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# Characterizations of SEP elements in a ring with involution

**Mengge Guan[a], Xinran Wang[a], Junchao Wei[a]**

[a] *School of Mathematical Sciences, Yangzhou University, Yangzhou, Jiangsu 225002, P.R.China*

**Abstract.** In this paper, we mainly give characterizations of SEP elements in terms of equations. In addition, some conditions involving powers of group and Moore-Penrose inverse are proposed to characterize SEP elements. Finally, we construct univariate equations, use the consistency of the equations and the solutions to the equations to characterize SEP elements.

## 1. Introduction

Let $R$ be an associative ring with unit 1. An involution $a \mapsto a^*$ in a ring $R$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a,\ (a+b)^* = a^* + b^*,\ (ab)^* = b^*a^* \text{ for } a,\ b \in R.$$

$R$ is called a $*$-ring if $R$ is a ring with involution $*$. In what follows, $R$ is a $*$-ring.

In 1958, Drazin proposed the Drazin inverse [2], that is, when $a \in R$, there exists $x \in R$ such that the following three equations hold:

$$xax = x,\ ax = xa,\ a^k = a^{k+1}x \text{ for some } k \geqslant 1.$$

The element $x$ above is unique if exists and is denoted by $a^D$. The least such $k$ is called the index of $a$, and denoted by ind($a$). In particular, when ind($a$)=1, the Drazin inverse $a^D$ is called the group inverse of $a$ [1] and it is denoted by $a^\#$. The set of all group invertible elements of $R$ is denoted by $R^\#$.

An element $a \in R$ is Moore-Penrose invertible if there exists $x \in R$ such that the following four equations hold:

$$a = axa,\ x = xax,\ (ax)^* = ax,\ (xa)^* = xa.$$

Such an $x$ is uniquely determined Moore-Penrose inverse (or MP-inverse) of $a$ [9], denoted by $x = a^+$. The set of all Moore-Penrese invertible elements of $R$ will be denoted by $R^+$.

Let $a, x \in R$. If

$$axa = a;\ xR = aR;\ Rx = Ra^*,$$

then $x$ is called a core inverse of $a$ and if such an element $x$ exists, then it is unique and denoted by $a^{\textcircled{\#}}$. The set of all core invertible elements in $R$ will be denoted by $R^{\textcircled{\#}}$ [12]. Xu, Chen and Zhang [13] characterized

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*Email addresses:* 2530374647@qq.com (Mengge Guan), 1092078512@qq.com (Xinran Wang), jcweiyz@126.com (Junchao Wei)

core invertible elements in $*$-rings by there equations. Let $a, x \in R$, then $a \in R^{\textcircled{\#}}$ and $a^{\textcircled{\#}} = x$ if and only if $a = xa^2, ax^2 = x$ and $(ax)^* = ax$. In particular, if $a \in R^\# \cap R^+$, then $a \in R^{\textcircled{\#}}$ and $a^{\textcircled{\#}} = a^\# aa^+$.

An element $a \in R$ is said to be EP if and only if $a \in R^\# \cap R^+$ and $a^\# = a^+$. Many authors have published papers on EP elements, see [3, 4, 6, 8, 10, 11] for example. In particular, Wang, Mosić and Gao [8] said that $a \in R$ is an EP element if and only if there exists $x \in R$ such that

$$a = axa\ ,\ (ax)^* = ax = xa.$$

We use the notation $R^{EP}$ to denote the set of all EP elements in $R$.

An element $a \in R$ satisfying $aa^*a = a$ is called a partial isometry. Some properties and equivalent characterizations of partial isometry elements are given in [15, 17]. The set of all partial isometry elements of $R$ is denoted by $R^{PI}$. We have that $a \in R$ is a partial isometry if and only if $a \in R^+$ and $a^* = a^+$ [10].

