ such that

$$|\theta_1(t, x(t), y(t))| \leq M_1, \quad |\theta_2(t, x(t), y(t))| \leq M_2.$$

For the ease of computational calculation, we pose

$$P_1 = \left[1 + \frac{|b_1|}{|a_1 + b_1|}\right]\frac{(logT)^{\gamma_1}}{\Gamma(\gamma_1 + 1)},$$
$$P_2 = \left[1 + \frac{|b_2|}{|a_2 + b_2|}\right]\frac{(logT)^{\delta_1}}{\Gamma(\delta_1 + 1)};$$
$$Q_1 = \frac{|c_1|}{|a_1 + b_1|} < 1 \quad and \qquad Q_2 = \frac{|c_2|}{|a_2 + b_2|} < 1.$$

(A4) From the assumptions in the above, we also consider $P_1(m_1 + m_2) + P_2(n_1 + n_2) < 1$.

In view of Lemma 2.5, we define an operator $\varphi : \mathfrak{W} \times \mathfrak{W} \to \mathfrak{W} \times \mathfrak{W}$ and (1)-(2) becomes

$$\varphi(z, \vartheta)(t) = \begin{pmatrix} \varphi_1(z, \vartheta)(t) \\ \varphi_2(z, \vartheta)(t) \end{pmatrix}, \tag{7}$$

where

$$\varphi_1(z, \vartheta)(t) = \frac{1}{\Gamma(\gamma_1)}\int_1^t \left(log\frac{t}{s}\right)^{\gamma_1 - 1}\theta_1(s)\frac{d}{ds} - \frac{b_1}{\Gamma(\gamma_1)(a_1 + b_1)}\int_1^T \left(log\frac{T}{s}\right)^{\gamma_1 - 1}\theta_1(s)\frac{d}{ds} + \frac{c_1}{a_1 + b_1}$$

and

$$\varphi_2(z, \vartheta)(t) = \frac{1}{\Gamma(\beta_1)}\int_1^t \left(log\frac{t}{s}\right)^{\beta_1 - 1}\theta_2(s)\frac{d}{ds} - \frac{b_2}{\Gamma(\beta_1)(a_2 + b_2)}\int_1^T \left(log\frac{T}{s}\right)^{\beta_1 - 1}\theta_2(s)\frac{d}{ds} + \frac{c_2}{a_2 + b_2}.$$

Theorem 3.1. *If (A1) to (A4) hold, then $\varphi\bar{\mathcal{B}}_r \subset \bar{\mathcal{B}}_r$, where $\bar{\mathcal{B}}_r = \{(z, \vartheta) \in \mathfrak{W} \times \mathfrak{W} : \|(z, \vartheta)\|_\infty \leq r\}$ is a closed ball with*

$$r = P_1(m_1 + m_1) + P_2(n_1 + n_2) < 1.$$

Moreover, (1) *and* (2) *have a unique solution on [1,T].*

Proof. Let $(z, \vartheta) \in \bar{\mathcal{B}}_r$ and $t \in [1, T]$, (A1) becomes

$$|\theta_1(t, z(t), \vartheta(t))| \leq |\theta_1(t, z(t), \vartheta(t)) - \theta_1(t, 0, 0)| \leq m_1\|z\|_\infty + m_2\|\vartheta\|_\infty.$$

Similarly, one can find that

$$|\theta_2(t, z(t), \vartheta(t))| \leq n_1\|z\|_\infty + n_2\|\vartheta\|_\infty.$$

Then we have

$$|\varphi_1(z, \vartheta)(t)| \leq \max_{t\in[1,T]}\left[\frac{1}{\Gamma(\gamma_1)}\int_1^t \left(log\frac{t}{s}\right)^{\gamma_1 - 1}\left|\theta_1(s, z(s), \vartheta(s)) - \theta_1(s, 0, 0)) + \theta_1(s, 0, 0)\right|\frac{d}{ds}\right.$$
$$\left. - \frac{|b_1|}{\Gamma(\gamma_1)|a_1 + b_1|}\int_1^T \left(log\frac{T}{s}\right)^{\gamma_1 - 1}|\theta_1(s, z(s), \vartheta(s)) - \theta_1(s, 0, 0) + \theta_1(s, 0, 0)|\frac{d}{ds} + \frac{|c_1|}{|a_1 + b_1|}\right]$$

$$\leq \frac{1}{\Gamma(\gamma_1)}\int_1^t \Big(log\frac{t}{s}\Big)^{\gamma_1-1}(m_1|z|+m_2|\vartheta|+\mathcal{N}_1)\frac{d}{ds}$$
$$+\frac{|b_1|}{\Gamma(\gamma_1)|a_1+b_1|}\int_1^T \Big(log\frac{T}{s}\Big)^{\gamma_1-1}(m_1|z|+m_2|\vartheta|+\mathcal{N}_1)\frac{d}{ds}+Q_1$$
$$\leq \frac{(logT)^{\gamma_1}}{\Gamma(\gamma_1+1)}\Big(1+\frac{|b_1|}{|a_1+b_1|}\Big)(m_1|z|+m_2|\vartheta|+\mathcal{N}_1)+Q_1.$$

