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On some m-symmetric generalized hypergeometric d-orthogonal polynomials

Mongi Blel[a]

[a] *Department of Mathematics, College of Science,King Saud University, Riyadh 11451, BP 2455*

Abstract. In [9] I. Lamiri and M.Ouni state some characterization theorems for d-orthogonal polynomials of Hermite, Gould-Hopper and Charlier type polynomials. In [3] Y.Ben Cheikh I. Lamiri and M.Ouni give a characterization theorem for some classes of generalized hypergeometric polynomials containing for example, Gegenbauer polynomials, Gould-Hopper polynomials, Humbert polynomials, a generalization of Laguerre polynomials and a generalization of Charlier polynomials. In this work, we introduce a new class $\mathcal{D}$ of generalized hypergeometric m-symmetric polynomial sequence containing the Hermite polynomial sequence and Laguerre polynomial sequence. Then we consider a characterization problem consisting in finding the d-orthogonal polynomial sequences in the class $\mathcal{D}$, $m \le d$. The solution provides new d-orthogonal polynomial sequences to be classified in d-Askey-scheme and also having a m-symmetry property with $m \le d$. This class contains the Gould-Hopper polynomial sequence, the class considered by Ben Cheikh-Douak, the class considered in [3]. This class contains new d-orthogonal polynomial sequences not belonging to the class $\mathcal{A}$. We derive also in this work the d-dimensional functional vectors ensuring the d-orthogonality of these polynomials. We also give an explicit expression of the d-dimensional functional vector.

1. Introduction and Main Results

To state our problem, we need to recall the meaning of the three keywords of the title.
The generalized hypergeometric functions ${}_pF_q(z)$ with p numerator and q denominator parameters are defined by ([Luke], for instance)

$$ {}_pF_q\left(\begin{matrix} (a_p); \\ (b_q); \end{matrix} \; z \right) := \sum_{m=0}^{\infty} \frac{[a_p]_m}{[b_q]_m} \frac{z^m}{m!}, $$

where (a_p) designates the set $\{a_1, a_2, \dots, a_p\}$, $[a_r]_p = \prod_{i=1}^{r}(a_i)_p$ and $(a)_p = \frac{\Gamma(a+p)}{\Gamma(a)}$. z being a complex variable.

Definition 1.1.
A generalized hypergeometric function ${}_pF_q$ is reduced to a polynomial called generalized hypergeometric polynomial if r numerator parameters take the form: $\Delta(r,-n)$ where $\Delta(r,\alpha)$ abbreviates the array of r parameters: $\frac{\alpha+j-1}{r}$; $j = 1, \dots, r$.

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Researchers Supporting Project RSPD2023R753, King Saud University, Riyadh, Saudi Arabia.
Email address: mblel@ksu.edu.sa (Mongi Blel)

The notion of d-orthogonality was introduced by Van Iseghem [6] and completed by Maroni [10] as follows. Let $\mathscr{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and $\mathscr{P}'$ its dual. A polynomial sequence $\{P_n\}_{n\geq 0}$ in $\mathscr{P}$ is called a polynomial set (PS, for shorter) if $\deg P_n = n$ for all integer n. We denote by $\langle u, f\rangle$ the effect of the linear functional $u \in \mathscr{P}'$ on the polynomial $f \in \mathscr{P}$.

Definition 1.2.
Let d be a positive integer and let $\{P_n\}_{n\geq 0}$ be a PS in $\mathscr{P}$. $\{P_n\}_{n\geq 0}$ is called a d-orthogonal polynomial set (d-OPS, for shorter) with respect to the d-dimensional functional vector $\Gamma = {}^t(\Gamma_0, \Gamma_1, \ldots, \Gamma_{d-1})$ if it satisfies the following conditions:

$$\begin{cases} \langle \Gamma_k, P_m P_n\rangle = 0 \;,\; m > nd + k \,, n \geq 0, \\ \langle \Gamma_k, P_n P_{nd+k}\rangle \neq 0 \,,\; n \geq 0, \end{cases}$$

for each integer $k \in \{0, 1, \ldots, d-1\}$.

For the particular case: $d = 1$, we meet the well known notion of orthogonality [5].

Definition 1.3.
Let m be a positive integer. A PS $\{P_n\}_n$ is called m-symmetric if
$P_n(wx) = w^n P_n(x)$ for all n, where $w = \exp\left(\frac{2i\pi}{m+1}\right)$.