If $a \in R^\# \cap R^+$, and $a^\# = a^+ = a^*$, then $a$ is called a strongly EP (for short SEP) element [14, 15]. We use the notation $R^{SEP}$ to denote all the SEP elements in $R$. Moreover, $a \in R$ is a SEP element if and only if $a$ is a partial isometry and EP. Mosić and Djordjević characterized SEP elements in $*$-rings by some equivalent conditions, see [5, 7]. Recently, Zhao, Wang and Wei [15], Zhao and Wei [16] by using solutions of certain equations, some characterizations of SEP elements in a ring with involution are discussed.

Motivated by these results, this paper is intended to provide further equivalent conditions for an element to be SEP.

## 2. Using equations to characterize SEP elements

In this section, we give new characterizations of SEP elements in terms of equations. We begin with some auxiliary theorems.

**Theorem 2.1.** *[8, Theorem 2.9] Let $R$ be a $*-$ring and $a \in R$. Then $a \in R^{EP}$ if and only if there exists $x \in R$ such that*

$$a = axa;\ (ax)^* = ax = xa.$$

**Theorem 2.2.** *[7, Theorem 1.5.3] Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $aa^\# = aa^*$ (or $a^\# a = a^* a$).*

**Theorem 2.3.** *[4] Let $R$ be a ring. Then $a \in R^\#$ if and only if $a \in a^2R \cap Ra^2$.*

Next, we will provide new characterizations of SEP elements.

**Theorem 2.4.** *Let $a \in R$. Then $a \in R^{SEP}$ if and only if there exists $x \in R$ such that*

$$a = axa;\ (ax)^* = xa = a^*a.$$

*Proof.* " $\Rightarrow$ " Since $a \in R^{SEP}$, $a^\# = a^+ = a^*$. Choose $x = a^\# = a^+ = a^*$. Then we are done.

" $\Leftarrow$ " From the assumption, we have $ax = (a^*a)^* = a^*a = (ax)^* = xa$. Hence, by Theorem 2.1, we have $a \in R^{EP}$ and $a = axa = aa^*a$, it follows that $a \in R^{PI}$. Hence $a \in R^{SEP}$. □

We find that this theorem can be simplified to the following corollary.

**Corollary 2.5.** *Let $a \in R$. Then $a \in R^{SEP}$ if and only if there exists $x \in R$ such that*

$$a = axa;\ ax = xa = a^*a.$$

In Corollary 2.5, the condition $ax = xa$ implies that $a = axa = xa^2 = a^2x$. From Theorem 2.3, it follows that the condition $a \in R$ can be replaced by $a \in R^\#$. Therefore we get the following theorem.

**Theorem 2.6.** *Let $a \in R^\#$. Then $a \in R^{SEP}$ if and only if there exists $x \in R$ such that*

$$a = axa;\ ax = a^*a.$$

*Proof.* " $\Rightarrow$ " It is clear. Indeed, we only have to choose $x = a^{\#}$.

" $\Leftarrow$ " From the assumption, we have $a = axa = a^*aa$. Since $a \in R^{\#}$, $a^{\#}a = aa^{\#} = a^*a^2a^{\#} = a^*a$. Hence $a \in R^{SEP}$. □

Consider the following question, there exists $x \in R$ such that $a = axa$ and $xa = a^*a \overset{?}{\Longrightarrow} a \in R^{SEP}$.

**Example 2.7.** *Let $R = M_3(Z_2)$, and set the involution of $R$ as the transpose of matrices. Take* $a = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Then* $a^{\#} = a$ *and* $a^+ = a^* = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. *Choose $x = a^+ = a^*$. Then $a = axa$ and $xa = a^*a$. But we can check that $a^* \neq a^{\#}$, which implies that $a$ is not SEP.*

Similarly, we can obtain the following results.

**Corollary 2.8.** *Let $a \in R^{\#}$. Then $a \in R^{SEP}$ if and only if there exists $x \in R$ such that*

$$a = axa;\ xa = aa^*.$$

**Theorem 2.9.** *Let $a \in R$. Then $a \in R^{SEP}$ if and only if there exists $x \in R$ such that*

$$a = axa;\ ax = a^*a;\ xa = aa^*.$$

*Proof.* " $\Rightarrow$ " It is obvious by Corollary 2.5.

