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Warped product pointwise semi-slant submanifolds of nearly Kaehler manifolds

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Abstract. In this paper, we study warped pointwise semi-slant submanifolds of nearly Kaehler manifolds. We prove that there do not exist non-trivial warped product pointwise semi-slant submanifolds of the form $N_\theta \times_f N_T$ in a nearly Kaehler manifold $\bar{M}$ but the geometry of warped products by reversing these two factors is similar to the case of general slant fiber.

1. Introduction

Warped products play very important role not only geometry but also in the theory of relativity and physics. In 1969, Bishop and O'Neill [4], introduced the notion of warped product manifolds to construct a large class of complete manifolds with negative curvature. These manifolds are natural generalizations of Riemannian product manifolds. They defined these manifolds as follows: Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and f, a positive differentiable function on N_1. Consider the product manifold $N_1 \times N_2$ with its canonical projections $\pi : N_1 \times N_2 \to N_1$ and $\rho : N_1 \times N_2 \to N_2$. The warped product $M = N_1 \times_f N_2$ is the product manifold $N_1 \times N_2$ equipped with the Riemannian structure such that

$$\|X\|^2 = \|\pi_*(X)\|^2 + f^2(\pi(p)) \left\|\rho_*(X)\right\|^2 \tag{1}$$

for all $X \in T_pM$, where $*$ denotes the maps on the tangent space. Consequently, we write $g = g_1 + f^2 g_2$, where the function f is called the warping function on M. In [5], O'Neill proved that for all $X \in TN_1$ and all $Z \in TN_2$, then

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z \tag{2}$$

where ∇ denote the Levi-Civita connection on M. A warped product manifold $M = N_1 \times_f N_2$ is called trivial or simply a Riemannian product if the warping function f is constant. Let $M = N_1 \times_f N_2$ be a warped product manifold then N_1 is totally geodesic and N_2 is a totally umbilical submanifold of M, respectively [5, 8].

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On the other hand, nearly Kaehler manifolds are Tachibana manifolds studied in [22]. It was shown in [1] that nearly Kaehler manifolds form an interesting class of manifolds admitting a metric connection with a parallel totally anti-symmetric torsion. A known example of a nearly Kaehler non-Kaehler manifold is the six dimensional sphere S^6. It has an almost complex structure J defined by the vector cross product in the space of purely imaginary Cayley numbers O which satisfies the nearly Kaehler structure. More examples of nearly Kaehler manifolds can be found in [26], namely, the homogeneous spaces G/K, where G is a compact semi-simple Lie group and K is the fixed point set of an automorphism of G of order 3. Strict nearly Kaehler manifolds obtained a lot of consideration in 1980s due to their relation to Killing spinors. Th. Friedrich and R. Grunewald showed in [16] that a 6-dimensional Riemannian manifold admits a Riemannian Killing spinor if and only if it is nearly Kaehler. The only known 6-dimensional strict nearly Kaehler manifolds are

$$S^6 = G_2/SU(3) \cdot Sp(2)/SU(2) \times U(1), \quad SU(3)/U(1) \times U(1), \quad S^3 \times S^3.$$

In fact, these are the only homogeneous nearly Kaehler manifolds in dimension six [6].

In [18], N. Papaghiuc introduced semi-slant submanifolds such that the class of CRsubmanifolds and the class of slant submanifolds are particular classes of semi-slant submanifolds. In [15], Etayo introduced pointwise slant submanifolds of almost Hermitian manifolds and then, in [10], these submanifolds have been studied by Chen and Garay. Using the idea of pointwise slant submanifolds, Sahin [21] extended the study of semi-slant submanifolds to pointwise semi-slant submanifolds and their warped products in Kaehler manifolds.

In [19], Sahin proved that there is no warped product semi-slant submanifolds of the forms $N_T \times {}_f N_\theta$ and $N_\theta \times_f N_T$ in a Kaehler manifold $\bar{M}$, where N_T is a holomorphic submanifold and N_θ is a proper slant submanifold of $\bar{M}$.

In this paper, we study warped product pointwise semi slant submanifolds of nearly Kaehler manifolds. We prove that the warped products of the form $N_\theta \times_f N_T$ are simply Riemannian products with a pointwise slant factor N_θ. On the other hand, the warped products obtained by reversing these two factors exist and we discuss their geometry in the last section.