For the particular case: $m = 1$, we get the symmetric PS [5]. We refer the reader to [4] for more properties of m-symmetric $d-$OPS.
The Askey-scheme contains orthogonal polynomials having generalized hypergeometric representations. Recently, some works [Ben Cheikh, Douak, Lamiri, Ouni, Zaghouani,] provided some generalizations of these polynomials. That may be used to look for a similar table to Askey-scheme in the context of the d-orthogonality notion, for which, we refer by d-Askey-scheme.
Ben Cheikh, Lamiri and Ouni [3] defined a general class of hypergeometric polynomials $\mathcal{A}$ containing all OPSs in Askey-scheme and found all d-OPSs in $\mathcal{A}$. Among them, we meet d-OPSs generalizing in a natural way all the OPSs in Askey-scheme. The other ones are defined only for $d \geq 2$. The authors showed that the possibility to obtain further generalized hypergeometric d-OPSs not belonging to $\mathcal{A}$ by solving characterization problems for special classes of generalized hypergeometric polynomials suggested by suitable transformations of generalized hypergeometric functions. That was done for a class $\mathcal{B}$ containing Gegenbauer PS and a class $\mathcal{C}$ containing Charlier PS.
The only m-symmetric d-OPSs in $\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}$ are the Humbert PS and Gould-Hopper PS which are d-symmetric. That are reduced respectively to Gegenbauer PS and Hermite PS for $d = 1$.
In this work, we introduce a further class $\mathcal{D}$ of generalized hypergeometric m-symmetric PSs containing the Hermite PS and Laguerre PS. Then we consider a characterization problem consisting in finding all d-OPSs in $\mathcal{D}$, $m \leq d$. The solution provides new d-OPSs to be classified in d-Askey-scheme and also having a m-symmetry property with $m \leq d$.

This paper is organized as follows: In Section 2, we introduce a new class of $\mathcal{D}$ of generalized hypergeometric m-symmetric polynomial sequences and we prove the main result 2.1 which consists in characterizing all d-orthogonal polynomial sequences in the class $\mathcal{D}$. We prove that the only d-orthogonal polynomial sequences in this class are those considered by Ben Cheikh and Blel [1]. The last section is devoted to give the dual sequence of the sequence $\{P_n\}_n$ and the dual sequences of the components.

Our analysis is based on a recurrent relation of order d of type

$$xP_n(x) = \beta_{n+1}P_{n+1}(x) + \sum_{k=0}^{d} \alpha_{k,n-d+k}P_{n-d+k}(x),$$

where $\beta_{n+1}\alpha_{0,n-d} \neq 0$, which characterizes the $d-$orthogonality of the polynomial set $\{P_n\}_n$. In the literature there are many polynomial sets satisfying recurrent relations of height order not necessarily satisfying the $d-$orthogonality property. We cite for instance [8, 12, 14–16].
We refer the reader to [13], where the author treats extensively various aspects of the classical orthogonal polynomials, hypergeometric functions and hypergeometric polynomials.

2. Main Result

2.1. Introduction to the class $\mathcal{D}$

In [1], the authors introduce a family of generalized hypergeometric $m-$symmetric $d-$orthogonal polynomials, $\{P_n\}_n$ defined by:

$$P_n(x) := x^n {}_{(m+1)(r+1)+p}F_q \left(\begin{matrix} \Delta(m+1,-n), (\Delta(m+1,-n-v\alpha_r)); & \\ & \left(\frac{1}{x}\right)^{m+1} \\ ; & \end{matrix} \right) \tag{1}$$

This family gives a generalized hypergeometric representation of Hermite polynomials and Laguerre polynomials [7]. Indeed

$$H_n(x) = P_n(1, 0, -; x) \quad \text{and} \quad L_n^{(\alpha)}(x) = \frac{1}{n!} P_n(0, 1, \alpha; x),$$

where

$$H_n(x) = x^n {}_2F_0 \left(\begin{matrix} \frac{-n}{2}, \frac{-n+1}{2}, & \\ & -\frac{4}{x^2} \\ -, & \end{matrix} \right)$$

and

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} {}_1F_1 \left(\begin{matrix} -n, & \\ & x \\ \alpha+1, & \end{matrix} \right) = \frac{x^n}{n!} {}_2F_0 \left(\begin{matrix} -n, -n-\alpha, & \\ & -\frac{1}{x} \\ -, & \end{matrix} \right).$$

This suggests to introduce a new class of m-symmetric generalized hypergeometric polynomials defined as follows:
Let d be a positive integer and $I_d = \{(m, r) \in \mathbb{N}^2;\ (m+1)(r+1) = d+1\}$. For $(m, r) \in I_d$ we define the set $\mathcal{D}_{m,r}$ of polynomials defined by

$$P_n\left(m, r, p, q, (\alpha_r), (a_p), (b_q); x\right) := x^n {}_{(m+1)(r+1)+p}F_q \left(\begin{matrix} \Delta(m+1,-n), (\Delta(m+1,-n-\alpha_r)), (a_p); & \\ & \left(\frac{1}{x}\right)^{m+1} \\ (b_q); & \end{matrix} \right).$$

Set $\mathcal{D}^{(d)} = \bigcup_{(m,r)\in I_d} \mathcal{D}_{m,r}$.