" $\Leftarrow$ "

$$a = axa = (a^*a)a = a^*a^2;$$

$$a = axa = a(aa^*) = a^2a^*.$$

Then $a \in R^{\#}$. Thus $a \in R^{SEP}$ by Theorem 2.6. □

**Theorem 2.10.** *Let $a \in R$. Then $a \in R^{SEP}$ if and only if there exists $x \in R$ such that*

$$a = a^2x = axa;\ ax = a^*a.$$

*Proof.* " $\Rightarrow$ " It is evident.

" $\Leftarrow$ " Since $a = axa = (a^*a)a = a^*a^2$ and $a = a^2x$. Then $a \in R^{\#}$. Thus $a \in R^{SEP}$ by Theorem 2.6. □

**Corollary 2.11.** *Let $a \in R$. Then $a \in R^{SEP}$ if and only if there exists $x \in R$ such that*

$$a = xa^2 = axa;\ xa = aa^*.$$

## 3. Using equivalent conditions to characterize SEP elements

In this section, SEP elements are characterized by conditions involving powers of their group and Moore-Penrose inverse. We use $Z^+$ to denote the set of positive integers.

**Lemma 3.1.** *[7, Theorem 1.2.2] Let $a \in R^{\#} \cap R^+$ and $n \in Z^+$. Then $a \in R^{EP}$ if and only if $(a^*)^n aa^{\#} = (a^*)^n$.*

**Theorem 3.2.** *Let $a \in R^{\#} \cap R^+$ and $2 \leq n \in Z^+$. Then $a \in R^{SEP}$ if and only if $(a^*)^{n+k}aa^{\#} = (a^+)^{n+k}, k = 0, 1$.*

*Proof.* " $\Rightarrow$ " It is an immediate result of Lemma 3.1.
" $\Leftarrow$ " From the assumption, we obtain

$$(a^*)^n aa^\# = (a^+)^n = (a^+)^n aa^+ = (a^*)^n aa^\# aa^+ = (a^*)^n aa^+ = (a^*)^n.$$

Then $a \in R^{EP}$ by Lemma 3.1. Now

$$(a^+)^{n+k} = (a^*)^{n+k} aa^\# = (a^*)^{n+k} aa^+ = (a^*)^{n+k},\ k = 0, 1.$$

$$(a^\#)^n = (a^+)^n = (a^*)^n = (a^*)^{n+1}(a^\#)^* = (a^+)^{n+1}(a^\#)^* = (a^\#)^{n+1}(a^\#)^*.$$

$$a = a^{n+1}(a^\#)^n = a^{n+1}(a^\#)^{n+1}(a^\#)^* = aa^\#(a^\#)^* = aa^\#(a^+)^* = (a^+)^* = (a^\#)^*.$$

Hence $a \in R^{SEP}$ by [7, Theorem 1.5.3]. □

From Lemma 3.1 and Theorem 3.2, we can obtain the following result.

**Theorem 3.3.** *Let $a \in R^\# \cap R^+$ and $2 \leq n \in Z^+$. Then $a \in R^{SEP}$ if and only if $a^*(a^\#)^{n-1}a^+ = a^\#(a^+)^n$.*

*Proof.* " $\Rightarrow$ " Since $a \in R^{SEP}$, $a^* = a^\# = a^+$, this gives $a^*(a^\#)^{n-1}a^+ = a^\#(a^+)^{n-1}a^+ = a^\#(a^+)^n$.
" $\Leftarrow$ " From the assumption, one gets

$$a^*(a^\#)^{n-1}a^+ = a^\#(a^+)^n = aa^+a^\#(a^+)^n = aa^+a^*(a^\#)^{n-1}a^+.$$

Multiplying the equality on the right by $a^{n+1}a^+$, one yields

$$a^* = aa^+a^*.$$

Hence $a \in R^{EP}$ by [7, Theorem 1.2.1], it follows that

$$a^* = a^*a^\#a = a^*(a^\#)^n a^n = a^*(a^\#)^{n-1}a^+a^n = a^\#(a^+)^n a^n = (a^\#)^{n+1}a^n = a^\#.$$