2. Preliminaries

Let $\bar{M}$ be an almost Hermitian manifold with an almost complex structure J and a Hermitian metric g such that

$$\text{(a) } J^2 = -I, \quad \text{(b) } g(JX, JY) = g(X, Y) \tag{3}$$

for all vector fields X, Y on $\bar{M}$. Also, let $\Gamma(T\bar{M})$ be the set of all vector fields tangent to $\bar{M}$ and $\bar{\nabla}$, the covariant differential operator on $\bar{M}$ with respect to g. By A. Gray [17], if the almost complex structure J satisfies

$$(\bar{\nabla}_X J)\, X = 0, \text{ equivalently } (\bar{\nabla}_X J)\, Y + (\bar{\nabla}_Y J)\, X = 0 \tag{4}$$

for all $X, Y \in \Gamma(T\bar{M})$, then the manifold $\bar{M}$ is called a nearly Kaehler manifold.

For a submanifold M of a Riemannian manifold $\bar{M}$, the Gauss and Weingarten formulas are respectively given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi \tag{5}$$

for all $X, Y \in \Gamma(TM)$, where ∇ is the induced Riemannian connection on M, ξ is a vector field normal to M, h is the second fundamental form of M, $\nabla^\perp$ is the normal connection in the normal bundle $T^\perp M$ and A_ξ is the shape operator of the second fundamental form. They are related by

$$g\,(A_\xi X, Y) = g(h(X, Y), \xi) \tag{6}$$

where g is the Riemannian metric on $\bar{M}$ as well as the metric induced on M. The mean curvature vector H of M is given by $H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$, where n is the dimension of M and $\{e_1, e_2, \cdots, e_n\}$ is a local orthonormal frame of vector fields on M. A submanifold M of an almost Hermitian manifold $\bar{M}$ is called totally umbilical if the second fundamental form satisfies $h(X, Y) = g(X, Y)H$, for all $X, Y \in \Gamma(TM)$. The submanifold M is totally geodesic if $h(X, Y) = 0$, for all $X, Y \in \Gamma(TM)$ and minimal if $H = 0$.

A submanifold M of an almost Hermitian manifold $\bar{M}$ is called holomorphic if, for any $p \in M$, we have $J\left(T_pM\right) = T_pM$, where T_pM denotes the tangent space of M at p. It is called totally real (or Lagrangian) if we have $J\left(T_pM\right) \subseteq T_p^{\perp}M$ for each $p \in M$, where $T_p^{\perp}M$ is the normal space of M in $\bar{M}$ at p.

For any $X \in \Gamma(TM)$ and $\xi \in \Gamma(T^{\perp}M)$, the transformations JX and $J\xi$ are decomposed into tangential and normal part as

$$JX = TX + FX, \quad J\xi = t\xi + f\xi. \tag{7}$$

A submanifold M of an almost Hermitian manifold $\tilde{M}^{2n}$ is called pointwise slant [10], if at each point $p \in M$, the Wirtinger angle $\theta(X)$ between JX and T_pM is independent of the choice of a non-zero vector $X \in T_pM$. In this case, the Wirtinger angle gives rise to a real-valued function $\theta : TM - \{0\} \to \mathbb{R}$ which is called the *Wirtinger function* or *slant function* of the pointwise slant submanifold.

We note that a pointwise slant submanifold is called *slant*, in the sense of [7], if its Wirtinger function θ is globally constant. We also note that every slant submanifold is a pointwise slant submanifold.

Moreover, complex and totally submanifolds are pointwise slant submanifolds with slant function $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A pointwise slant submanifold M of a Kaehler manifold $\tilde{M}$ is called a *proper pointwise slant* if it is neither complex nor totally real nor θ is globally constant on M.

Furthermore, we know that from Lemma 2.1 of [10], a submanifold M is a pointwise slant submanifold of an almost Hermitian manifold $\tilde{M}$ if and only if

$$T^2(X) = -\cos^2\theta_p(X), \tag{8}$$

where θ_p is the slant function of M at $p \in M$ (see [10]). As a consequence of the formula (8), we have

$$g(TX, TY) = \cos^2\theta_p g(X, Y) \tag{9}$$

$$g(FX, FY) = \sin^2\theta_p g(X, Y) \tag{10}$$

for any $X, Y \in \Gamma(TM)$.

