Filomat 38:4 (2024), 1203–1219
https://doi.org/10.2298/FIL2404203F

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Landau-type theorems for some polyharmonic mappings and log-p-harmonic mappings

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Abstract. In this paper, we first establish a sharp version of Landau-type theorem of polyharmonic mappings. Then, we establish two versions of Landau-type theorems of polyharmonic mappings by applying Cauchy-inequality, which improve the corresponding theorems given in Luo et al.(Computational Methods and Function Theory, 23(2):303-325, 2023). Finally, three new Landau-type theorems of log-p-harmonic mappings are established, one of which improves upon a result given in Bai et al. (Complex Analysis and Operator Theory, 13(2):321-340, 2019).

1. Introduction

Suppose $F(z) = u(z) + iv(z)$ is a $2p$ times continuously differentiable complex-valued mapping in a domain $D \subseteq \mathbb{C}$, where p is a positive integer. Then $F(z)$ is said to be polyharmonic (or p-harmonic) in D if $F(z)$ satisfies the p-harmonic equation

$$\Delta^p F = \Delta(\Delta^{p-1})F = 0,$$

where $\Delta := \Delta^1$ represents the usual complex Laplacian operator

$$\Delta := 4\frac{\partial^2}{\partial z \partial \overline{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Obviously, for $p = 1$ (resp. $p = 2$), we obtain the usual class of harmonic (resp. biharmonic) mappings. A complex-value function $f(z)$ is a harmonic mapping in a simply connected domain D if and only if $f(z)$ has the following representation $f(z) = h(z) + \overline{g(z)}$ with $f(0) = h(0)$, $g(z)$ and $h(z)$ being analytic in D (for details see [4]).

It is well-known (cf.[11]) that a mapping $F(z)$ is polyharmonic in a simply connected domain $D \subseteq \mathbb{C}$ if and only if $F(z)$ has the following representation

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z),$$

2020 *Mathematics Subject Classification*. Primary 30C99; Secondary 30C62.
Keywords. Landau-type theorem; Polyharmonic mappings; log-p-harmonic mappings; Univalence.
Received: 28 May 2023; Accepted: 18 August 2023
Communicated by Miodrag Mateljević
Research supported by Guangdong Natural Science Foundation (Grant No. 2021A1515010058).
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where $G_{p-k+1}(z)$ is harmonic on D for each $k \in \{1, \ldots, p\}$. In particular, $F(z)$ is a biharmonic mapping in a simply connected domain D if and only if $F(z)$ has the following representation

$$F(z) = |z|^2 g(z) + h(z),$$

where $g(z)$, $h(z)$ are harmonic on D (cf.[1]).

A mapping $F(z)$ is called a log-p-harmonic mapping if and only if $\log F(z)$ is a p-harmonic mapping. When $p = 1$, $F(z)$ is called a log-harmonic mapping. When $p = 2$, $F(z)$ is called a log-biharmonic mapping. Hence, $F(z)$ is called a log-p-harmonic mapping in a simply connected domain $D \subseteq \mathbb{C}$ if and only if $F(z)$ has the following representation

$$F(z) = \prod_{k=1}^{p} g_{p-k+1}(z)^{|z|^{2(k-1)}},$$

where $g_{p-k+1}(z)$ is log-harmonic on D for each $k \in \{1, \ldots, p\}$(cf. [14]).

For a continuously differentiable mapping $F(z)$ in D, we define the maximum dilation and minimum dilation respectively as follows:

$$\Lambda_F(z) = \max_{0\le\theta\le2\pi} |e^{i\theta}F_z(z) + e^{-i\theta}F_{\bar{z}}(z)| = |F_z(z)| + |F_{\bar{z}}(z)|,$$

and

$$\lambda_F(z) = \min_{0\le\theta\le2\pi} |e^{i\theta}F_z(z) + e^{-i\theta}F_{\bar{z}}(z)| = ||F_z(z)| - |F_{\bar{z}}(z)||.$$

Denote the Jacobian of F by

$$J_F = |F_z(z)|^2 - |F_{\bar{z}}(z)|^2.$$

Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk, and $\mathbb{U}_r$ be the disk with center at the origin and radius $r > 0$. The classical Landau's theorem states that if f is an analytic function in the unit disk $\mathbb{U}$ with $f(0) = f'(0) - 1 = 0$ and $|f(z)| < M$ for $z \in \mathbb{U}$, then f is univalent in the disk $\mathbb{U}_{\rho_0}$ with $\rho_0 = \frac{1}{M+\sqrt{M^2-1}}$ and $f(\mathbb{U}_{\rho_0})$ contains a disk $|w| < R_0$ with $R_0 = M\rho_0^2$. This result is sharp, with the extremal function $f_0(z) = Mz\frac{1-Mz}{M-z}$. Furthermore, the Bloch theorem asserts the existence of a positive constant number b such that if f is an analytic function on the unit disk $\mathbb{U}$ with $f'(0) = 1$, then $f(\mathbb{U})$ contains a schlicht disk of radius b, that is, a disk of radius b which is the univalent image of some region in $\mathbb{U}$. The supremum of all such constants b is called the Bloch constant (for the detail see [6, 12]).

Since Landau's theorems of harmonic mappings were given by Chen et al.([6]) in 2000, many authors are keen on Landau-type theorems for harmonic mappings, biharmonic mappings and polyharmonic mappings ([3, 7, 9, 10, 15–20, 22, 23, 27]). Meanwhile, there are many Bloch's theorems for different functions. In 2002, Mateljević [24] gave a version of Bloch's theorems for quasiregular harmonic mappings. And in 2017, Chen et al. [8] obtained a Landau-Bloch type theorem for harmonic functions in hardy spaces.

There are many good results, but the sharp results are rarely seen. Recently, Luo and Liu ([23]) established following theorem for polyharmonic mappings, which improved the related result of Bai and Liu in [3].

Theorem A([23]) Suppose $F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$ is a polyharmonic mapping in the unit disk $\mathbb{U}$, with $F(0) = \lambda_F(0) - 1 = 0$, and satisfying following conditions:

(i) $G_{p-k+1}(z)$ is harmonic in $\mathbb{U}$, and $G_{p-k+1}(0) = 0$ for $k \in \{1, \cdots, p\}$;

(ii) for $k \in \{2, 3, \cdots, p\}$, $|G_{p-k+1}(z)| \le M_{p-k+1}$, and $\Lambda_{G_p}(z) \le \Lambda_p$ for $z \in \mathbb{U}$.

Then $M_{p-k+1} \ge 0$, $\Lambda_p \ge 1$, $F(z)$ is univalent in $\mathbb{U}_{\rho_1}$, and $F(\mathbb{U}_{\rho_1})$ contains a schlicht disk $\mathbb{U}_{\rho_1'}$, where ρ_1 is the minimum root in $(0, 1)$ of the equation

$$\frac{\Lambda_p(1 - \Lambda_p r)}{\Lambda_p - r} - \sum_{k=2}^{p} r^{2(k-1)} \Big[\frac{4M_{p-k+1}}{\pi(1 - r^2)} + \frac{8(k - 1)M_{p-k+1}}{\pi}\Big] = 0, \tag{1}$$

and

$$\rho_1' = \Lambda_p^2 \rho_1 + (\Lambda_p^3 - \Lambda_p) \log(1 - \frac{\rho_1}{\Lambda_p}) - \sum_{k=2}^{p} \rho_1^{2k-1} \frac{4M_{p-k+1}}{\pi}. \tag{2}$$

When $M_{p-k+1}=0, k=2,\dots,p$, the result is sharp.

Meanwhile, another two new theorems for polyharmonic mappings were established.

Theorem B([23]) Suppose $F(z)=\sum\limits_{k=1}^{p}|z|^{2(k-1)}G_{p-k+1}(z)$ is a polyharmonic mapping in the unit disk $\mathbb{U}$, with $F(0)=0$, and satisfying following conditions:
(i) for $k\in\{1,\cdots,p\}$, $G_{p-k+1}(z)$ is harmonic in $\mathbb{U}$, and $\lambda_{G_{p-k+1}}(0)-1=G_{p-k+1}(0)=0$;
(ii) for $k\in\{2,\cdots,p\}$, $|G_{p-k+1}(z)|\le M_{p-k+1}$, $\Lambda_{G_p}(z)\le\Lambda_p$ for all $z\in\mathbb{U}$.