Thus $a \in R^{SEP}$. □

Let $m, n, d \in Z^+$, we denote the maximum common divisor of $m$ and $n$ as $(m, n) = d$. Especially when $d = 1$, we say that $m$ and $n$ are coprime.

**Theorem 3.4.** *Let $a \in R^\# \cap R^+$ and $m, n \in Z^+$, such that $(m, n) = 1$. Then $a \in R^{SEP}$ if and only if $(a^*)^k aa^\# = (a^+)^k$, $k = m, n$.*

*Proof.* " $\Rightarrow$ " It is clear.
" $\Leftarrow$ " Since $(m, n) = 1$, there exist $s, t \in Z$, such that $sm + tn = 1$. We can assume $s > 0$ and $t < 0$. Noting that

$$(a^*)^m aa^\# = (a^+)^m = (a^+)^m aa^+ = (a^*)^m aa^\# aa^+ = (a^*)^m.$$

Then $a \in R^{EP}$ by [7, Theorem 1.2.2]. This induces

$$(a^*)^k = (a^*)^k aa^+ = (a^*)^k aa^\# = (a^+)^k = (a^\#)^k,\ k = m, n.$$

Now we have

$$(a^\#)^{ms-1} = (a^\#)^{-nt} = (a^\#)^{n|t|} = (a^*)^{n|t|} = (a^*)^{-nt} = (a^*)^{ms-1}.$$

$$(a^\#)^{ms} = (a^*)^{ms} = (a^*)^{ms-1}a^* = (a^\#)^{ms-1}a^*.$$

$$a^\#a = a^\#a^{ms+1}(a^\#)^{ms} = a^\#a^{ms+1}(a^\#)^{ms-1}a^* = a^\#a^2a^* = aa^*.$$

Hence $a \in R^{SEP}$ by [7, Theorem 1.5.3]. □

**Theorem 3.5.** *Let $a \in R^\# \cap R^+$, $2 \leq n \in Z^+$, $(a^*)^{n+k} = (a^\#)^{n+k-1}a^*, k = 0, 1$. Then $a \in R^{SEP}$.*

*Proof.* " ⇒ " It is clear.

" ⇐ " Using the equality $(a^*)^{n+k} = (a^\#)^{n+k-1}a^*$, we obtain

$$(a^*)^{n+1} = (a^\#)^n a^* = aa^+(a^\#)^n a^* = aa^+(a^*)^{n+1},$$

$$a^* = (a^*)^{n+1}((a^\#)^*)^n = aa^+(a^*)^{n+1}((a^\#)^*)^n = aa^+a^*.$$

Hence $a \in R^{EP}$ by [7, Theorem 1.2.1].

$$(a^*)^n = (a^*)^n aa^+ = (a^*)^n a^+a = (a^*)^{n+1}(a^+)^* = (a^\#)^n a^*(a^+)^* = (a^\#)^n a^+a = (a^\#)^n,$$

$$(a^*)^{n-1} = (a^*)^{n-1}a^+a = (a^*)^n(a^+)^* = (a^\#)^{n-1}a^*(a^+)^* = (a^\#)^{n-1},$$

$$a^\# = a^{n-1}(a^\#)^n = a^{n-1}(a^*)^n = a^{n-1}(a^*)^{n-1}a^* = a^{n-1}(a^\#)^{n-1}a^* = aa^\#a^* = a^*.$$

Thus $a \in R^{SEP}$. □

## 4. Using the solution of univariate equations to characterize SEP elements

In this section, we construct the equation $a^*xa = a^+$ and consider the consistence of the equation. Firstly, we start with a lemma.