Definition 2.1. Let $\bar{M}$ be an almost Hermitian manifold and M is a real submanifold of $\bar{M}$. Then, M is a pointwise semi-slant submanifold if there exist two orthogonal distributions D^T and D^{θ} on M such that

(a) TM admits the orthogonal direct decomposition $TM = D^T \oplus D^{\theta}$.

(b) The distribution D^T is holomorphic i.e., $JD^T = D^T$.

(c) The distribution D^{θ} is pointwise slant with slant function θ.

In this case, we call the angle θ the slant function of the pointwise slant distribution D^{θ}. The holomorphic distribution D^T of a pointwise semi-slant submanifold is a pointwise slant distribution with slant function $\theta = 0$.

3. Warped product pointwise semi-slant submanifolds

In this section, we study warped product submanifolds of a nearly Kaehler manifold $\bar{M}$, either in the form $N_{\theta} \times {}_fN_T$ or $N_T \times {}_fN_{\theta}$, where N_T and N_{θ} are holomorphic and proper pointwise slant submanifolds of $\bar{M}$, respectively. These two types of warped products are the products in between the holomorphic and proper pointwise slant submanifolds of $\bar{M}$,

Theorem 3.1. *Let $\bar{M}$ be a nearly Kaehler manifold and $M = N_\theta \times_f N_T$ be a warped product submanifold of $\bar{M}$. Then M is Riemannian product of N_T and N_θ, where N_T and N_θ are holomorphic and proper pointwise slant submanifolds of $\bar{M}$, respectively.*

Proof. For any $X, Y \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$, we have

$$\begin{aligned} g(h(X,Y),FZ) &= g\left(\bar{\nabla}_X Y, JZ - TZ\right) \\ &= g\left((\bar{\nabla}_X J)Y, Z\right) - g\left(\bar{\nabla}_X JY, Z\right) + g\left(Y, \bar{\nabla}_X TZ\right) \\ &= g\left((\bar{\nabla}_X J)Y, Z\right) + Z(\ln f) g\left(X, JY\right) + TZ(\ln f) g\left(X, Y\right). \end{aligned} \tag{11}$$

Interchanging X and Y in (11), we get

$$g(h(X,Y),FZ) = g\left((\bar{\nabla}_Y J)X, Z\right) - Z(\ln f) g\left(X, JY\right) + TZ(\ln f) g\left(X, Y\right). \tag{12}$$

Then, from (11) and (12) together with the nearly Kaehler characteristic equation (4), we derive

$$g(h(X,Y),FZ) = TZ(\ln f) g\left(X, Y\right). \tag{13}$$

Also, we have

$$\begin{aligned} g(h(X,JY),FZ) &= g\left(\bar{\nabla}_X JY, JZ\right) - g\left(\bar{\nabla}_X JY, TZ\right) \\ &= g\left((\bar{\nabla}_X J)Y, JZ\right) + g\left(J\bar{\nabla}_X Y, JZ\right) - g\left((\bar{\nabla}_X J)Y, TZ\right) + g\left(\bar{\nabla}_X Y, T^2 Z\right) + g\left(\bar{\nabla}_X Y, FTZ\right) \\ &= g\left((\bar{\nabla}_X J)Y, FZ\right) - Z(\ln f) g(X,Y) - g\left(Y, \bar{\nabla}_X(-\cos^2\theta) Z\right) + g\left(h(X,Y), FTZ\right) \\ &= g\left((\bar{\nabla}_X J)Y, FZ\right) - Z(\ln f) g(X,Y) + \cos^2\theta Z(\ln f) g\left(X, Y\right) - \sin 2\theta X(\theta) g(Y,Z) \\ &\quad + g\left(h(X,Y), FTZ\right) \\ &= g\left((\bar{\nabla}_X J)Y, FZ\right) - \sin^2\theta Z(\ln f) g(X,Y) + g\left(h(X,Y), FTZ\right). \end{aligned} \tag{14}$$

Replacing Z by TZ in (14), we have

$$g(h(X,JY),FTZ)) = g\left((\bar{\nabla}_X J)Y, FTZ\right) - \sin^2\theta TZ(\ln f) g(X,Y) - \cos^2\theta g(h(X,Y),FZ). \tag{15}$$