Thus

$$\begin{aligned}\|\varphi_1(z,\vartheta)(t)\|_\infty &\leq \frac{(logT)^{\gamma_1}}{\Gamma(\gamma_1+1)}\Big(1+\frac{|b_1|}{|a_1+b_1|}\Big)(m_1\|z\|_\infty+m_2\|\vartheta\|_\infty+\mathcal{N}_1)+Q_1\\ &\leq P_1(m_1\|z\|_\infty+m_2\|\vartheta\|_\infty+\mathcal{N}_1)+Q_1\\ &\leq (P_1m_1+P_2m_2)r+P_1\mathcal{N}_1+Q_1\\ &\leq P_1(m_1+m_2)r+P_1\mathcal{N}_1+Q_1.\end{aligned}$$

In a similar way, one can derive that

$$\|\varphi_2(z,\vartheta)(t)\|_\infty \leq [P_2(n_1+n_2)]r+P_2\mathcal{N}_2+Q_2.$$

From the foregoing estimates for φ_1 and φ_2, it follows that $\|\varphi(z,\vartheta)(t)\|_\infty \leq r$. Next, for $(z_1,\vartheta_1),(z_2,\vartheta_2)\in\mathfrak{W}\times\mathfrak{W}$ and $t\in[1,T]$, we get

$$\begin{aligned}\Big|\varphi_1(z_2,\vartheta_2)(t)-\varphi_1(z_1,\vartheta_1)(t)\Big| &\leq \frac{1}{\Gamma(\gamma_1)}\int_1^t (log\frac{t}{s})^{\gamma_1-1}\,|\theta_1(s,z_2(s),\vartheta_2(s))-\theta_1(s,z_1(s),\vartheta_1(s))|\,\frac{d}{ds}\\ &\quad+\frac{|b_1|}{\Gamma(\gamma_1)|a_1+b_1|}\int_1^T (log\frac{T}{s})^{\gamma_1-1}\,|\theta_1(s,z_2(s),\vartheta_2(s))-\theta_1(s,z_1(s),\vartheta_1(s))|\,\frac{d}{ds}\\ &\leq \Big[1+\frac{b_1}{a_1+b_1}\frac{(logT)^{\gamma_1}}{\Gamma(\gamma_1+1)}\Big]\Big[m_1\|z_2-z_1\|_\infty+m_2\|\vartheta_2-\vartheta_1\|_\infty\Big]\\ &= P_1m_1\|z_2-z_1\|_\infty+P_1m_2\|\vartheta_2-\vartheta_1\|_\infty\end{aligned}$$

which implies that

$$\Big\|\varphi_1(z_2,\vartheta_2)(t)-\varphi_1(z_1,\vartheta_1)(t)\Big\|_\infty \leq P_1(m_1+m_2)\Big[\|z_2-z_1\|_\infty+\|\vartheta_2-\vartheta_1\|_\infty\Big] \tag{8}$$

In a similar way,

$$\Big\|\varphi_2(z_2,\vartheta_2)(t)-\varphi_2(z_1,\vartheta_1)(t)\Big\|_\infty \leq P_2(n_1+n_2)\Big[\|z_2-z_1\|_\infty+\|\vartheta_2-\vartheta_1\|_\infty\Big]. \tag{9}$$

From (8) and (9), we deduce that

$$\Big\|\varphi(z_2,\vartheta_2)(t)-\varphi(z_1,\vartheta_1)(t)\Big\|_\infty \leq [P_1(m_1+m_2)+P_2(n_1+n_2)]\left(\|z_2-z_1\|_\infty+\|\vartheta_2-\vartheta_1\|_\infty\right).$$

In view of condition $P_1(m_1+m_1)+P_2(n_1+n_2)<1$, it follows that the operator φ possesses a unique fixed point. This leads to the conclusion that the problems (1)-(2) have a unique solution on $[1,T]$. This completes the proof. □

Theorem 3.2. *Let the hypothesis (A1) and (A2) hold. Then* (1)-(2) *has at least one solution on [1,T].*

Proof. The proof will be given in several steps.

Step I: The operator $\varphi:\mathfrak{W}\times\mathfrak{W}\to\mathfrak{W}\times\mathfrak{W}$ is continuous.