If $d+1$ is a prime number $\mathcal{D}^{(d)} = \mathcal{D}_{d,0} \cup \mathcal{D}_{0,d}$. For example

$$\mathcal{D}^{(3)} = \mathcal{D}_{3,0} \cup \mathcal{D}_{1,1} \cup \mathcal{D}_{0,3}.$$

$$\mathcal{D}^{(5)} = \mathcal{D}_{5,0} \cup \mathcal{D}_{3,2} \cup \mathcal{D}_{2,3} \cup \mathcal{D}_{0,5}.$$

It is easy to verify that the class $\mathcal{D}$ contains the following PS $\{Q_n\}_{n\geq 0}$ defined by the generating function (cf [4]):

$$e^{t^{m+1}} {}_0F_r \left(\begin{matrix} - & \\ & -xt \\ \alpha_1+1, \ldots, \alpha_r+1 & \end{matrix} \right) = \sum_{n=0}^{\infty} Q_n(x) t^n.$$

In fact, we need the following identity [Srivastava-Manocha ([11]) p. 100]:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k),$$

to prove that

$$\begin{aligned} e^{t^{m+1}}{}_0F_r\left(\begin{matrix} - \\ \alpha_1+1,\ldots,\alpha_r+1 \end{matrix}\ -xt\right) &= \sum_{k=0}^{\infty}\frac{t^{k(m+1)}}{k!}\sum_{n=0}^{\infty}\frac{(-xt)^n}{(\alpha_1+1)_n\cdots(\alpha_r+1)_n n!} \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{[\frac{n}{m+1}]}\frac{(-x)^{n-(m+1)j}}{j!(n-(m+1)j)!(\alpha_1+1)_{n-(m+1)j}\cdots(\alpha_r+1)_{n-(m+1)j}}t^n. \end{aligned}$$

From which we deduce that

$$Q_n(x)=\sum_{j=0}^{[\frac{n}{m+1}]}\frac{(-x)^{n-(m+1)j}}{j!(n-(m+1)j)!\prod_{s=1}^{r}(1+\alpha_s)_{n-(m+1)j}}.$$

Using the Definition 1.1 and the following identities [See Srivastava-Manocha Book ([11]), p.22-23, for instance]:

$$(\lambda)_{mn}=m^{mn}\prod_{i=1}^{m}\left(\frac{\lambda+i-1}{m}\right)_n,\qquad \forall\, n\in\mathbb{N}\cup\{0\},$$

$$(\lambda)_n=(\lambda)_k(\lambda+k)_{n-k},\quad (n-k)!=\frac{(-1)^k n!}{(-n)_k},\qquad \forall\, 0\le k\le n$$

and $(-n)_k=0$, for $k>n$.

We obtain

$$Q_n(x)=C_n x^n{}_{(m+1)(r+1)}F_0\left(\begin{matrix}\Delta(m+1,-n),(\Delta(m+1,-n-\alpha_r))\,; \\ -;\end{matrix}\ \left(\frac{C}{x}\right)^{m+1}\right).$$

where

$$C_n=\frac{(-1)^n}{n!\prod_{i=1}^{r}(\alpha_i+1)_n}\quad\text{and}\quad C=(-1)^r(m+1)^{(r+1)}$$

Then

$$Q_n(x)=\frac{(-1)^{n(r+1)}(m+1)^{n(r+1)}}{n!\prod_{i=1}^{r}(\alpha_i+1)_n}P_n\left(\frac{(-1)^r x}{(m+1)^{r+1}}\right),\tag{2}$$

$$P_n(x)=\frac{(-1)^{n(r+1)}n!\prod_{i=1}^{r}(\alpha_i+1)_n}{(m+1)^{n(r+1)}}Q_n((-1)^r(m+1)^{r+1}x).\tag{3}$$