Then for $k\in\{2,\cdots,p\}$, $M_{p-k+1}\ge 1, \Lambda_p\ge 1$, $F(z)$ is univalent in $\mathbb{U}_{\rho_2}$, and $F(\mathbb{U}_{\rho_2})$ contains the schlicht disk $\mathbb{U}_{\rho_2'}$, where ρ_2 is the minimum positive root in $(0,1)$ of the following equation

$$\frac{\Lambda_p(1-\Lambda_p r)}{\Lambda_p-r}-\sum_{k=2}^{p}\left[(2k-1)K_1(M_{p-k+1})r^{2k-2}+K_2(M_{p-k+1})r^{2k-1}\frac{2k-(2k-1)r}{(1-r)^2}\right]=0. \tag{3}$$

and

$$\rho_2'=\Lambda_p^2\rho_2+(\Lambda_p^3-\Lambda_p)\log(1-\frac{\rho_2}{\Lambda_p})-\sum_{k=2}^{p}\left[K_1(M_{p-k+1})\rho_2^{2k-1}+K_2(M_{p-k+1})\frac{\rho_2^{2k}}{1-\rho_2}\right], \tag{4}$$

where

$$K_1(M_{p-k+1})=\min\left\{\sqrt{2M_{p-k+1}^2-1},\frac{4M_{p-k+1}}{\pi}\right\}, K_2(M_{p-k+1})=\min\left\{\sqrt{2M_{p-k+1}^2-2},\frac{4M_{p-k+1}}{\pi}\right\}.$$

When $M_{p-k+1}=1, k=2,\dots,p$, the result is sharp.

Theorem C([23]) Let $F(z)=\sum\limits_{k=1}^{p}|z|^{2(k-1)}G_{p-k+1}(z)$ be a polyharmonic mapping in the unit disk $\mathbb{U}$, with $F(0)=\lambda_F(0)-1=0$, and satisfying the following conditions:
(i) for $k\in\{1,\cdots,p\}$, $G_{p-k+1}(z)$ is harmonic in $\mathbb{U}$, and $G_{p-k+1}(0)=0$;
(ii) for $k\in\{2,3,\cdots,p\}$, $\Lambda_{G_{p-k+1}}(z)\le\Lambda_{p-k+1}$, and $|G_p(z)|\le M_p$ for $z\in\mathbb{U}$.

Then $\Lambda_{p-k+1}\ge 0, M_p\ge 1$, $F(z)$ is univalent in $\mathbb{U}_{\rho_3}$, and $F(\mathbb{U}_{\rho_3})$ contains a schlicht disk $\mathbb{U}_{\rho_3'}$, where ρ_3 is the unique positive root in $(0,1)$ of the following equation:

$$1-K_2(M_p)\frac{2r-r^2}{(1-r)^2}-\sum_{k=2}^{p}(2k-1)\Lambda_{p-k+1}r^{2(k-1)}=0, \tag{5}$$

and

$$\rho_3'=\rho_3-K_2(M_p)\frac{\rho_3^2}{1-\rho_3}-\sum_{k=2}^{p}\rho_3^{2k-1}\Lambda_{p-k+1}. \tag{6}$$

When $M_p=1$, the result is sharp.

On the other hand, Liu and Luo obtained the sharp results for Landau's theorem of polyharmonic mappings with conditions $\Lambda_{G_p}(z)\le 1$, and $\Lambda_{G_{p-k+1}}(z)\le\Lambda_{p-k+1}$, $k\in\{2,3,\cdots,p\}$.

Theorem D([20]) Suppose that p is a positive integer, $p\ge 2$, $\Lambda_1,\cdots,\Lambda_{p-1}\ge 0$. Let $F(z)=\sum_{k=1}^{p}|z|^{2(k-1)}G_{p-k+1}(z)$ be a polyharmonic mapping of $\mathbb{U}$, where all G_{p-k+1} are harmonic on $\mathbb{U}$, satisfying $G_{p-k+1}(0)=\lambda_F(0)-1=0$ for $k=1,2,\cdots,p$. If $\Lambda_{G_p}(z)\le 1$, and $\Lambda_{G_{p-k+1}}(z)\le\Lambda_{p-k+1}$, $k\in\{2,3,\cdots,p\}$ for all $z\in\mathbb{U}$. Then $F(z)$ is univalent in $\mathbb{U}_{\rho_4}$, and $F(\mathbb{U}_{\rho_4})$ contains a schlicht disk $\mathbb{U}_{\rho_4'}$, where

$$\rho_4=\begin{cases}1, & \text{if } \sum\limits_{k=1}^{p-1}(2k+1)\Lambda_{p-k}\le 1,\\ \rho_4'', & \text{if } \sum\limits_{k=1}^{p-1}(2k+1)\Lambda_{p-k}>1,\end{cases} \tag{7}$$

and ρ_4'' is the unique root in $(0,1)$ of the equation

$$1-\sum_{k=1}^{p-1}(2k+1)\Lambda_{p-k}r^{2k}=0, \tag{8}$$

and $\rho_4'=\rho_4-\sum_{k=1}^{p-1}\Lambda_{p-k}\rho_4^{2k+1}$. Moreover, these estimates are sharp, with an extremal function given by

$$F_1'(z)=z-\sum_{k=1}^{p-1}\Lambda_{p-k}|z|^{2k}z. \tag{9}$$

In 2012, Li and Wang firstly obtained the following Landau's theorem for log-p-harmonic mappings with condition of $J_f(0)=1$.

Theorem E([14]) Let $f(z)=\prod\limits_{k=1}^{p} g_{p-k+1}(z)^{|z|^{2(k-1)}}$ be a log-p-harmonic mapping of the unit disk $\mathbb{U}$, where $g_{p-k+1}(z)$ is log-harmonic with $g_{p-k+1}(0)=g_p(0)=J_f(0)=1$, $|g_{p-k+1}(z)|<M_1$, for $k\in\{2,\cdots,p\}$, and $|g_p(z)|<M_2$, where $M_i\geq 1$ $(i=1,2)$ are positive constants. Then there exists $\rho_5\in(0,1)$ such that $f(z)$ is univalent in $\mathbb{U}_{\rho_5}$, where ρ_5 satisfies the following equation

$$\lambda_0(M_2^*)-\frac{T(M_2^*)\rho_5(2-\rho_5)}{(1-\rho_5)^2}-\frac{4M_1^*}{\pi(1-\rho_5)^2}\sum_{k=1}^{p-1}\rho_5^{2k}-2M_1^*\sum_{k=1}^{p-1}k\rho_5^{2k-1}=0, \tag{10}$$

where $M_i^*=\log M_i+\pi$ $(i=1,2)$.

Moreover, the range $F(\mathbb{U}_{\rho_5})$ contains a univalent disk $\mathbb{U}(z_5,\rho_5'')$, where

$$z_5=\cosh\Big(\frac{\rho_5'}{\sqrt{2}}\Big),\quad \rho_5''=\min\Big\{\sinh\Big(\frac{\rho_5'}{\sqrt{2}}\Big),\cosh\Big(\frac{\rho_5'}{\sqrt{2}}\Big)\sin\Big(\frac{\rho_5'}{\sqrt{2}}\Big)\Big\}, \tag{11}$$

$$\rho_5'=\rho_5\Big[\lambda_0(M_2^*)-\frac{T(M_2^*)\rho_5}{(1-\rho_5)}-\frac{4M_1^*}{\pi(1-\rho_5)}\sum_{k=1}^{p-1}\rho_5^{2k}\Big]. \tag{12}$$

In 2019, Bai and Liu improved the Landau theorem of log-p-harmonic mapping with the condition of $\lambda_f(0)=1$.

Theorem F([3]) Let $F(z)=\prod\limits_{k=1}^{p} g_{p-k+1}(z)^{|z|^{2(k-1)}}$ be a log-p-harmonic mapping of the unit disk $\mathbb{U}$, satisfying $f(0)=g_p(0)=\lambda_f(0)=1$. Suppose that for each $k\in\{1,\cdots,p\}$, we have

(i) $g_{p-k+1}(z)$ is log-harmonic in $\mathbb{U}$,

(ii) $|g_{p-k+1}(z)|\leq M_{p-k+1}$, Let $G_p=\log g_p$ and $\Lambda_{G_p}(z)\leq\Lambda_p$, where $M_{p-k+1}\geq 1,\Lambda_p>1$.

Then there is a positive number ρ_6 such that $F(z)$ is univalent in $\mathbb{U}_{\rho_6}$, where ρ_6 $(0<\rho_6<1)$ satisfies the following equation

$$1-\frac{4}{\pi(1-r^2)}\sum_{k=1}^{p-1}r^{2k}M_{p-k}^*-\sum_{k=1}^{p-1}kM_{p-k}^*r^{2k}\frac{8}{\pi(1-r)}-\frac{\Lambda_p^2-1}{\Lambda_p}\frac{r}{1-r}=0, \tag{13}$$

where $M_{p-k+1}^*=\log M_{p-k+1}+\pi, k=2,3,\cdots,p$. Moreover, the range $F(\mathbb{U}_{\rho_6})$ contains a univalent disk $\mathbb{U}(z_6,\rho_6'')$, where

$$z_6=\cosh\Big(\frac{\rho_6'}{\sqrt{2}}\Big),\quad \rho_6''=\min\Big\{\sinh\Big(\frac{\rho_6'}{\sqrt{2}}\Big),\cosh\Big(\frac{\rho_6'}{\sqrt{2}}\Big)\sin\Big(\frac{\rho_6'}{\sqrt{2}}\Big)\Big\}, \tag{14}$$

$$\rho_6' = \rho_6 + \frac{\Lambda_p^2 - 1}{\Lambda_p}[\rho_6 + \log(1-\rho_6)] - \sum_{k=1}^{p-1} \rho_6^{2k} \frac{4M_{p-k}^* \rho_6}{\pi(1-\rho_6)}. \tag{15}$$

However, Theorem A is not sharp for $M_{p-k+1} > 0, k = 2, 3, ..., p$, and Theorem F is also not sharp. In this paper, we first establish a sharp version of Landau-type theorem for polyharmonic mappings with extremal function given by Example 3.2. For Example 3.2 satisfying with the hypothesis of Theorems A, it is natural to pose a Conjecture. Next, we establish two versions of Landau-type theorems of polyharmonic mappings by applying Cauchy-inequality, which improve the correspondent results for Theorems B and C, respectively. Finally, three new Landau-type theorems of log-p-harmonic mappings are established, where Theorems 3.9, 3.10 and 3.11 are the corresponding forms of Theorems 3.4, A and 3.5, respectively.