By the definition of θ_1 and θ_2, the operator $\varphi \subset \mathfrak{W} \times \mathfrak{W}$ is bounded. Let (z_n, ϑ_n) be a sequence of points in $\mathfrak{W} \times \mathfrak{W}$ converging to a point $(z, \vartheta) \in \mathfrak{W} \times \mathfrak{W}$. By Lebesgue Dominated Convergence Theorem,

$$\begin{aligned}|\varphi_1(z_n, \vartheta_n)(t) - \varphi_1(z, \vartheta)(t)| &\leq \frac{1}{\Gamma(\gamma_1)} \int_1^t \Big(log\frac{t}{s}\Big)^{\gamma_1-1} |\theta_1(s, z_n(s), \vartheta_n(s)) - \theta_1(s, z(s), \vartheta(s))| \frac{d}{ds} \\ &\quad - \frac{|b_1|}{\Gamma(\gamma_1)|a_1+b_1|} \int_1^T \Big(log\frac{T}{s}\Big)^{\gamma_1-1} |\theta_1(s, z_n(s), \vartheta_n(s)) - \theta_1(s, z(s), \vartheta(s))| \frac{d}{ds} \\ &\leq \Big[1 + \frac{|b_1|}{|a_1+b_1|}\Big] \frac{(logT)^{\gamma_1}}{\Gamma(\gamma_1+1)} \|\theta_1(., z_n(.), \vartheta_n(.)) - \theta_1(., z(.), \vartheta(.))\|_\infty.\end{aligned}$$

For all $t \in [1, T]$, θ_1 is continuous, we have $\|\varphi_1(z_n, \vartheta_n) - \varphi_1(z, \vartheta)\|_\infty \to 0$ *as* $n \to \infty$.

Similarly, we can prove $\|\varphi_2(z_n, \vartheta_n) - \varphi_2(z, \vartheta)\|_\infty \to 0$ *as* $n \to \infty$ for all $t \in [1, T]$. Hence, it follows from the foregoing inequalities satisfied by φ_1 and φ_2 that the operator φ is continuous.

Step II : Let φ: $C([1, T] \times R \times R \to R)$, there exist positive constants $\mathcal{L}_1$ and $\mathcal{L}_2$ such that for each

$$(z, \vartheta) \in \mathcal{B}_{\nu_1^*} := \{(z, \vartheta) \in C([1, T] \times R \times R, R) : \|z\|_\infty \leq \nu_1^*\},$$

certainly for any $\nu_1^* > 0$, we have

$$\begin{aligned}\Big|\varphi_1(z, \vartheta)(t)\Big| &\leq \frac{1}{\Gamma(\gamma_1)} \int_1^t \Big(log\frac{t}{s}\Big)^{\gamma_1-1} |\theta_1(s, z(s), \vartheta(s))| \frac{d}{ds} \\ &\quad + \frac{|b_1|}{\Gamma(\gamma_1)|a_1+b_1|} \int_1^T \Big(log\frac{T}{s}\Big)^{\gamma_1-1} |\theta_1(s, z(s), \vartheta(s))| \frac{d}{ds} + \frac{c_1}{a_1+b_1}\end{aligned}$$

and

$$\|\varphi_1(z, \vartheta)(t)\|_\infty \leq \Big[1 + \frac{|b_1|}{|a_1+b_1|}\Big] \frac{(logT)^{\gamma_1}}{\Gamma(\gamma_1+1)} M_1 + \frac{|c_1|}{|a_1+b_1|} := \mathcal{L}_1.$$

Thus we deduce that $\|\varphi_1(z, \vartheta)(t)\|_\infty \leq \mathcal{L}_1$. In a similar fashion, it can be found that $\|\varphi_2(z, \vartheta)(t)\|_\infty \leq \mathcal{L}_2$. Hence it follows from the foregoing inequalities that φ_1 and φ_2 are uniformly bounded and hence φ is uniformly bounded.

Step III : Next we prove that φ: $C([1, T] \times R \times R \to R)$ is equicontinuous. Let $r_1, r_2 \in [1, T]$ with $r_1 < r_2$.

$$\begin{aligned}\Big|\varphi_1(z(r_2), \vartheta(r_2) - \varphi_1(z(r_1), \vartheta(r_1))\Big| &\leq \frac{1}{\Gamma(\gamma_1)} \int_1^{r_1} \Big[(log\frac{r_2}{s})^{\gamma_1-1} - (log\frac{r_1}{s})^{\gamma_1-1}\Big] |\theta_1(s, z(s), \vartheta(s))| \frac{d}{ds} \\ &\quad + \frac{1}{\Gamma(\gamma_1)} \int_{r_1}^{r_2} (log\frac{r_2}{s})^{\gamma_1-1} |\theta_1(s, z(s), \vartheta(s))| \frac{d}{ds} \\ &\leq \frac{M_1}{\Gamma(\gamma_1+1)} \Big[(logr_2)^{\gamma_1} - (logr_1)^{\gamma_1}\Big] \\ &\to 0 \quad as \quad r_1 \to r_2.\end{aligned}$$

Analogously, we can obtain that

$$\Big|\varphi_2(z(r_2), \vartheta(r_2) - \varphi_2(z(r_1), \vartheta(r_1))\Big| \leq \frac{M_2}{\Gamma(\delta_1+1)} \Big[(logr_2)^{\delta_1} - (logr_1)^{\delta_1}\Big].$$

Therefore the operator φ is equicontinuous and hence the operator $\varphi(z, \vartheta)$ is completely continuous.