**Lemma 4.1.** *[7, Theorem 1.5.6] Suppose that $a \in R^\# \cap R^+$, $b \in R$ and $a = aba$. Then $a \in R^{EP}$ if and only if $a^+ = a^+ba$.*

**Theorem 4.2.** *Let $a \in R^\# \cap R^+$, $b \in R$ and $a = aba$. Then $a \in R^{SEP}$ if and only if $a^+ = a^*ba$.*

*Proof.* " ⇒ " It is an immediate result of Lemma 4.1.

" ⇐ " Since $a^+ = a^*ba$, $a = aa^+a = aa^*ba^2$, one yields

$$aa^\# = aa^*ba^2a^\# = aa^*ba = aa^+.$$

Then $a \in R^{EP}$. This gives

$$a^\# = a^+ = a^*ba = (a^*a^+a)ba = a^*a^+a = a^*.$$

Thus $a \in R^{SEP}$. □

**Corollary 4.3.** *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if the following equations has at least one solution.*

$$\begin{cases} axa = a; \\ a^*xa = a^+. \end{cases} \tag{1}$$

Naturally, we investigate the following equation

$$a^*xa = a^+ \tag{2}$$

**Lemma 4.4.** *Let $a \in R^\# \cap R^+$. Then $a \in R^{EP}$ if and only if Eq.(4.2) is consistent.*

*Proof.* " ⇒ " Assume that $a \in R^{EP}$. Then $a^+ = a^\# = a^\#a^+a = a^+a^+a$. Hence $x = (a^+)^*a^+a^+$ is a solution to Eq.(4.2).

" ⇐ " From the assumption, one gets $a^*x_0a = a^+$ for some $x_0 \in R$. This gives

$$a^+a^+a = (a^*x_0a)a^+a = a^*x_0a = a^+.$$

Then $a \in R^{EP}$. □

**Remark 4.5.** *If Eq.(4.2) is consistent, then the general solution is given by*

$$x = (a^+)^*a^+a^+ + u - aa^+uaa^+,\ \text{where } u \in R. \tag{3}$$

*Proof.* First, by Lemma 4.4, $a \in R^{EP}$, this induces the formula (4.3) is the solution to Eq.(4.2). Now let $x = x_0$ be any solution to Eq.(4.2). Then

$$a^* x_0 a = a^+.$$

Choose $u = x_0$. Then $aa^+uaa^+ = (a^+)^*(a^*x_0a)a^+ = (a^+)^*a^+a^+$, one yields

$$x_0 = (a^+)^*a^+a^+ + x_0 - aa^+uaa^+ = (a^+)^*a^+a^+ + u - aa^+uaa^+.$$

Thus the general solution to Eq.(4.2) is given by (4.3). □

**Theorem 4.6.** *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(4.2) is consistent and the general solution is given by*

$$x = aa^+a^+ + u - aa^+uaa^+, \ u \in R. \tag{4}$$

*Proof.* " ⇒ " Since $a \in R^{SEP}$, $a \in R^{EP}$ and $(a^+)^* = a$. By Remark 4.5, we are done.

" ⇐ " Noting that Eq.(4.2) is consistent. Then $a \in R^{EP}$. By the hypothesis, we have

$$a^*a^+a = a^*(aa^+a^+ + u - aa^+uaa^+)a = a^+.$$

Since $a \in R^{EP}$, $a^*a^+a = a^*$, one has $a^* = a^+$. Thus $a \in R^{SEP}$. □

Finally, we construct equation as follows, which has the general solution as (4.4).

$$(aa^\#)^*xaa^+ = a^+. \tag{5}$$

It is clear that we have the following theorem.

**Theorem 4.7.** *Let $a \in R^\# \cap R^+$. Then the general solution to Eq.(4.5) is given by (4.4).*

Theorem 4.6 and Theorem 4.7 infer the following theorem.