2. Preliminaries

In order to establish our main results, we need the following lemmas.

Lemma 2.1 ([5]) Suppose that $f(z) = f_1(z) + \overline{f_2(z)}$ is a harmonic mapping with $f_1(z) = \sum_{n=1}^{\infty} a_n z^n$ and $f_2(z) = \sum_{n=1}^{\infty} b_n z^n$ being analytic in $\mathbb{U}$. If $|f(z)| \leq M$ for all $z \in \mathbb{U}$, then

$$\Lambda_f(z) \leq \frac{4M}{\pi(1-|z|^2)}. \tag{1}$$

Lemma 2.2 ([6]) Let f be a harmonic mapping of the unit disk $\mathbb{U}$ with $f(0) = 0$ and $f(\mathbb{U}) \subset \mathbb{U}$. Then

$$|f(z)| \leq \frac{4}{\pi} \arctan |z| \leq \frac{4}{\pi}|z|, \ \text{ for } z \in \mathbb{U}.$$

Lemma 2.2 is called Schwarz type Lemma of complex-valued harmonic functions with $f(0) = 0$. Later, Hethcote[13] obtained sharp inequality by removing the assumption $f(0) = 0$, and then Mateljević et al. [25][26] gave the improvements of Hethcote's result.

Lemma 2.3 ([22]) Suppose that $f(z) = f_1(z) + \overline{f_2(z)}$ is a harmonic mapping of the unit disk $\mathbb{U}$ with $f_1(z) = \sum_{n=1}^{\infty} a_n z^n$ and $f_2(z) = \sum_{n=1}^{\infty} b_n z^n$. If f satisfies $|f(z)| \leq M$ for all $z \in \mathbb{U}$ and $\lambda_f(0) = 1$, then $M \geq 1$, and

$$|a_1| + |b_1| \leq K_1(M) = \min\{\sqrt{2M^2 - 1}, \frac{4M}{\pi}\}. \tag{2}$$

The inequality (2) is sharp for $M = 1$, with $f_0(z) = z$ being an extremal mapping.

Lemma 2.4 ([27]) Suppose that $f(z) = f_1(z) + \overline{f_2(z)}$ is a harmonic mapping of the unit disk $\mathbb{U}$ with $f_1(z) = \sum_{n=1}^{\infty} a_n z^n$ and $f_2(z) = \sum_{n=1}^{\infty} b_n z^n$.

(1) If f satisfies $|f(z)| \leq M$ for all $z \in \mathbb{U}$, then

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|)^2 \leq 2M^2. \tag{3}$$

(2) If f satisfies $|f(z)| \leq M$ for all $z \in \mathbb{U}$ and $J_f(0) = 1$, then

$$\Big(\sum_{n=2}^{\infty} (|a_n| + |b_n|)^2\Big)^{\frac{1}{2}} \leq \sqrt{M^4 - 1} \cdot \lambda_f(0), \tag{4}$$

where

$$\lambda_f(0) \geq \lambda_0(M) = \begin{cases} \frac{\sqrt{2}}{\sqrt{M^2-1} + \sqrt{M^2+1}}, & 1 \leq M \leq M_0 = \frac{\pi}{2\sqrt[4]{2\pi^2 - 16} \approx 1.1296}, \\ \frac{\pi}{4}, & M > M_0. \end{cases} \tag{5}$$

(3) If f satisfies $|f(z)| \leq M$ for all $z \in \mathbb{U}$ and $\lambda_f(0) = 1$, then

$$\left(\sum_{n=2}^{\infty}(|a_n| + |b_n|)^2\right)^{\frac{1}{2}} \leq \sqrt{2M^2 - 2}. \tag{6}$$

Lemma 2.5 ([21]) Suppose $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk $\mathbb{U}$ with $h(z)$, $g(z)$ are holomorphic in $\mathbb{U}$, $h(0) = g(0) = \lambda_f(0) - 1 = 0$, $\Lambda_f(z) < \Lambda$ for all $z \in \mathbb{U}$. Then

(i) For two distinct points $z_1, z_2 \in \mathbb{U}_r$ ($r < \frac{1}{\Lambda}$),

$$|f(z_1) - f(z_2)| \geq \frac{\Lambda(1 - \Lambda r)}{\Lambda - r}|z_1 - z_2|.$$

(ii) For $z = re^{i\theta} \in \partial\mathbb{U}_r$,

$$|f(z)| \geq \Lambda^2 r + (\Lambda^3 - \Lambda)\ln(1 - \frac{r}{\Lambda}).$$

Lemma 2.6 ([23]) For $z_1, z_2 \in \mathbb{U}_r, k, j \in\in N_+$, we have

$$\left| |z_1|^{2k} z_1^j - |z_2|^{2k} z_2^j \right| \leq (2k + j) r^{2k+j-1} |z_1 - z_2|.$$

Lemma 2.7 ([23]) Suppose $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk $\mathbb{U}$ with $\lambda_f(0) = 1$ and $f(0) = 0$. Then $|f(z)| \leq 1$ for all $z \in \mathbb{U}$ if and only if $\Lambda_f(z) \leq 1$ for all $z \in \mathbb{U}$.

Lemma 2.8 ([20]) Suppose that p is a positive integer and $0 < \sigma < 1, 0 < \rho \leq 1$. Let $f(z)$ be a log-p-harmonic mapping of $\mathbb{U}$ satisfying $f(0) = \lambda_f(0) = 1$. Suppose that $f(z)$ is univalent in $\mathbb{U}_\rho$ and $F(\mathbb{U}_\rho) \supset \mathbb{U}_\sigma$, where $F(z) = \log f(z)$. Then the range $F(\mathbb{U}_\rho)$ contains a schlicht disk $\mathbb{U}(w_0, r_0) = \{w \in \mathbb{C} : |w - w_0| < r_0\}$, where

$$w_0 = \cosh\sigma, \quad r_0 = \sinh\sigma.$$

Moreover, if ρ is the biggest univalent radius of $f(z)$, then the radius $r_0 = \sinh\sigma$ is sharp.

3. Main Results

Applying Lemma 2.6, we first establish a sharp version of Landau-type theorem for polyharmonic mappings.

Theorem 3.1 Suppose that $\Lambda_p \geq 1$, $M_{p-k+1} \geq 0$, $|G_{p-k+1}| \leq M_{p-k+1}$ for $k \in \{2, \cdots, p\}$ and $|G_p| = \Lambda_p$. Let

$$F_1(z) = \sum_{k=2}^{p} G_{p-k+1}|z|^{2(k-1)}z + G_p \int_0^z \frac{\zeta - \frac{1}{\Lambda_p}}{1 - \frac{\zeta}{\Lambda_p}} d\zeta$$

be a polyharmonic mapping of the unit disk $\mathbb{U}$. Then $F_1(z)$ is univalent in the disk $\mathbb{U}_{r_1}$, where r_1 is the unique positive root in $(0, 1)$ of the equation

$$\frac{\Lambda_p(1 - \Lambda_p r)}{\Lambda_p - r} - \sum_{k=2}^{p}(2k - 1)M_{p-k+1}r^{2(k-1)} = 0,$$

and $F_1(\mathbb{U}_{r_1})$ contains a schlicht disk $\mathbb{U}_{R_1}$, with

$$R_1 = \Lambda_p^2 r_1 + (\Lambda_p^3 - \Lambda_p)\log(1 - \frac{r_1}{\Lambda_p}) - \sum_{k=2}^{p} M_{p-k+1} r_1^{2k-1}.$$

Both of r_1 and R_1 are sharp.