Using (13), we find

$$g(h(X,JY),FTZ) = g\left((\bar{\nabla}_X J)Y, FTZ\right) - TZ(\ln f) g(X,Y). \tag{16}$$

Interchanging X and Y, we derive

$$g(h(JX,Y),FTZ) = g\left((\bar{\nabla}_Y J)X, FTZ\right) - TZ(\ln f) g(X,Y). \tag{17}$$

Then, from (16) and (17), we obtain

$$g(h(X,JY) + h(JX,Y), FTZ) = -2TZ(\ln f) g(X,Y). \tag{18}$$

Replacing X by JX and Z by TZ in (18), we get

$$-\cos^2\theta g(h(JX,JY) - h(X,Y), FZ) = 2\cos^2\theta Z(\ln f) g(JX,Y). \tag{19}$$

Again, using (13), we derive

$$-g(h(JX,JY),FZ) + TZ(\ln f) g(X,Y) = -2Z(\ln f) g(X,JY). \tag{20}$$

Interchanging X with JX and Y with JY, we find

$$g(h(X,Y),FZ) = TZ(\ln f) g(JX,JY) - 2Z(\ln f) g(JX,Y). \tag{21}$$

Again, using (13) with (1), we get

$$2Z(\ln f) g(X,JY) = 0, \tag{22}$$

which implies that f is constant and hence the proof is complete. □

Now, we study the warped product pointwise semi-slant submanifolds of the form $N_T \times_f N_\theta$ in a nearly Kaehler manifold $\bar{M}$ such that N_T is a holomorphic submanifold and N_θ is a proper pointwise slant submanifold of $\bar{M}$.

First, we give the following non-trivial example of pointwise semi-slant warped product submanifolds in an Euclidean space.

Example 3.2. Consider a 4-dimensional submanifold M^4 of $\mathbb{C}^5$ defined by

$$\psi(u, v, , r, s) = (u\cos r, u\sin r, u\cos s, u\sin s, r - s, v\cos r, v\sin r, v\cos s, v\sin s, r + s), \quad u, v > 0.$$

Then the tangent bundle TM^5 is spanned by

$$X_u = (\cos r, \sin r, \cos s, \sin s, 0; 0, 0, 0, 0, 0\Big), \;\; X_v = \Big(0, 0, 0, 0, 0; \cos r, \sin r, \cos s, \sin s, 0),$$
$$X_r = (-u\sin r, u\cos r, 0, 0, 1; -v\sin r, v\cos r, 0, 0, 1\Big), \;\; X_s = \Big(0, 0, -u\sin s, u\cos s, -1; 0, 0, -v\sin s, v\cos s, 1).$$

We put

$$\mathfrak{D}^T = \text{Span}\{X_u, X_v\}, \quad \mathfrak{D}^\theta = \text{Span}\{X_r,\ X_s\} \tag{23}$$

where the slant function θ satisfies $\theta = \cos^{-1}\left(\frac{2}{u^2+v^2+2}\right)$. Thus, M is a pointwise semi-slant submanifold. In fact, M is a pointwise semi-slant warped product submanifold of the form $M_T \times_f M_\theta$ with the metric structure

$$g = 2\left(du^2 + dv^2\right) + \left(u^2 + v^2 + 2\right)\left(dr^2 + ds^2\right). \tag{24}$$

Lemma 3.3. *Let $M = N_T \times N_\theta$ be a warped product pointwise semi-slant submanifold of nearly Kaehler manifold $\bar{M}$. Then, we have*

(i) $g(h(X, Y), FZ) = 0,$

(ii) $g(h(X, W), FZ) = -JX(\ln f)g(Z, W) + \frac{1}{3}X(\ln f)g(Z, TW).$

for any $X, Y \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$.

Proof. For any $X, Y \in TN_T$ and $Z \in TN_\theta$, we have:

$$g(h(X, Y), FZ) = g\left(\bar{\nabla}_X Y, JZ\right) - g\left(\bar{\nabla}_X Y, TZ\right) = g\left((\bar{\nabla}_X J)Y, Z\right) - g\left(\bar{\nabla}_X JY, Z\right) - g\left(\nabla_X Y, TZ\right).$$

Since N_T is totally geodesic in M, using this fact with the orthogonality of vector fields, we derive

$$g(h(X, Y), FZ) = g\left((\bar{\nabla}_X J)Y, Z\right). \tag{25}$$

Interchanging X with Y in (25), we get

$$g(h(X, Y), FZ) = g\left((\bar{\nabla}_Y J)X, Z\right). \tag{26}$$

Hence, (i) following from (25) and (26) with the nearly Kaehler characteristic equation (4).