2.2. Characterization Theorem

Ben Cheikh and Douak [2] showed that the PS $\{Q_n\}_{n\geq 0}$ is $((m+1)(r+1)-1)$-orthogonal. It contains the particular cases of the Gould-Hopper PS ($r=0$) and Ben Cheikh-Douak PS ($m=0$) which also belongs to the class $\mathcal{A}$ considered in [3]. It appears that the class $\mathcal{D}$ contains new d-OPSs not belonging to $\mathcal{A}$. It is significant to put the following problem: find all $d-$OPS in $\mathcal{D}_{m,r}$. The case $\mathcal{D}_{d,0}$ was considered by Lamiri Ouni and the case $\mathcal{D}_{0,d}$ was considered by Ben Cheikh, Douak and Ben Cheikh, Ouni. We state the following

Theorem 2.1.
The only $d-$OPs in $\mathcal{D}_{m,r}$ are the polynomials (1).

Proof
Let $\{P_n\}_n$ be a sequence in the class $\mathcal{D}$,

$$\begin{aligned} P_n(x) &:= x^n{}_{(m+1)(r+1)+p}F_q\left(\begin{matrix}\Delta(m+1,-n),(\Delta(m+1,-n-\alpha_r)),(a_p); & \\ & \left(\frac{1}{x}\right)^{m+1} \\ (b_q); & \end{matrix}\right) \\ &= \sum_{k=0}^{[\frac{n}{m+1}]}\gamma_n(k)x^{n-k(m+1)}. \end{aligned} \tag{4}$$

The sequence $\{P_n\}_n$ is $m-$symmetric, then according to the notations in [4], if the sequence is d-OPS, then they satisfy a $d+1-$order recurrence relation of type:

$$XP_n = \sum_{j=0}^{d_1}\alpha_j(n)P_{n-j(m+1)+1},$$

with $\alpha_{d_1}(n)\neq 0$, $\alpha_0(n)=1$ and $d+1=d_1(m+1)$. Then

$$\begin{aligned} \sum_{k=0}^{[\frac{n}{m+1}]}\gamma_n(k)x^{n+1-k(m+1)} &= \sum_{j=0}^{d_1}\alpha_j(n)\sum_{k=0}^{[\frac{n+1}{m+1}]-j}\gamma_{n+1-j(m+1)}(k)x^{n+1-(j+k)(m+1)} \\ &= \sum_{j=0}^{d_1}\alpha_j(n)\sum_{k=j}^{[\frac{n+1}{m+1}]}\gamma_{n+1-j(m+1)}(k-j)x^{n+1-k(m+1)} \\ &= \sum_{k=0}^{[\frac{n+1}{m+1}]}\left(\sum_{j=0}^{\inf(k,d_1)}\alpha_j(n)\gamma_{n+1-j(m+1)}(k-j)\right)x^{n+1-k(m+1)} \end{aligned}$$

It follows that if $n+1=\ell(m+1)$,

$$\sum_{j=0}^{\inf(\ell,d_1)}\alpha_j(n)\gamma_{n+1-j(m+1)}(\ell-j)=0.$$

In what follows, we assume that $n+1\not\equiv 0[m+1]$, then

$$\gamma_n(k)=\sum_{j=0}^{\inf(k,d_1)}\alpha_j(n)\gamma_{n+1-j(m+1)}(k-j). \tag{5}$$

We derive from (4) that

$$\begin{aligned}\gamma_n(k) &= \frac{[a_p]_k \Delta(m+1,-n)_k \prod_{j=1}^{r} \Delta(m+1,-n-\alpha_j)_k}{k![b_q]_k} \qquad (6)\\ &= \frac{[a_p]_{k-d_1}[a_p+k-d_1]_{d_1} n! \prod_{s=1}^{r}(1+\alpha_s)_n}{[b_q]_{k-d_1}[b_q+k-d_1]_{d_1}(k-d_1)!(k-d_1+1)_{d_1}} \\ &\cdot \frac{(-1)^{k(r+1)(m+1)}}{(m+1)^{k(r+1)(m+1)}(n-k(m+1))! \prod_{s=1}^{r}(1+\alpha_s)_{n-k(m+1)}}\end{aligned}$$

and

$$\begin{aligned}\gamma_{n+1-j(m+1)}(k-j) &= \frac{[a_p]_{k-d_1}[a_p+k-d_1]_{d_1-j}}{[b_q]_{k-d_1}[b_q+k-d_1]_{d_1-j}(k-d_1)!(k-d_1+1)_{d_1-j}} \qquad (7)\\ &\cdot \frac{(-1)^{(k-j)(r+1)(m+1)}(n+1-j(m+1))! \prod_{s=1}^{r}(1+\alpha_s)_{n+1-j(m+1)}}{(n+1-k(m+1))! \prod_{s=1}^{r}(1+\alpha_s)_{n+1-k(m+1)}(m+1)^{(k-j)(r+1)(m+1)}}\end{aligned}$$