Proof Firstly, we prove $F_1(z)$ is univalent in the disk $\mathbb{U}_{r_1}$. To this end, we choose two distinct points z_1, z_2 in the disk $\mathbb{U}_r (r < r_1)$. Then, applying Lemma 2.6, we have

$$\begin{aligned}
& |F_1(z_1) - F_1(z_2)| \\
= & \Big| \sum_{k=2}^{p} G_{p-k+1} |z_1|^{2(k-1)} z_1 + G_p \int_0^{z_1} \frac{\zeta - \frac{1}{\Lambda_p}}{1 - \frac{\zeta}{\Lambda_p}} d\zeta - \\
& \sum_{k=2}^{p} G_{p-k+1} |z_2|^{2(k-1)} z_2 - G_p \int_0^{z_2} \frac{\zeta - \frac{1}{\Lambda_p}}{1 - \frac{\zeta}{\Lambda_p}} d\zeta \Big| \\
\geq & \Lambda_p \Big| \int_{z_1}^{z_2} \frac{\zeta - \frac{1}{\Lambda_p}}{1 - \frac{\zeta}{\Lambda_p}} d\zeta \Big| - \sum_{k=2}^{p} M_{p-k+1} \Big| |z_1|^{2(k-1)} z_1 - |z_2|^{2(k-1)} z_2 \Big| \\
\geq & \Lambda_p \frac{\frac{1}{\Lambda_p} - r}{1 - \frac{r}{\Lambda_p}} |z_1 - z_2| - \sum_{k=2}^{p} (2k-1) M_{p-k+1} r^{2(k-1)} |z_1 - z_2| \\
= & \Big[\frac{\Lambda_p (1 - \Lambda_p r)}{\Lambda_p - r} - \sum_{k=2}^{p} (2k-1) M_{p-k+1} r^{2(k-1)} \Big] |z_1 - z_2| > 0.
\end{aligned}$$

Thus, we have $F_1(z_1) \neq F_1(z_2)$, which proves the univalence of $F_1(z)$ in the disk $\mathbb{U}_{r_1}$.

Next, we prove the sharpness of r_1. Considering the real function

$$f(x) = -\sum_{k=2}^{p} M_{p-k+1} x^{2k-1} - \Lambda_p \int_0^x \frac{\zeta - \frac{1}{\Lambda_p}}{1 - \frac{\zeta}{\Lambda_p}} d\zeta, x \in [0, 1].$$

Then

$$f'(x) = \frac{\Lambda_p (1 - \Lambda_p x)}{\Lambda_p - x} - \sum_{k=2}^{p} (2k-1) M_{p-k+1} x^{2(k-1)}.$$

Because $f'(x)$ is strictly monotone decreasing on $[0, 1]$, and

$$f'(0) = 1, f'(1) = -\Lambda_p - \sum_{k=2}^{p} (2k-1) M_{p-k+1} < 0,$$

so $f'(x) = 0$ for $x \in (0, 1)$ if and only if $x = r_1$. Hence $f(x)$ is strictly monotone increasing on $[0, r_1]$ and strictly monotone decreasing on $[r_1, 1]$. For every fixed $r' \in (r_1, 1)$, there exists two distinct points $x_1, x_2 \in (0, r')$, $f(x_1) = f(x_2)$. Thus, r_1 cannot be replaced by any bigger number.

And for any point $z = r_1 e^{i\theta}$ on $\partial\mathbb{U}_{r_1}$, we have

$$\begin{aligned}
|F_1(z)| &= \Big|\sum_{k=2}^{p} G_{p-k+1}|z|^{2(k-1)}z + G_p \int_0^z \frac{\zeta - \frac{1}{\Lambda_p}}{1-\frac{\zeta}{\Lambda_p}} d\zeta\Big| \\
&\geq \Lambda_p \Big|\int_0^z \frac{\zeta - \frac{1}{\Lambda_p}}{1-\frac{\zeta}{\Lambda_p}} d\zeta\Big| - \sum_{k=2}^{p} M_{p-k+1} r_1^{2k-1} \\
&\geq \Lambda_p \int_0^{r_1} \frac{\frac{1}{\Lambda_p} - t}{1-\frac{t}{\Lambda_p}} dt - \sum_{k=2}^{p} M_{p-k+1} r_1^{2k-1} \\
&= -\Lambda_p \int_0^{r_1} \frac{t - \frac{1}{\Lambda_p}}{1-\frac{t}{\Lambda_p}} dt - \sum_{k=2}^{p} M_{p-k+1} r_1^{2k-1} \\
&= \Lambda_p^2 r_1 + (\Lambda_p^3 - \Lambda_p)\log(1-\frac{r_1}{\Lambda_p}) - \sum_{k=2}^{p} M_{p-k+1} r_1^{2k-1} = R_1, \\
f(r_1) &= -\Lambda_p \int_0^{r_1} \frac{\zeta - \frac{1}{\Lambda_p}}{1-\frac{\zeta}{\Lambda_p}} d\zeta - \sum_{k=2}^{p} M_{p-k+1} r_1^{2k-1} \\
&= \Lambda_p^2 r_1 + (\Lambda_p^3 - \Lambda_p)\log(1-\frac{r_1}{\Lambda_p}) - \sum_{k=2}^{p} M_{p-k+1} r_1^{2k-1} = R_1.
\end{aligned}$$

Hence R_1 is sharp. This completes the proof. □

By the proof of Theorem 3.1, we obtain the extremal function $F_2(z)$ by the following example.

Example 3.2 Suppose that $\Lambda_p \geq 1, M_{p-k+1} \geq 0, k \in \{2, \cdots, p\}$. Let

$$F_2(z) = -\sum_{k=2}^{p} M_{p-k+1}|z|^{2(k-1)}z - \Lambda_p \int_0^z \frac{\zeta - \frac{1}{\Lambda_p}}{1-\frac{\zeta}{\Lambda_p}} d\zeta$$

be a polyharmonic mapping of the unit disk $\mathbb{U}$. Then $F_2(z)$ is univalent in the disk $\mathbb{U}_{r_1}$, and $F_2(\mathbb{U}_{r_1})$ contains a schlicht disk $\mathbb{U}_{R_1}$, where r_1 and R_1 are given by Theorem 3.1. Both of r_1 and R_1 are sharp.

We note that the polyharmonic mappings in Example 3.2 satisfying the hypothesis of Theorem A, it is natural to pose a conjecture as follows:

Conjecture 3.3 Under the hypothesis of Theorem A, $F(z)$ is univalent in $\mathbb{U}_{r_1}$ and $F(\mathbb{U}_{r_1})$ contains a schlicht disk $\mathbb{U}_{R_1}$. This result is sharp, with r_1, R_1, and the extremal mapping are given by Example 3.2.

Next, we establish a new version Landau-type theorem by adding extra conditions $\lambda_{G_{p-k+1}}(0) = 1, k \in \{2, 3, \cdots, p\}$ to Theorem A, which is sharp when $M_{p-k+1} = 1\,(k = 2, 3, \cdots, p)$. We prove the following result with a method of proof of [27].

Theorem 3.4 Suppose $F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$ is a polyharmonic mapping in the unit disk $\mathbb{U}$, with $F(0) = 0$, and satisfying following conditions:

(i) for $k \in \{1, \cdots, p\}$, $G_{p-k+1}(z)$ is harmonic in $\mathbb{U}$, and $\lambda_{G_{p-k+1}}(0) - 1 = G_{p-k+1}(0) = 0$;

(ii) for $k \in \{2, \cdots, p\}$, $|G_{p-k+1}(z)| \leq M_{p-k+1}$, $\Lambda_{G_p}(z) \leq \Lambda_p$ for all $z \in \mathbb{U}$.

Then for $k \in \{2, \cdots, p\}$, $M_{p-k+1} \geq 1, \Lambda_p \geq 1$, $F(z)$ is univalent in $\mathbb{U}_{r_2}$, and $F(\mathbb{U}_{r_2})$ contains the schlicht disk

$\mathbb{U}_{R_2}$, where r_2 is a unique root in $(0,1)$ of the equation $A_1(r)=0$, $A_1(r)$ is defined by the following equation

$$\begin{aligned} A_1(r) &= \frac{\Lambda_p(1-\Lambda_p r)}{\Lambda_p - r} - \sum_{k=2}^{p} r^{2(k-1)}\Big[(2k-1)K_1(M_{p-k+1}) \\ &\quad + \sqrt{2M_{p-k+1}^2 - 2}\Big(\frac{2(k-1)r}{\sqrt{1-r^2}} + \frac{r\sqrt{4-3r^2+r^4}}{(1-r^2)^{\frac{3}{2}}}\Big)\Big], \end{aligned} \tag{1}$$

$$K_1(M_{p-k+1}) = \min\Big\{\sqrt{2M_{p-k+1}^2 - 1}\frac{4M_{p-k+1}}{\pi}\Big\}, \tag{2}$$

and

$$R_2 = \Lambda_p^2 r_2 + (\Lambda_p^3 - \Lambda_p)\log(1-\frac{r_2}{\Lambda_p}) - \sum_{k=2}^{p} r_2^{2k-1}\Big[K_1(M_{p-k+1}) + \sqrt{2M_{p-k+1}^2 - 2}\cdot\frac{r_2}{\sqrt{1-r_2^2}}\Big]. \tag{3}$$

When $M_{p-k+1}=1, k=2,\dots,p$, the result is sharp, with an extremal function given by

$$F_3(z) = \Lambda_p\int_0^z \frac{\frac{1}{\Lambda_p}-\zeta}{1-\frac{\zeta}{\Lambda_p}}d\zeta - \sum_{k=2}^{p}|z|^{2(k-1)}z = \Lambda_p^2 z + (\Lambda_p^3-\Lambda_p)\log(1-\frac{z}{\Lambda_p}) - \sum_{k=2}^{p}|z|^{2(k-1)}z. \tag{4}$$

Proof By the hypothesis of Theorem 3.4 and Lemma 2.3, we have that $M_{p-k+1}\ge 1$ for $k\in\{2,\dots,p\}$, and $\Lambda_p \ge \Lambda_{G_p}(0) \ge \lambda_{G_p}(0) = 1$.