For the second part (ii), for any $X \in \Gamma(TN_T)$ and $Z, W \in \Gamma(TN_\theta)$, we have

$$g(h(X, Z), FW) = g(\bar{\nabla}_X Z, JW) - g(\bar{\nabla}_X Z, TW) = g((\bar{\nabla}_X J)Z, W) - g(\bar{\nabla}_X JZ, W) - X(\ln f)g(Z, TW). \tag{27}$$

Then, from (5) and (7) with (2), we derive

$$\begin{aligned} g(h(X, Z), FW) &= g((\bar{\nabla}_X J)Z, W) - g(\bar{\nabla}_X TZ, W) - g(\bar{\nabla}_X FZ, W) - X(\ln f)g(Z, TW) \\ &= g((\bar{\nabla}_X J)Z, W) + g(h(X, W), FZ). \end{aligned} \tag{28}$$

On the other hand, we have

$$\begin{aligned} g(h(X,Z),FW) &= g(\bar{\nabla}_Z X, JW) - g(\bar{\nabla}_Z X, TW) \\ &= g((\bar{\nabla}_Z J)X, W) - g(\bar{\nabla}_Z JX, W) - X \ln f g(Z, TW) \\ &= g((\bar{\nabla}_Z J)X, W) - JX(\ln f)g(Z,W) - X(\ln f)g(Z,TW). \end{aligned} \tag{29}$$

Then, from (28) and (29) with (4), we derive

$$2g\left(h\left(X,Z\right),FW\right) = g\left(h\left(X,W\right),FZ\right) - JX(\ln f)g\left(Z,W\right) - X(\ln f)g\left(Z,TW\right). \tag{30}$$

Interchanging Z and W in (30), we obtain

$$2g\left(h\left(X,W\right),FZ\right) = g\left(h\left(X,Z\right),FW\right) - JX(\ln f)g\left(Z,W\right) + X(\ln f)g\left(Z,TW\right). \tag{31}$$

Then, from (31) and (30), we get the required result (ii). Hence, the proof is complete. □

4. Inequality for warped product pointwise semi-slant submanifolds

In this section, we establish Chen's inequality for the squared norm of the second fundamental form h of $M = N_T \times N_\theta$, where N_T and N_θ holomorphic and proper pointwise slant submanifolds of a nearly kaehler manifold $\bar{M}$.

For this we assume the following fame fields for the warped product submanifold $M = N_T \times {}_f N_\theta$. Let $2m = \dim_{\mathbf{R}} \bar{M}$, $2p = \dim N_T$ and $2q = \dim N_\theta$, then $n = 2p + 2q$. Let us denote by D^T and D^θ, the tangent bundles on N_T and N_θ, respectively and let $\left\{e_1, \ldots, e_p, e_{p+1} = Je_1, \ldots, e_{2p} = Je_p\right\}$ and $\left\{e_{2p+1} = e_1^*, \ldots, e_{p+q} = e_q^*, e_{2p+q+1} = \sec\theta T e_1^*, \ldots, e_n = \sec\theta T e_q^*\right\}$ be the local orthonormal frames of D^T and D^θ, respectively. Then, the orthonormal frames of FD^θ and μ are $\{e_{n+1} = \bar{e}_1 = \csc\theta F e_1^*, \ldots, \bar{e}_q = \csc\theta F e_q^*, \bar{e}_{q+1} = \csc\theta \sec\theta FT e_1^*, \ldots, \bar{e}_{2q} = \csc\theta\sec\theta FT e_q^*\}$ and $\left\{e_{n+2q+1} \ \ldots, e_{2m}\right\}$, respectively, where $e_{n+2q+1} \ldots, e_{2m}$ are orthonormal vectors in the invariant normal subbundle μ of $T^\perp M$. The dimensions of FD^θ and μ will be $2q$ and $2m - n - 2q$, respectively.