In what follows, we take k and n large enough, we derive from the relations (5), (6) and (7) that

$$\begin{aligned}&\frac{[a_p+k-d_1]_{d_1}(n+1+-k(m+1))\prod_{s=1}^{r}(n+1+\alpha_s-k(m+1))}{[b_q+k-d_1]_{d_1}(k-d_1+1)_{d_1}(m+1)^{d_1(r+1)(m+1)}}.(-1)^{d_1(r+1)(m+1)}n!\prod_{s=1}^{r}(1+\alpha_s)_n\\ &= \sum_{j=0}^{d_1}\alpha_j(n)\frac{[a_p+k-d_1]_{d_1-j}(-1)^{(d_1-j)(r+1)(m+1)}(n+1-j(m+1))!}{[b_q+k-d_1]_{d_1-j}(k-d_1+1)_{d_1-j}(m+1)^{(d_1-j)(r+1)(m+1)}}.\prod_{s=1}^{r}(1+\alpha_s)_{n+1-j(m+1)}\end{aligned}$$

It follows that

$$\begin{aligned}&[a_p+k-d_1]_{d_1}n!(n+1-k(m+1))\prod_{s=1}^{r}(n+1+\alpha_s-k(m+1))\prod_{s=1}^{r}(1+\alpha_s)_n\\ &= \sum_{j=0}^{d_1}\alpha_j(n)[a_p+k-d_1]_{d_1-j}[b_q+k-j]_j(k-j+1)_j(-1)^{j(r+1)(m+1)}.\prod_{s=0}^{r}(1+\alpha_s)_{n+1-j(m+1)}(m+1)^{j(r+1)(m+1)}\end{aligned}$$

Let

$$R(x) = [a_p+x-d_1]_{d_1}(n+1-x(m+1)n!\prod_{s=1}^{r}(1+\alpha_s)_n\prod_{s=1}^{r}(n+1+\alpha_s-x(m+1))$$

and $S(x) = \sum_{j=0}^{d_1}\alpha_j(n)[a_p+x-d_1]_{d_1-j}[b_q+x-j]_j(x-j+1)_j A(j,n)$, with

$A(j,n) = (-m-1)^{j(r+1)(m+1)}(n+1-j(m+1))!\prod_{s=1}^{r}(1+\alpha_s)_{n+1-j(m+1)}$.

If we take n and k large enough, we find that the polynomials R and S are equal. Furthermore $\deg R = d_1p+1+r$ and $\deg S \le \max_{0\le j\le d_1}(p(d_1-j)+jq+j) = \max_{0\le j\le d_1}(pd_1+j(q+1-p))$.

- If $q+1-p\le 0$, then $\deg R = d_1p+1+r$ and $\deg S \le pd_1$, which is impossible. It results that $q \ge p$.
- If $p \ne 0$, then $R(d_1-a_1)=0$ and $S(d_1-a_1)\ne 0$. Hence $p=0$.
- Since $\alpha_{d_1}(n)\ne 0$, we get $\deg S = d_1(q+1)$, hence $d_1(q+1)=1+r$.

We deduce that

$$R(x) = n!(n+1-x(m+1))\prod_{s=1}^{r}(1+\alpha_s)_n\prod_{s=1}^{r}(n+1+\alpha_s-x(m+1))$$

and

$$S(x) = \sum_{j=0}^{d_1}\alpha_j(n)A(j,n)[b_q+x-j]_j(x-j+1)_j.$$

We denote $\alpha_0 = 0$, then for $x = 0$, $R(0) = \prod_{s=0}^{r}(1+\alpha_s)_{n+1}$ and $S(0) = \alpha_0(n)\prod_{s=0}^{r}(1+\alpha_s)_{n+1}$, then $\alpha_0(n) = 1$. Furthermore

$$\begin{aligned} R(x) - R(0) &= \prod_{s=0}^{r}(1+\alpha_s)_n\left(\prod_{s=0}^{r}(n+1+\alpha_s-x(m+1)) - \prod_{s=0}^{r}(n+1+\alpha_s)\right) \\ &= \sum_{j=1}^{d_1}\alpha_j(n)[b_q+x-j]_j(x-j+1)_jA(j,n) \end{aligned}$$