In order to prove the univalence of F, we choose two distinct points $z_1, z_2 \in \mathbb{U}_r (0<r<1)$. Then we have

$$\begin{aligned} |F(z_2)-F(z_1)| &= \Big|\sum_{k=2}^{p}|z_2|^{2(k-1)}G_{p-k+1}(z_2) + G_p(z_2) - \sum_{k=2}^{p}|z_2|^{2(k-1)}G_{p-k+1}(z_1) - G_p(z_1)\Big| \\ &\ge \Big|G_p(z_2)-G_p(z_1)\Big| - \Big|\sum_{k=2}^{p}|z_2|^{2(k-1)}G_{p-k+1}(z_2) - \sum_{k=2}^{p}|z_2|^{2(k-1)}G_{p-k+1}(z_1)\Big|. \end{aligned}$$

Since $\lambda_F(0) = \Big||(G_p)_z(0)| - |(G_p)_{\bar{z}}(0)|\Big| = \lambda_{G_p}(0) = 1$, $\Lambda_{G_p}(z) < \Lambda_p$, by Lemma 2.5, we have

$$\Big|G_p(z_2)-G_p(z_1)\Big| \ge \frac{\Lambda_p(1-\Lambda_p r)}{\Lambda_p - r}|z_2 - z_1|.$$

For any $k\in\{2,\dots,p\}$, we give the series form of G_{p-k+1} as follow:

$$G_{p-k+1}(z) = \sum_{j=1}^{\infty} a_{j,p-k+1}z^j + \sum_{j=1}^{\infty}\overline{b_{j,p-k+1}}\bar{z}^j.$$

Using Lemmas 2.3, 2.4 and 2.6, we have

$$\begin{aligned} &\Big|\sum_{k=2}^{p}|z_2|^{2(k-1)}G_{p-k+1}(z_2) - \sum_{k=2}^{p}|z_1|^{2(k-1)}G_{p-k+1}(z_1)\Big| \\ =\ &\Big|\sum_{k=2}^{p}\sum_{j=1}^{\infty}\Big(a_{j,p-k+1}(|z_2|^{2(k-1)}z_2^j - |z_1|^{2(k-1)}z_1^j) + b_{j,p-k+1}(|z_2|^{2(k-1)}\overline{z_2}^j - |z_1|^{2(k-1)}\overline{z_1}^j)\Big)\Big| \\ \le\ &\sum_{k=2}^{p}\sum_{j=1}^{\infty}(|a_{j,p-k+1}| + |b_{j,p-k+1}|)||z_2|^{2(k-1)}z_2^j - |z_1|^{2(k-1)}z_1^j| \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=2}^{p}\sum_{j=1}^{\infty}(|a_{j,p-k+1}|+|b_{j,p-k+1}|)(2k+j-2)r^{2k+j-3}|z_1-z_2| \\
&\leq \sum_{k=2}^{p} r^{2(k-1)}\Big[(2k-1)K_1(M_{p-k+1})+2(k-1)\Big(\sum_{j=2}^{\infty}\big(|a_{j,p-k+1}|+|b_{j,p-k+1}|\big)\Big)^{\frac{1}{2}}\Big(\sum_{j=2}^{\infty}r^{2(j-1)}\Big)^{\frac{1}{2}} \\
&+ \Big(\sum_{j=2}^{\infty}\big(|a_{j,p-k+1}|+|b_{j,p-k+1}|\big)\Big)^{\frac{1}{2}}\Big(\sum_{j=2}^{\infty}j^2r^{2(j-1)}\Big)^{\frac{1}{2}}\Big]|z_1-z_2| \\
&= \sum_{k=2}^{p} r^{2(k-1)}\Big[(2k-1)K_1(M_{p-k+1})+\sqrt{2M_{p-k+1}^2-2}\Big(\frac{2(k-1)r}{\sqrt{1-r^2}}+\frac{r\sqrt{4-3r^2+r^4}}{(1-r^2)^{\frac{3}{2}}}\Big)\Big]|z_1-z_2|.
\end{aligned}
$$

Hence,

$$|F(z_1)-F(z_2)| \geq A_1(r)|z_1-z_2|,$$

where $A_1(r)$ is defined by (1).

It is not difficult to verify that $A_1(r)$ is strictly decreasing in $(0,1)$, and

$$\lim_{r\to 0}A_1(r)=1,\quad \lim_{r\to 1}A_1(r)=-\infty.$$

Hence there exists a unique root r_2 in $(0,1)$ of the equation $A_1(r)=0$. This shows that

$$|F(z_2)-F(z_1)|>0$$

for any two distinct points $z_1, z_2 \in \mathbb{U}_{r_2}$. Thus F is univalent in $\mathbb{U}_{r_2}$.

Next, for any point $z=r_2e^{i\theta}$ on $\partial\mathbb{U}_{r_2}$, by Lemmas 2.3, 2.4 and 2.5, we have

$$
\begin{aligned}
|F(z)| &= \Big|G_p(z)+\sum_{k=2}^{p}|z|^{2(k-1)}G_{p-k+1}(z)\Big| \\
&= \Big|G_p(z)+\sum_{k=2}^{p}|z|^{2(k-1)}\sum_{j=1}^{\infty}a_{j,p-k+1}z^j+\sum_{j=1}^{\infty}\overline{b_{j,p-k+1}}\bar{z}^j\Big| \\
&\geq \Lambda_p^2r_2+(\Lambda_p^3-\Lambda_p)\log\Big(1-\frac{r_2}{\Lambda_p}\Big) \\
&\quad -\sum_{k=2}^{p}|z|^{2(k-1)}\Big[\big(|a_{1,p-k+1}z|+|b_{1,p-k+1}z|\big)+\sum_{j=2}^{\infty}\big(|a_{j,p-k+1}z^j|+|b_{j,p-k+1}z^j|\big)\Big] \\
&\geq \Lambda_p^2r_2+(\Lambda_p^3-\Lambda_p)\log\Big(1-\frac{r_2}{\Lambda_p}\Big) \\
&\quad -\sum_{k=2}^{p}r_2^{2(k-1)}\Big[K_1(M_{p-k+1})r_2+\sqrt{2M_{p-k+1}^2-2}\cdot\frac{r_2^2}{\sqrt{1-r_2^2}}\Big]=R_2.
\end{aligned}
$$

Hence, $F(\mathbb{U}_{r_2})$ contains a schlicht disk $\mathbb{U}_{R_2}$.

When $M_{p-k+1}=1, \Lambda_p\geq 1$ for $k=2,\ldots,p$, the result is sharp with an extremal function $F_3(z)$, which is given by (4). This completes the proof. □

The equation $A_1(r)=0$ which $A_1(r)$ is defined by (1) cannot be solved explicitly. The Computer Algebra System Mathematica has calculated the numerical solutions to equations (1), (3), (3) and (4). Table 1 shows the approximate values of r_2, R_2 and ρ_2, ρ_2' that correspond to different choice of the constants M_1 and Λ_2, which shows that $r_2>\rho_2$ and $R_2>\rho_2'$, that is, Theorem 3.4 is an improvement of Theorem B.

Table 1: The values of ρ_2, ρ_2' and r_2, R_2 are in Theorems B and Theorem 3.4

	$M_1 = \Lambda_2 = 1.1$	$M_1 = 1.5, \Lambda_2 = 2$	$M_1 = \Lambda_2 = 2$	$M_1 = 2.5, \Lambda_2 = 3$	$M_1 = \Lambda_2 = 3$
ρ_2	0.397736	0.261255	0.234962	0.190024	0.180374
r_2	0.422555	0.268498	0.241163	0.192773	0.182519
ρ_2'	0.275692	0.161787	0.147208	0.112778	0.107824
R_2	0.286601	0.164292	0.149431	0.113631	0.108473

And then, changing some hypothesis of Theorem 3.4, we establish a new version of Landau-type theorems of polyharmonic mappings as follows.

Theorem 3.5 Let $F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$ be a polyharmonic mapping in the unit disk $\mathbb{U}$, with $F(0) = \lambda_F(0) - 1 = 0$, and satisfying the following conditions:
(i) for $k \in \{1, \cdots, p\}$, $G_{p-k+1}(z)$ is harmonic in $\mathbb{U}$, and $G_{p-k+1}(0) = 0$;
(ii) for $k \in \{2, 3, \cdots, p\}$, $\Lambda_{G_{p-k+1}}(z) \leq \Lambda_{p-k+1}$, and $|G_p(z)| \leq M_p$ for $z \in \mathbb{U}$.