Step IV : We show that the set

$$\mathcal{P} = \{(z,\vartheta) \in \mathfrak{W} \times \mathfrak{W} : (z,\vartheta) = \lambda_1 \varphi(z,\vartheta),\ 0 < \lambda_1 < 1\}$$

is bounded. Let $(z,\vartheta) \in \mathcal{P}$ and $t \in [1,T]$. Then it follows from $z(t) = \lambda_1\varphi_1(z,\vartheta)(t)$ that $\vartheta(t) = \lambda_1\varphi_2(z,\vartheta)(t)$ that

$$\begin{aligned}|z(t)| &\le \frac{1}{\Gamma(\gamma_1)}\int_1^t \Big(log\frac{t}{s}\Big)^{\gamma_1-1}|\theta_1(s,z(s),\vartheta(s))|\frac{d}{ds} - \frac{|b_1|}{\Gamma(\gamma_1)|a_1+b_1|}\int_1^T \Big(log\frac{T}{s}\Big)^{\gamma_1-1}|\theta_1(s,z(s),\vartheta(s))|\frac{d}{ds} + \frac{c_1}{a_1+b_1}\\ &\le \Big[1+\frac{|b_1|}{|a_1+b_1|}\Big]\frac{(logT)^{\gamma_1}}{\Gamma(\gamma_1+1)}M_1 + \frac{|c_1|}{|a_1+b_1|} := R,\end{aligned}$$

$$\|z(t)\|_\infty \le R \tag{10}$$

and

$$\|\vartheta(t)\|_\infty \le \Big[1+\frac{|b_2|}{|a_2+b_2|}\Big]\frac{(logT)^{\delta_1}}{\Gamma(\delta_1+1)}M_1 + \frac{|c_2|}{|a_2+b_2|} := R. \tag{11}$$

Hence, from (10) and (11), we obtain

$$\|z\|_\infty + \|\vartheta\|_\infty \le R$$

which implies that

$$\|(z,\vartheta)\|_\infty \le R.$$

Hence $\mathcal{P}$ is bounded and therefore by Theorem 3.2, φ has a fixed point. Then the problem (1)-(2) has at least one solution on [0,T]. Thus the proof is completed. □

4. An example

Example 4.1. *Consider the system of coupled fractional differential equations:*

$$\begin{aligned}&{}^c_H D^{1/2}(z(t)) = \frac{2}{53}z(t) + \frac{2}{9}\frac{\vartheta(t)}{1+\vartheta(t)} + \frac{2}{7},\\ &{}^c_H D^{1/2}(\vartheta(t)) = \frac{3}{40}\frac{|\cos z(t)|}{1+|\cos z(t)|} + \frac{1}{26}\sin\vartheta(t) + \frac{5}{7},\end{aligned} \tag{12}$$

$$\begin{aligned}&z(1)+z(e)=0,\\ &\vartheta(1)+\vartheta(e)=0,\end{aligned} \tag{13}$$

Here $\gamma_1 = \delta_1 = \frac{1}{2}$, *T=e,*$a_1 = b_1 = a_2 = b_2 = 1, c_1 = c_2 = 0$, *and*

$$\theta_1(t,z(t).\vartheta(t)) = \frac{2}{53}z(t) + \frac{2}{9}\frac{\vartheta(t)}{1+\vartheta(t)} + \frac{2}{7},$$

$$\theta_2(t,z(t),\vartheta(t)) = \frac{3}{40}\frac{|\cos z(t)|}{1+|\cos z(t)|} + \frac{1}{26}\sin\vartheta(t) + \frac{5}{7},$$

$$m_1 = \frac{2}{53},\ \ m_2 = \frac{2}{9},\ \ n_1 = \frac{3}{40},\ \ n_2 = \frac{1}{26}.$$

From the given data, we find that $P_1 = P_2 = 1.6930$. *Therefore* $P_1(m_1+m_2) + P_2(n_1+n_2) = 0.632157735 < 1$. *By Theorem 3.1, the problem* (12)-(13) *with the given* $\theta_1(t,z,\vartheta)$ *and* $\theta_2(t,z,\vartheta)$ *has at least one solution on* $[1,T]$.

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