If $q \geq 1$ and there exists $b_j \neq 1$, we can suppose that $j = 1$ and we take $x = 1 - b_1$, then from the relation $(1-j)_j = 0$, for all $j \geq 1$. Since $\alpha_{d_1}(n) \neq 0$, we deduce that

$$\prod_{s=0}^{r}(n+1+\alpha_s-(1-b_1)(m+1)) = \prod_{s=0}^{r}(n+1+\alpha_s).$$

We define the two monic polynomials R_1 and R_2 of degree $r+1$ by:

$$R_1(z) = \prod_{s=0}^{r}(z+\alpha_s-(1-b_1)(m+1)), \qquad R_2(z) = \prod_{s=0}^{r}(z+\alpha_s).$$

These two polynomials are equal for $z = n \in \mathbb{N}$ large enough, and $\deg R_1 = \deg R_2 = r+1$, this is impossible.

If $b_j = 1$, $1 \leq j \leq q$, then 0 is a root of the polynomial S_1 of order at least $q+1$.

$$S_1(x) = \sum_{j=1}^{d_1}\alpha_j(n)(x-j+1)_j^{q+1}A(j,n) = R(x) - R(0).$$

On the other hand $\dfrac{R'(x)}{R(x)} = \sum_{j=0}^{r}\dfrac{1}{n+1+\alpha_s-x(m+1)}$, which is non zero for n large enough near 0. Then $q = 0$ and the polynomial P_n is equal to

$$P_n(x) = x^n{}_{(m+1)(r+1)}F_0\left(\begin{matrix}\Delta(m+1,-n),(\Delta(m+1,-n-\alpha_r)); & \left(\frac{1}{x}\right)^{m+1} \\ -; & \end{matrix}\right).$$

The sequence $\{P_n\}_n$ is m−symmetric d−OPS and classical. □

Remark 2.2.

1. *Let $O_{m,r}$ be the set of d−OPSs in $\mathcal{D}_{m,r}$. If $\{P_n^{(k)}\}_{n\geq 0}$ are the components of $\{P_n\}_{n\geq 0}$, $0 \leq k \leq m$, the polynomials $\{P_n\}_{n\geq 0}$ are in $O_{0,d}$.*
2. *If the polynomials $\{P_n\}_{n\geq 0}$ are in $O_{m,r}$, then the polynomials $\{P'_{n+1}\}_{n\geq 0}$ are in $O_{m,r}$.*
3. *We deduce then if $\{P_n\}_{n\geq 0}$ is in $O^{(d)} = \cup_{(m,r)\in I_d}O_{m,r}$, then $\{P_n\}_{n\geq 0}$ is classical d−OPS and all the components $\{P_n^{(k)}\}_{n\geq 0}$, $0 \leq k \leq m$ are d−OPS.*

3. Properties of the Obtained Polynomials

3.1. Some Properties of m-Symmetric PSs

In this section we state the dual sequence of the sequence $\{P_n\}_n$ and since this sequence is $m-$symmetric, we derive the dual sequences of the components. First we recall some properties of $m-$symmetric PSs. (We refer the reader to [4] for more details.)

Definition 3.1.
Let m be a nonnegative integer. A PS $\{P_n\}_n$ is called m-symmetric if
$P_n(wx) = w^n P_n(x)$ *for all n, where* $w = e^{\frac{2i\pi}{m+1}}$ *a* $(m+1)$*-root of the unity.*

For the particular case: $m = 1$, we meet the well known notion of symmetric PS [5]. A characteristic property of m-symmetric PS is given by the following.

Proposition 3.2.
A PS $\{P_n\}_{n\geq 0}$ is m-symmetric if and only if there exist $(m+1)$ PSs $\{P_n^k\}_{n\geq 0}$; $k = 0, 1, \ldots, m$; such that

$$P_{(m+1)n+k}(x) \; = \; x^k \, P_n^k(x^{m+1}); \qquad n \geq 0.$$

Definition 3.3.