Then $\Lambda_{p-k+1} \geq 0, M_p \geq 1$, $F(z)$ is univalent in $\mathbb{U}_{r_3}$, and $F(\mathbb{U}_{r_3})$ contains a schlicht disk $\mathbb{U}_{R_3}$, where r_3 is the unique positive root in $(0, 1)$ of the following equation:

$$1 - \sqrt{2M_p^2 - 2} \cdot \frac{r\sqrt{r^4 - 3r^2 + 4}}{(1 - r^2)^{\frac{3}{2}}} - \sum_{k=2}^{p} (2k-1)\Lambda_{p-k+1} r^{2(k-1)} = 0, \tag{5}$$

and

$$R_3 = r_3 - \sqrt{2M_p^2 - 2} \cdot \frac{r_3^2}{\sqrt{1 - r_3^2}} - \sum_{k=2}^{p} r_3^{2k-1} \Lambda_{p-k+1}. \tag{6}$$

When $M_p = 1$, the result is sharp, with an extremal function $F_1'(z)$, which is given by (9).

Proof By the hypothesis of Theorem 3.5 and Lemma 2.4, we have $\Lambda_{p-k+1} \geq 0$ and $M_p \geq 1$ for $k \in \{2, \cdots, p\}$, and $G_p(z)$ has the following series form

$$G_p(z) = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n}.$$

Then we have $\lambda_{G_p}(0) = \Big||(G_p)_z(0)| - |(G_p)_{\overline{z}}(0)|\Big| = \Big||a_1| - |b_1|\Big| = \lambda_F(0) = 1$.

By Lemma 2.4, we have $\left(\sum_{n=2}^{\infty} (|a_n| + |b_n|)^2\right)^{\frac{1}{2}} \leq \sqrt{2M_p^2 - 2}, n \geq 2$.

In order to prove the univalence of F, we choose two distinct points $z_1, z_2 \in \mathbb{U}_r (0 < r < 1)$. Then we have

$$\begin{aligned} |F(z_1) - F(z_2)| &= \left| \int_{[z_1, z_2]} F_z(z) dz + F_{\overline{z}}(z) d\overline{z} \right| \\ &\geq \left| \int_{[z_1, z_2]} (G_p)_z(0) dz + (G_p)_{\overline{z}}(0) d\overline{z} \right| \end{aligned}$$

$$
\begin{aligned}
&-\Big|\int_{[z_1,z_2]}[(G_p)_z(z)-(G_p)_z(0)]dz+[(G_p)_{\overline{z}}(z)-(G_p)_{\overline{z}}(0)]d\overline{z}\Big|\\
&-\Big|\sum_{k=2}^{p}\int_{[z_1,z_2]}|z|^{2(k-1)}[(G_{p-k+1})_z(z)dz+(G_{p-k+1})_{\overline{z}}(z)d\overline{z}]\Big|\\
&-\Big|\sum_{k=2}^{p}\int_{[z_1,z_2]}(k-1)G_{p-k+1}(z)(\overline{z}^{k-1}z^{k-2}dz+\overline{z}^{k-2}z^{k-1}d\overline{z})\Big|\\
\geq\ & |z_1-z_2|\Big(\lambda_{G_p}(0)-\sum_{n=2}^{\infty}n(|a_n|+|b_n|)r^{n-1}-\sum_{k=2}^{p}r^{2k-1}\Lambda_{G_{p-k+1}}\Big)\\
&-\sum_{k=2}^{p}\int_{[z_1,z_2]}(k-1)|G_{p-k+1}(z)|(|\overline{z}^{k-1}z^{k-2}||dz|+|\overline{z}^{k-2}z^{k-1}||d\overline{z})|)\\
\geq\ & |z_1-z_2|\Big[1-\Big(\sum_{n=2}^{\infty}(|a_n|+|b_n|)^2\Big)^{\frac{1}{2}}\Big(\sum_{n=2}^{\infty}n^2r^{2(n-1)}\Big)^{\frac{1}{2}}\\
&-\sum_{k=2}^{p}(2k-1)\Lambda_{p-k+1}r^{2(k-1)}\Big]\\
\geq\ & |z_1-z_2|\Big[1-\sqrt{2M_p^2-2}\cdot\frac{r\sqrt{r^4-3r^2+4}}{(1-r^2)^{\frac{3}{2}}}-\sum_{k=1}^{p-1}(2k+1)\Lambda_{p-k}r^{2k}\Big]\\
=\ & A_2(r)|z_1-z_2|.
\end{aligned}
$$

It is not difficult to verify that $A_2(r)$ is strictly decreasing in $(0,1)$, and

$$\lim_{r\to 0}A_2(r)=1,\quad \lim_{r\to 1}A_2(r)=-\infty.$$

Hence there exists an unique root r_3 in $(0,1)$ of the equation $A_2(r)=0$. This shows that $|F(z_1)-F(z_2)|>0$ for any two distinct points $z_1,z_2\in\mathbb{U}_{r_3}$. Then $F(z)$ is univalent in $\mathbb{U}_{r_3}$.

Next, we prove $F(\mathbb{U}_{r_3})\supset\mathbb{U}_{R_3}$. For $z=r_3e^{i\theta}\in\partial\mathbb{U}_{r_3}$, we have

$$
\begin{aligned}
|F(z)| &= \Big|\sum_{n=1}^{\infty}(a_nz^n+\overline{b_n}\overline{z^n})+\sum_{k=2}^{p}|z|^{2(k-1)}G_{p-k+1}(z)\Big|\\
&\geq |a_1z+\overline{b_1}\overline{z}|-\Big|\sum_{n=2}^{\infty}(a_nz^n+\overline{b_n}\overline{z^n})\Big|-\sum_{k=2}^{p}|z|^{2(k-1)}|G_{p-k+1}(z)|\\
&\geq r_3-\sqrt{2M_p^2-2}\cdot\frac{r_3^2}{\sqrt{1-r_3^2}}-\sum_{k=2}^{p}r_3^{2k-1}\Lambda_{p-k+1}=R_3.
\end{aligned}
$$

Finally, when $M_p=1$, $\sqrt{2M_p^2-2}=0$. Since $\lambda_{G_p}(0)-1=G_p(0)=0$, it follows from Lemma 2.7 that $\Lambda_{G_p}(z)\leq 1$ for all $z\in U$. Thus, by using Theorem D, we obtain that the result is sharp. This completes the proof. □

The equation $A_2(r)=0$ which $A_2(r)$ is defined by (5) cannot be solved explicitly. The Computer Algebra System Mathematica has calculated the numerical solutions to equations (5), (6), (5) and (6). Table 2 shows the approximate values of r_3, R_3 and ρ_3, ρ_3' that correspond to different choice of the constants M_2 and Λ_1 when $p=2$, which shows that $r_3>\rho_3$ and $R_3>\rho_3'$, that is, Theorem 3.5 is an improvement of Theorems C.

Table 2: The values of ρ_3, ρ_3' and r_3, R_3 are in Theorems C and Theorems 3.5 when $p = 2$

	$M_2 = 1.1, \Lambda_1 = 1.1$	$M_2 = 1.1, \Lambda_1 = 0.1$	$M_2 = 2, \Lambda_1 = 2$	$M_2 = 3, \Lambda_1 = 2$	$M_2 = 3, \Lambda_1 = 3$
ρ_3	0.304897	0.365167	0.14212	0.103741	0.101139
r_3	0.365621	0.504695	0.165365	0.113638	0.109897
ρ_3'	0.187046	0.224169	0.0787076	0.0556412	0.0545667
R_3	0.21878	0.300625	0.0884032	0.0587119	0.0573114

Using the analogous proof of Theorem 3.4 and 3.5, we can obtain the following corollaries.

Corollary 3.6 Let $F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$ be a p-harmonic mapping of the unit disk $\mathbb{U}$, with $F(0) = J_F(0) - 1 = 0$, and satisfying the following conditions:
(i) $G_{p-k+1}(z)$ is harmonic in $\mathbb{U}$, and $G_{p-k+1}(0) = 0$ for $k \in \{1, \cdots, p\}$;
(ii) $\Lambda_{G_{p-k+1}}(z) \leq \Lambda_{p-k+1}$ for $k \in \{2, \cdots, p\}$ and $|G_p(z)| \leq M_p$.

Then $\Lambda_{p-k+1} \geq 0, M_p \geq 1$, $F(z)$ is univalent in $\mathbb{U}_{\tau_1}$, and $F(\mathbb{U}_{\tau_1})$ contains a univalent disk $\mathbb{U}_{\tau_1'}$, where τ_1 is the unique positive root in $(0, 1)$ of the equation

$$\lambda_0(M_p) - \lambda_0(M_p)\sqrt{M_p^4 - 1} \cdot \frac{r\sqrt{r^4 - 3r^2 + 4}}{(1 - r^2)^{\frac{3}{2}}} - \sum_{k=2}^{p} (2k - 1)\Lambda_{p-k+1} r^{2(k-1)} = 0, \tag{7}$$

$\lambda_0(M_p)$ is defined by (5), and

$$\tau_1' = \lambda_0(M_p)\Big[\tau_1 - \sqrt{M_p^4 - 1} \cdot \frac{\tau_1^2}{\sqrt{1 - \tau_1^2}}\Big] - \sum_{k=2}^{p} \Lambda_{p-k+1} \tau_1^{2(k-1)}. \tag{8}$$

When $M_p = 1$, the result is sharp.