**Theorem 4.8.** *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(4.2) has the same solution as Eq.(4.5).*

## 5. Using core invertible elements to characterize SEP elements

**Theorem 5.1.** *Let $a \in R$. Then the followings are equivalent:*

*(1) $a \in R^{SEP}$;*

*(2) $a \in R^{\text{\textcircled{\#}}}$ and $a^* = a^{\text{\textcircled{\#}}}$;*

*(3) $a \in R^{\text{\textcircled{\#}}}$ and $aa^* = a^{\text{\textcircled{\#}}}a$.*

*Proof.* Suppose that $a \in R^{SEP}$, we have $a \in R^{\text{\textcircled{\#}}}$ and $a^{\text{\textcircled{\#}}} = a^\#aa^+$. Then (1)⇒(2) and (1)⇒(3) are easy to prove.

(2)⇒(1) Since $a \in R^{\text{\textcircled{\#}}}$, we can check that $a \in R^\#$ and $a^\# = (a^{\text{\textcircled{\#}}})^2a$, by direct computation. Then

$$aa^\#a^{\text{\textcircled{\#}}} = a^{\text{\textcircled{\#}}}aa^{\text{\textcircled{\#}}} = a^{\text{\textcircled{\#}}}.$$

This gives

$$a^* = a^{\text{\textcircled{\#}}} = aa^\#a^{\text{\textcircled{\#}}} = aa^\#a^*.$$

Hence $a \in R^{EP}$ and $a^\# = a^\#aa^+ = a^{\text{\textcircled{\#}}} = a^*$. Thus $a \in R^{SEP}$.

(3)⇒(1) Since $a \in R^{\text{\textcircled{\#}}}$, $a \in R^\#$ and $a^\# = (a^{\text{\textcircled{\#}}})^2a$, then $aa^\# = a^{\text{\textcircled{\#}}}a$. Hence $aa^* = a^{\text{\textcircled{\#}}}a = aa^\#$. Thus $a \in R^{SEP}$ by [7, Theorem 1.5.3]. □

Now we establish the following equation

$$xa^* = a^{\textcircled{\#}}x. \tag{6}$$

**Theorem 5.2.** *Let $a \in R^{\textcircled{\#}}$. Then $a \in R^{SEP}$ if and only if Eq.(5.1) has at least one solution in $G_a = \{a, a^\#, a^*, (a^\#)^*\}$.*

*Proof.* " $\Rightarrow$ " It is obvious by Theorem 5.1 (3).

" $\Leftarrow$ " (1) If $x = a$, then $aa^* = a^{\textcircled{\#}}a$. By Theorem 5.1, $a \in R^{SEP}$.

(2) If $x = a^\#$, then $a^\# a^* = a^{\textcircled{\#}}a^\# = (a^{\textcircled{\#}}a)a^\# a^\# = (aa^\#)a^\# a^\# = a^\# a^\#$. One yields

$$aa^* = aaa^\# a^* = aaa^\# a^\# = aa^\#.$$

Hence $a \in R^{SEP}$ [7, Theorem 1.5.3].

(3) If $x = a^*$, then $a^* a^* = a^{\textcircled{\#}}a^* = aa^\# a^{\textcircled{\#}}a^* = aa^\# a^* a^*$. One gets

$$a^* = a^* a^* (a^\#)^* = aa^\# a^* a^* (a^\#)^* = aa^\# a^*.$$

Hence $a \in R^{EP}$ [7, Theorem 1.2.1]. This gives $a^{\textcircled{\#}} = a^\#$ and so $a^* a^* = a^{\textcircled{\#}}a^* = a^\# a^*$. Thus $a \in R^{SEP}$ [7, Theorem 1.5.3].