1. *For any PS $\{P_n\}_n$ there exists a sequence of linear functionals $(\mathscr{L}_n)_n$ defined by: $\langle \mathscr{L}_n, P_m\rangle = \delta_{n,m}$ called the dual sequence of the sequence $\{P_n\}_n$.*
2. *For $L \in \mathscr{P}'$ and $q \in \mathbb{N}$, we define the linear functional $X^q L$ by: $\langle X^q L, Q\rangle = \langle L, X^q Q\rangle$, $Q \in \mathscr{P}$, where $(X^q Q)(x) = x^q Q(x)$, the multiplication operator by x^q is $\mathscr{P}$.*
3. *If $L \in \mathscr{P}'$, we define $(\tau_w)_* L \in \mathscr{P}'$ by:*

$$\langle (\tau_w)_* L, P\rangle = \langle L, (\tau_w)^* P\rangle, \quad P \in \mathscr{P}$$

   with $(\tau_w)^* P(x) = P(wx)$ and $w = e^{\frac{2i\pi}{m+1}}$.
4. *A linear functional sequence $\mathscr{L} = \{\mathscr{L}_n\}_n$ is called m-symmetric if*

$$\langle \mathscr{L}_k, x^j\rangle = 0,$$

   for all integers j and k such that $k \not\equiv j[m+1]$.

A characterization of $m-$symmetric linear functional sequence and $m-$symmetric PS is given by the following.

Proposition 3.4.
Let $\mathscr{L} = \{\mathscr{L}_n\}_n$ be a linear functional sequence. $\mathscr{L}$ is m-symmetric if and only if

$$(\tau_w)_* \mathscr{L}_k = w^k \mathscr{L}_k, \quad \forall\, k \geq 0.$$

Let $\{P_n\}_n$ be a PS and let $\mathscr{L} = \{\mathscr{L}_n\}_n$ be its dual sequence. Then the following properties are equivalent:

1. *The sequence $\{P_n\}_n$ is m-symmetric.*
2. *The linear functional sequence $\mathscr{L}$ is m-symmetric.*

Our purpose now is to derive the dual sequence of the components of an $m-$symmetric PS. First, we give the following notations:
Let q be a nonnegative integer and let $\mathscr{L} \in \mathscr{P}'$ be a linear functional. Put
• $(\sigma_q)^* : \mathscr{P} \to \mathscr{P}$ the linear mapping defined by: $(\sigma_q)^*(P)(x) = P(x^q)$, $P \in \mathscr{P}$.
• $(\sigma_q)_* \mathscr{L}$ the linear mapping defined by: $\langle (\sigma_q)_* \mathscr{L}, Q\rangle = \langle \mathscr{L}, (\sigma_q)^* Q\rangle$, $Q \in \mathscr{P}$.

Proposition 3.5.
Let $\{P_n\}_n$ be a m-symmetric PSs, and let $\{P_n^k\}_n$; $k = 0, 1, \ldots, m$ be its components. If $\{\mathscr{L}_n\}_{n\geq 0}$ is the dual sequence of $\{P_n\}_n$, then $\{\mathscr{L}_{n,k}\}_{n\geq 0}$, the dual sequence of $\{P_n^k\}_n$, $0 \leq k \leq m$ is given by:

$$\mathscr{L}_{n,k} = (\sigma_{m+1})_* \left(X^k \mathscr{L}_{n(m+1)+k} \right).$$

3.2. Components of the sequence $\{Q_n\}_n$

We recall the generator function of the sequence $\{Q_n\}_n$ is given by:

$$e^{t^{m+1}}{}_0F_r\begin{pmatrix} - & \\ 1+\alpha_1,\dots,1+\alpha_r & ;-xt\end{pmatrix} = \sum_{n=0}^{\infty} Q_n(x)t^n \tag{8}$$

The sequence $\{Q_n\}_n$ is classical $m-$symmetric $((m+1)(r+1)-1)$-orthogonal. Moreover the corresponding $m+1$ components are also classical $((m+1)(r+1)-1)$-orthogonal.

We recall the identity 2, Problem 7, page 213 in [11]

$$\begin{aligned}{}_pF_q\begin{pmatrix} a_1, & \dots & ,a_p & \\ b_1, & \dots & ,b_q & ;x\end{pmatrix} &= \sum_{k=0}^{m} \frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{x^k}{k!} \\ &\quad {}_{(m+1)p}F_{(m+1)q+m}\begin{pmatrix} (\Delta(m+1,a_p+k)) & \\ \Delta^*(m+1,k+1), \quad (\Delta(m+1,b_q+k)) & ;z\end{pmatrix},\end{aligned} \tag{9}$$

where $z=\dfrac{x^{m+1}}{(m+1)^{(1-p+q)(m+1)}}$, $\Delta(m+1,\lambda)$ the array of $m+1$ parameters

$$\frac{\lambda}{m+1}, \frac{\lambda+1}{m+1},\dots,\frac{\lambda+m}{m+1}$$

and $\Delta^*(m+1,k+1)$ is the array of only m parameters

$$\frac{k+1}{m+1}, \frac{k+2}{m+1},\dots,\frac{k+m+1}{m+1}, \quad 0\le k\le m$$

where we omit the therm $\dfrac{m+1}{m+1}$.