Corollary 3.7 Suppose $F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$ is a polyharmonic mapping in the unit disk $\mathbb{U}$, with $F(0) = \lambda_F(0) - 1 = 0$. and satisfying following conditions:
(i) for $k \in \{1, \cdots, p\}$, $G_{p-k+1}(z)$ is harmonic in $\mathbb{U}$, and $\lambda_{G_{p-k+1}}(0) - 1 = G_{p-k+1}(0) = 0$;
(ii) for $k \in \{2, \cdots, p\}$, $|G_{p-k+1}(z)| \leq M_{p-k+1}$, $|G_p(z)| \leq M_p$ for all $z \in \mathbb{U}$.

Then for $k \in \{2, \cdots, p\}$, $M_{p-k+1} \geq 1, M_p \geq 1$, $F(z)$ is univalent in $\mathbb{U}_{\tau_2}$, and $F(\mathbb{U}_{\tau_2})$ contains the schlicht disk $\mathbb{U}_{\tau_2'}$, where τ_2 is a unique root in $(0, 1)$ of the equation

$$\begin{aligned} 1 - \sqrt{2M_p^2 - 2} \cdot \frac{r\sqrt{r^4 - 3r^2 + 4}}{(1 - r^2)^{\frac{3}{2}}} - \sum_{k=2}^{p} r^{2(k-1)} \Big[&(2k - 1)K_1(M_{p-k+1}) \\ &+ \sqrt{2M_{p-k+1}^2 - 2}\Big(\frac{2(k-1)r}{\sqrt{1 - r^2}} + \frac{r\sqrt{4 - 3r^2 + r^4}}{(1 - r^2)^{\frac{3}{2}}}\Big)\Big] = 0, \end{aligned}$$

and

$$\tau_2' = \tau_2 - \sqrt{2M_p^2 - 2} \cdot \frac{\tau_2^2}{\sqrt{1 - \tau_2^2}} - \sum_{k=2}^{p} \tau_2^{2k-1} \Big[K_1(M_{p-k+1}) + \sqrt{2M_{p-k+1}^2 - 2} \cdot \frac{\tau_2}{\sqrt{1 - \tau_2^2}}\Big],$$

and $K_1(M_{p-k+1})$ is defined by (2).

When $M_{p-k+1} = 1, k = 1, 2, \ldots, p$, the result is sharp.

Corollary 3.8 Suppose $F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$ is a polyharmonic mapping in the unit disk $\mathbb{U}$, with $F(0) = J_F(0) - 1 = 0$, and satisfying

(i) for $k \in \{1, \cdots, p\}$, $G_{p-k+1}(z)$ is harmonic in $\mathbb{U}$, and $\lambda_{G_{p-k+1}}(0) - 1 = G_{p-k+1}(0) = 0$;
(ii) for $k \in \{2, \cdots, p\}$, $|G_{p-k+1}(z)| \le M_{p-k+1}$, $|G_p(z)| \le M_p$ for all $z \in \mathbb{U}$.

Then for $k \in \{2, \cdots, p\}$, $M_{p-k+1} \ge 1, M_p \ge 1$, $F(z)$ is univalent in $\mathbb{U}_{\tau_3}$, and $F(\mathbb{U}_{\tau_3})$ contains the schlicht disk $\mathbb{U}_{\tau_3'}$, where τ_3 is a unique root in $(0,1)$ of the equation

$$\lambda_0(M_p) - \lambda_0(M_p)\sqrt{M_p^4 - 1} \cdot \frac{r\sqrt{r^4 - 3r^2 + 4}}{(1-r^2)^{\frac{3}{2}}}$$
$$- \sum_{k=2}^{p} r^{2(k-1)}\Big[(2k-1)K_1(M_{p-k+1}) + \sqrt{2M_{p-k+1}^2 - 2}\Big(\frac{2(k-1)r}{\sqrt{1-r^2}} + \frac{r\sqrt{4-3r^2+r^4}}{(1-r^2)^{\frac{3}{2}}}\Big)\Big] = 0,$$

and

$$\tau_3' = \lambda_0(M_p)\Big[\tau_3 - \sqrt{M_p^4 - 1} \cdot \frac{\tau_3^2}{\sqrt{1-\tau_3^2}}\Big] - \sum_{k=2}^{p} \tau_3^{2k-1}\Big[K_1(M_{p-k+1}) + \sqrt{2M_{p-k+1}^3 - 2} \cdot \frac{\tau_3}{\sqrt{1-\tau_3^2}}\Big],$$

and $K_1(M_{p-k+1})$ is defined by (2), $\lambda_0(M_p)$ is defined by (5).

When $M_{p-k+1} = 1, k = 1, 2, \ldots, p$, the result is sharp.

Meanwhile, we establish three forms of Landau-type theorems for some log-p-harmonic mappings. Firstly, We establish one form of Landau-type theorems for certain log-p-harmonic mappings by applying the method of our proof of Theorem 3.4 in[20].

Theorem 3.9 Suppose $f(z) = \prod_{k=1}^{p} g_{p-k+1}(z)^{|z|^{2(k-1)}}$ is a log-p-harmonic mapping in the unit disk $\mathbb{U}$, with $f(0) = \lambda_f(0) = 0$, and satisfying
(i) for $k \in \{1, \cdots, p\}$, $g_{p-k+1}(z)$ is log-harmonic in $\mathbb{U}$, $g_{p-k+1}(0) = 1$,
(ii) let $G_{p-k+1} = \log g_{p-k+1}$, for $k \in \{2, \cdots, p\}$, $\lambda_{G_{p-k+1}}(0) - 1 = G_{p-k+1}(0) = 0$, and $|G_{p-k+1}(z)| \le M_{p-k+1}$, $\Lambda_{G_p}(z) \le \Lambda_p$ for all $z \in \mathbb{U}$.

Then for $k \in \{2, \cdots, p\}$, $M_{p-k+1} \ge 1, \Lambda_p \ge 1$, $f(z)$ is univalent in $\mathbb{U}_{r_2}$, where r_2 is the unique root in $(0,1)$ of the equation $A_1(r) = 0$, $A_1(r)$ is defined by (1). Moreover, the range $F(\mathbb{U}_{r_2})$ contains a univalent disk $\mathbb{U}(w_2, r_2')$, where R_2 is given by (3), and

$$w_2 = \cosh R_2, \quad r_2' = \sinh R_2. \tag{9}$$

When $M_{p-k+1} = 1, k = 2, \ldots, p$, these estimates are sharp with $r_2 = \widetilde{r_2}$, $r_2' = \sinh R_2 = \sinh \widetilde{R_2}$, where $\widetilde{r_2}$ is the unique root in $(0,1)$ of the equation

$$\frac{\Lambda_p(1-\Lambda_p r)}{\Lambda_p - r} - \sum_{k=2}^{p}(2k-1)r^{2(k-1)} = 0, \tag{10}$$

and

$$\widetilde{R_2} = \Lambda_p^2\widetilde{r_2} + (\Lambda_p^3 - \Lambda_p)\log(1 - \frac{\widetilde{r_2}}{\Lambda_p}) - \sum_{k=2}^{p} \widetilde{r_2}^{2k-1}. \tag{11}$$

Proof Let $F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$, for each $k \in \{1, 2, \cdots, p\}$.
Then it follows from the hypothesis of Theorem 3.9 and the definition of log-harmonic mappings that $G_{p-k+1}(z) = \log g_{p-k+1}(z)$ is harmonic mappings in $\mathbb{U}$ for each $k \in \{1, 2, \cdots, p\}$.
Thus $F = \log f$ is a polyharmonic mapping in $\mathbb{U}$.

We know that

$$\lambda_f(0) = \Big||f_z(0)| - |f_{\bar{z}}(0)|\Big| = |f(0)|\Big||F_z(0)| - |F_{\bar{z}}(0)|\Big|,$$

and $f(0) = 1$, so it follows from $g_p(0) = \lambda_f(0) = 1$, we have $G_p(0) = \lambda_F(0) - 1 = 0$.

In order to prove the univalence of f, we fix r with $0 < r < 1$ and choose two distinct points $z_1, z_2 \in \mathbb{U}_r$. Let $\Gamma = \{(z_1 - z_2)t + z_2 : 0 \le t \le 1\}$.
Then it follows from our proof of Theorem 3.4 and the hypothesis of Theorem 3.9 that

$$\begin{aligned} |\log f(z_1) - \log f(z_2)| &= |F(z_1) - F(z_2)| = \left|\int_\Gamma F_z(z)dz + F_{\overline{z}}(z)d\overline{z}\right| \\ &\ge |z_1 - z_2|\Big\{\frac{\Lambda_p(1-\Lambda_p r)}{\Lambda_p - r} - \sum_{k=2}^{p} r^{2(k-1)}\Big[(2k-1)K_1(M_{p-k+1}) \\ &\quad + \sqrt{2M_{p-k+1}^2 - 2}\Big(\frac{2(k-1)r}{\sqrt{1-r^2}} + \frac{r\sqrt{4-3r^2+r^4}}{(1-r^2)^{\frac{3}{2}}}\Big)\Big]\Big\} > 0. \end{aligned}$$

From the proof of Theorem 3.4, we know that there is a unique $r_2 \in (0, 1)$ satisfying the equation $A_1(r) = 0$, $A_1(r)$ is defined by (1), such that

$$|\log f(z_1) - \log f(z_2)| > 0$$

for any two distinct points z_1, z_2 in $|z| < r_2$, which shows that f is univalent in $\mathbb{U}_{r_2}$.