(4) If $x = (a^\#)^*$, then $(a^\#)^* a^* = a^{\textcircled{\#}}(a^\#)^*$.

$$(aa^\#)^* = a^{\textcircled{\#}}(a^\#)^* = aa^\# a^{\textcircled{\#}}(a^\#)^* = aa^\#(aa^\#)^*.$$

Hence $a \in R^{EP}$ [7, Theorem 1.1.3]. It follows that $aa^\# = (aa^\#)^* = a^{\textcircled{\#}}(a^\#)^* = a^\#(a^\#)^*$. Then

$$a = aaa^\# = aa^\#(a^\#)^* = aa^\#(a^+)^* = (a^+)^*.$$

Thus $a \in R^{SEP}$. $\square$

Furtherly, we construct the following equation.

$$xa^* + a^\# = a^{\textcircled{\#}}x + a^+. \tag{7}$$

**Theorem 5.3.** *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(5.2) has at least one solution in $H_a = \{a^{\textcircled{\#}}, (a^{\textcircled{\#}})^*, a^+, (a^+)^*\}$.*

*Proof.* First $a^{\textcircled{\#}} = a^\# aa^+$.

" $\Rightarrow$ " If $a \in R^{SEP}$, then $x = a^+ = a^\# = a^*$ is a solution.

" $\Leftarrow$ " (1) If $x = a^{\textcircled{\#}} = a^\# aa^+$, then

$$a^\# aa^+ a^* + a^\# = a^\# aa^+ a^\# aa^+ + a^+ = a^\# a^+ + a^+.$$

Multiplying the equality on the left by $aa^\#$, one has $a^\# = aa^\# a^+$. Hence $a \in R^{EP}$ [7, Theorem 1.2.1]. This gives $a^{\textcircled{\#}} = a^\# = a^+$ and $a^\# a^* = a^{\textcircled{\#}}a^\#$. By Theorem 5.2, $a \in R^{SEP}$.

(2) If $x = (a^{\textcircled{\#}})^* = aa^+(a^\#)^*$, then

$$aa^+ + a^\# = aa^+(a^\#)^* a^* + a^\# = a^\# aa^+ aa^+(a^\#)^* + a^+ = a^\# aa^+(a^\#)^* + a^+.$$

Multiplying the equality on the left by $aa^\#$, one gets

$$a^+ = aa^\# a^+.$$

Then $a \in R^{EP}$ and $a^{\#} = a^{+}$. From the assumption, we obtain

$$aa^{\#} = aa^{+} = a^{\#}aa^{+}(a^{\#})^{*} = a^{\#}(a^{\#})^{*},$$

$$a = aaa^{\#} = aa^{\#}(a^{\#})^{*} = aa^{+}(a^{+})^{*} = (a^{+})^{*}.$$

Hence $a \in R^{SEP}$.

(3) If $x = a^{+}$, then $a^{+}a^{*} + a^{\#} = a^{\#}aa^{+}a^{+} + a^{+}$. Multiplying the equality on the right by $aa^{+}$, one yields

$$a^{\#}aa^{+} = a^{\#}.$$

Then $a \in R^{EP}$ [7, Theorem 1.2.1], this induces

$$a^{+}a^{*} = a^{\#}aa^{+}a^{+} = a^{+}a^{+}.$$

By [16, Corollary 2.10], $a \in R^{PI}$. Thus $a \in R^{SEP}$.

(4) If $x = (a^{+})^{*}$, then $aa^{+} + a^{\#} = (a^{+})^{*}a^{*} + a^{\#} = a^{\#}aa^{+}(a^{+})^{*} + a^{+} = a^{\#}(a^{+})^{*} + a^{+}$. Multiplying the equality on the left by $aa^{\#}$, one has

$$a^{+} = aa^{\#}a^{+}.$$

Then $a \in R^{EP}$ [7, Theorem 1.2.1], one gets $a^{+} = a^{\#}$, $(a^{+})^{*} = (a^{\#})^{*}$. Now we have

$$a^{\#}a = aa^{+} = a^{\#}(a^{+})^{*} = a^{\#}(a^{\#})^{*}.$$

Hence $a \in R^{SEP}$ by (2). □

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