We prove that

$$\Delta(m+1,\lambda)_j = \frac{(\lambda)_{j(m+1)}}{(m+1)^{j(m+1)}},$$

$$\Delta^*(m+1,k)_j = \frac{(k)_{j(m+1)}}{j!(m+1)^{jm}}.$$

For $z=\dfrac{(-xt)^{m+1}}{(m+1)^{(1+r)(m+1)}}$,

$${}_0F_{(m+1)r+m}\begin{pmatrix} - & \\ \Delta^*(m+1,k+1), \quad (\Delta(m+1,\alpha_r+k+1)) & ;z\end{pmatrix} = \sum_{n=0}^{+\infty} \frac{(-xt)^{n(m+1)}}{(m+1)^n(k+1)_{n(m+1)}\prod_{s=1}^r(\alpha_s+k+1)_{n(m+1)}}.$$

It results from (9) that

$$e^{t^{m+1}}{}_0F_r\begin{pmatrix} - & \\ 1+\alpha_1,\dots,1+\alpha_r & ;-xt\end{pmatrix} = \sum_{n=0}^{+\infty}\sum_{k=0}^{m} x^k\left(\sum_{j=0}^{n}\frac{(-1)^{k+j(m+1)}x^{j(m+1)}}{\prod_{s=1}^r(\alpha_s+1)_{j(m+1)+k}(m+1)^j(k+j(m+1))!}\right)t^{n(m+1)+k}$$

Then the k^{th} component of the polynomial Q_n is

$$Q_{k,n}(x)=\sum_{j=0}^{n}\frac{(-1)^{k+j(m+1)}x^{j}}{\prod_{s=1}^{r}(\alpha_s+1)_{j(m+1)+k}(m+1)^{j}(k+j(m+1))!}.$$

We recall that $x^kQ_{k,n}(x^{m+1})=Q_{n(m+1)+k}(x)$.

From the relations (2) and (3)

$$P_{k,n}(x)=\sum_{j=0}^{n}\frac{(-1)^{(r+1)(k+j(m+1))}(m+1)^{r(k+j(m+1))}x^{j}}{\prod_{s=1}^{r}(\alpha_s+1)_{j(m+1)+k}(m+1)^{j}(k+j(m+1))!}.$$

3.3. Dual sequence of $\{P_n\}_n$

Now we give the dual sequence of the sequence $\{P_n\}_n$ and for its components.

From the relation (8), we derive that

$${}_0F_r\left(\begin{matrix}-\\ 1+\alpha_1,\ldots,1+\alpha_r\end{matrix};-xt\right)=e^{-t^{m+1}}\sum_{n=0}^{\infty}Q_n(x)t^n$$

Then

$$\sum_{n=0}^{+\infty}\frac{(-xt)^n}{n!\prod_{s=1}^{r}(\alpha_s+1)_n}=\sum_{n=0}^{+\infty}\frac{(-t^{(m+1)})^n}{n!}\sum_{n=0}^{\infty}Q_n(x)t^n$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{[\frac{n}{m+1}]}\frac{(-1)^kQ_{n-k(m+1)}(x)}{(n-k)!k(m+1)!}\right)t^n$$

Then

$$x^n=\sum_{k=0}^{[\frac{n}{m+1}]}\frac{(-1)^{n+k}\prod_{s=1}^{r}(1+\alpha_s)_n}{(n-k)!k(m+1)!}Q_{n-k(m+1)}(x).$$

and

$$x^n=\sum_{k=0}^{[\frac{n}{m+1}]}\frac{(-1)^{n+k(r(m+1)+1)}\prod_{s=1}^{r}(1+\alpha_s)_n}{(m+1)^{kr(m+1)}(n-k)!k(m+1)!\prod_{s=1}^{r}(1+\alpha_s)_{n-k(m+1)}}P_{n-k(m+1)}(x).$$

The dual sequence of the sequence $\{P_n\}$ is given by:

$$\langle\mathscr{L}_j,x^n\rangle=\sum_{k=0}^{[\frac{n}{m+1}]}\frac{(-1)^{n+k(r(m+1)+1)}\prod_{s=1}^{r}(1+\alpha_s)_n}{(m+1)^{kr(m+1)}(n-k)!k(m+1)!\prod_{s=1}^{r}(1+\alpha_s)_{n-k(m+1)}}\delta_{j,n-k(m+1)}.$$

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