Next, for any point $z = r_2 e^{i\theta}$ on $\partial\mathbb{U}_{r_2}$, by our proof of Theorem 3.4, we have

$$\begin{aligned} |\log f(z)| &= |F(z)| = \left|G_p(z) + \sum_{k=2}^{p} |z|^{2(k-1)} G_{p-k+1}(z)\right| \\ &\ge \Lambda_p^2 r_2 + (\Lambda_p^3 - \Lambda_p)\log(1 - \frac{r_2}{\Lambda_p}) \\ &\quad - \sum_{k=2}^{p} r_2^{2(k-1)}\Big[K_1(M_{p-k+1})r_2 + \sqrt{2M_{p-k+1}^2 - 2}\cdot\frac{r_2^2}{\sqrt{1-r_2^2}}\Big] = R_2, \end{aligned}$$

where R_2 is given by (3).

By Lemma 2.8, we obtain that the range $f(U_{r_2})$ contains a schlicht disk $\mathbb{U}(w_2, r_2')$, where w_2 and r_2' are defined by (9).

Next, we prove that the univalent radius r_2 and $r_2' = \sinh R_2$ are sharp when $M_{p-k+1} = 1, k = 2, \ldots, p$, by means of the method as in the proof of Theorem 3.4 in [20]. For the convenience of readers, we give the detail of the proof.

Firstly, we consider the log-p harmonic mapping $f_3(z) = e^{F_3(z)}$, where $F_3(z)$ is given by (4). It is easy to verify that $f_3(z)$ satisfies the hypothesis of Theorem 3.9, thus we obtain that $f_3(z)$ is univalent in the disk U_{r_2}, and the range $f_3(U_{r_2})$ contains a univalent disk $\mathbb{U}(w_2, r_2')$.

To prove that the univalent radius r_2 is sharp with $r_2 = \widetilde{r_2}$, we need to prove that $f_3(z)$ is not univalent in U_r for each $r \in (\widetilde{r_2}, 1]$. In fact, if we fix $r \in (\widetilde{r_2}, 1]$, by our proof of Theorem 3.1, we know that $F_3(z)$ is is not univalent in U_r, thus there exist two distinct points $z_1, z_2 \in U_r$ such that $F_3(z_1) = F_3(z_2)$, which implies that $f_3(z_1) = e^{F_3(z_1)} = e^{F_3(z_2)} = f_3(z_2)$, that is $f_3(z)$ is not univalent in U_r for each $r \in (\widetilde{r_2}, 1]$. Hence, the univalent radius r_2 is sharp.

Next, we prove that the radius $r_2' = \sinh R_2$ is sharp with $R_2 = \widetilde{R_2}$.

For $r \in [0, 1]$, considering the continuous function

$$g_1(r) = \frac{\Lambda_p(1-\Lambda_p r)}{\Lambda_p - r} - \sum_{k=2}^{p}(2k-1)r^{2(k-1)},$$

it is easy to verify that $g_1(r)$ is strictly decreasing on $[0, 1]$, $g_1(0) = 1 > 0$ and

$$g_1(\frac{1}{\Lambda_p}) = -\sum_{k=2}^{p}(2k-1)(\frac{1}{\Lambda_p})^{2(k-1)} \le 0.$$

Thus we have $0 < \widetilde{r_2} \le \frac{1}{\Lambda_p}$.

By (10) and (11), it is easy to verify that $\widetilde{R_2} > 0$. Next we can prove $\widetilde{R_2} < 1$.

Let $h(r) = \Lambda_p^2 r + (\Lambda_p^3 - \Lambda_p)\log(1 - \frac{r}{\Lambda_p}), 0 < r \le \frac{1}{\Lambda_p}$, then

$$h'(r) = \Lambda_p^2 + \frac{1 - \Lambda_p^2}{1 - \frac{r}{\Lambda_p}} = \Lambda_p \frac{\frac{1}{\Lambda_p} - r}{1 - \frac{r}{\Lambda_p}} \ge 0, , 0 < r \le \frac{1}{\Lambda_p},$$

which implies that $h(r)$ is increasing in $(0, \frac{1}{\Lambda_p}]$. Therefore,

$$\begin{aligned} \widetilde{R_2} &= \Lambda_p^2 \widetilde{r_2} + (\Lambda_p^3 - \Lambda_p)\log(1 - \frac{\widetilde{r_2}}{\Lambda_p}) - \sum_{k=2}^{p} \widetilde{r_2}^{2k-1} \\ &\le h(\widetilde{r_2}) \le h(\frac{1}{\Lambda_p}) = \Lambda_p + (\Lambda_p^3 - \Lambda_p)\log(1 - \frac{1}{\Lambda_p^2}) \\ &< \Lambda_p + (\Lambda_p^3 - \Lambda_p) \cdot (-\frac{1}{\Lambda_p^2}) = \frac{1}{\Lambda_p} < 1. \end{aligned}$$

Hence, $0 < \widetilde{R_2} < 1$.

Because the univalent radius r_2 is sharp with $r_2 = \widetilde{r_2}$ when $M_{p-k+1} = 1, k = 2, \ldots, p$, the sharpness of the radius $r_2' = \sinh R_2 = \sinh \widetilde{R_2}$ follows from Lemma 2.8 and the fact that $0 < \widetilde{R_2} < 1$. The proof is complete. □

By means of Theorem 1 in [23] and the same method as the proof of Theorem 3.4 in [20], applying the same method as the proof of Theorem 3.9, it is not difficult to prove following Theorem.

Theorem 3.10 Suppose $f(z) = \prod_{k=1}^{p} g_{p-k+1}(z)^{|z|^{2(k-1)}}$ is a log-p-harmonic mapping in the unit disk $\mathbb{U}$, with $f(0) = \lambda_f(0) = 0$, and satisfying

(i) for $k \in \{1, \cdots, p\}$, $g_{p-k+1}(z)$ is log-harmonic in $\mathbb{U}$, $g_{p-k+1}(0) = 1$,

(ii) let $G_{p-k+1} = \log g_{p-k+1}$, for $k \in \{2, \cdots, p\}$, $|G_{p-k+1}(z)| \le M_{p-k+1}$, $\Lambda_{G_p}(z) \le \Lambda_p$ for all $z \in \mathbb{U}$.

Then for $k \in \{2, \cdots, p\}$, $M_{p-k+1} \ge 0, \Lambda_p \ge 1$, $f(z)$ is univalent in $\mathbb{U}_{\rho_1}$, where ρ_1 is the unique root in $(0, 1)$ of the equation which is defined by (1). Moreover, the range $F(\mathbb{U}_{\rho_1})$ contains a univalent disk $\mathbb{U}(w_1', \widetilde{\rho_1'})$, where ρ_1' is given by (2), and

$$w_1' = \cosh \rho_1', \ \ \widetilde{\rho_1'} = \sinh \rho_1'.$$

When $M_{p-k+1} = 0, k = 2, \ldots, p$, the radii ρ_1 and $\widetilde{\rho_1'} = \sinh \rho_1'$ are sharp.

By means of Theorem 3.5 and the same method as the proof of Theorem 3.2 and Theorem 3.5 in [20], applying the same method as the proof of Theorem 3.9, we have following Theorem.

Theorem 3.11 Suppose $f(z) = \prod_{k=1}^{p} g_{p-k+1}(z)^{|z|^{2(k-1)}}$ is a log-p-harmonic mapping in the unit disk $\mathbb{U}$, with $f(0) = \lambda_f(0) = 0$, and satisfying

(i) for $k \in \{1, \cdots, p\}$, $g_{p-k+1}(z)$ is log-harmonic in $\mathbb{U}$, $g_{p-k+1}(0) = 1$,

(ii) and let $G_{p-k+1} = \log g_{p-k+1}$, for $k \in \{2, \cdots, p\}$, $\Lambda_{G_{p-k+1}}(z) \le \Lambda_{p-k+1}$, $|G_p(z)| \le M_p$ for all $z \in \mathbb{U}$.

Then for $k \in \{2, \cdots, p\}$, $\Lambda_{p-k+1} \ge 0, M_p \ge 1$, $F(z)$ is univalent in $\mathbb{U}_{r_3}$, where r_3 is the unique positive root in $(0, 1)$ of the equation which is defined by (5). Moreover, the range $F(\mathbb{U}_{r_3})$ contains a univalent disk $\mathbb{U}(w_3, r_3')$, where R_3 is given by (6), and

$$w_3 = \cosh R_3, \ \ r_3' = \sinh R_3.$$

When $M_p = 1$, the radii r_3 and $r_3' = \sinh R_3$ are sharp.

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