We use the above mentioned frame fields and some basic results from the previous sections to establish the following inequality.

Theorem 4.1. *Let $M = N_T \times_f N_\theta$ be a warped product pointwise semi-slant submanifold of a nearly Kaehler manifold $\bar{M}$ such that N_T and N_θ are holomorphic and pointwise slant submanifolds of $\bar{M}$, respectively. Then*

(i) *The squared norm of the second fundamental form h of M satisfies*

$$\|h\|^2 \geq \frac{2q}{9}\left\{10\csc^2\theta - 1\right\}\|\nabla^T(\ln f)\|^2 \tag{32}$$

where $\nabla^T \ln f$ is the gradient of $\ln f$ along N_T and $2q = \dim N_\theta$.

(ii) *If the equality sign in (32) holds identically, then N_T is totally geodesic and N_θ is totally umbilical in $\bar{M}$. Moreover, M is a minimal submanifold of $\bar{M}$.*

Proof. From the definition, we have

$$\|h\|^2 = \sum_{i,j=1}^{n} g\left(h\left(e_i, e_j\right), h\left(e_i, e_j\right)\right) = \sum_{r=n+1}^{2m}\sum_{i,j=1}^{n} g\left(h\left(e_i, e_j\right), e_r\right)^2.$$

On the decomposition for the frames of FD^θ and μ the above relation will be

$$\|h\|^2 = \sum_{r=n+1}^{n+2q}\sum_{i,j=1}^{n} g\left(h\left(e_i, e_j\right), e_r\right)^2 + \sum_{r=n+2q+1}^{2m}\sum_{i,j=1}^{n} g\left(h\left(e_i, e_j\right), e_r\right)^2. \tag{33}$$

In the first term of right hand side in (33), e_r belongs to FD^θ while in the second term of right hand side e_r belongs to μ. We shall equate only the first term in right hand side, then we get

$$||h||^2 \geq \sum_{r=n+1}^{n+2q} \sum_{i,j=1}^{n} g\left(h\left(e_i, e_j\right), e_r\right)^2 .$$

Then for the orthonormal frames of D^T and D^θ, the above equality takes the form

$$||h||^2 = \sum_{r=1}^{2q} \sum_{i,j=1}^{2p} g\left(h\left(e_i, e_j\right), \bar{e}_r\right)^2 + 2\sum_{r=1}^{2q} \sum_{i=1}^{2p} \sum_{j=1}^{2q} g\left(h\left(e_i, e_j^*\right), \bar{e}_r\right)^2 + \sum_{r=1}^{2q} \sum_{i,j=1}^{2q} g\left(h\left(e_i^*, e_j^*\right), \bar{e}_r\right)^2 . \tag{34}$$

Thus by Lemma 3.3 (i), the first term of the right hand side in (34) is identically zero and we shall compute just the next term and leave the third term, then we get

$$||h||^2 \geq 2\sum_{r=1}^{2q} \sum_{i=1}^{2p} \sum_{j=1}^{2q} g\left(h\left(e_i, e_j^*\right), \bar{e}_r\right)^2 .$$

Then using the orthonormal frame fields of D, D_θ and FD_θ, we derive

$$\begin{aligned} ||h||^2 \geq\ & 2\csc^2\theta \sum_{r=1}^{q} \sum_{i=1}^{p} \sum_{j=1}^{q} g\left(h\left(e_i, e_j^*\right), Fe_r^*\right)^2 + 2\csc^2\theta \sum_{i=1}^{p} \sum_{r,j=1}^{q} g\left(h\left(Je_i, e_j^*\right), Fe_r^*\right)^2 \\ &+ 2\csc^2\theta\sec^2\theta \sum_{i=1}^{p} \sum_{r,j=1}^{q} g\left(h\left(e_i, Te_j^*\right), Fe_r^*\right)^2 + 2\csc^2\theta\sec^2\theta \sum_{i=1}^{p} \sum_{r,j=1}^{q} g\left(h\left(Je_i, Te_j^*\right), Fe_r^*\right)^2 \\ &+ 2\csc^2\theta\sec^2\theta \sum_{i=1}^{p} \sum_{r,j=1}^{q} g\left(h\left(e_i, e_j^*\right), FTe_r^*\right)^2 + 2\csc^2\theta\sec^2\theta \sum_{i=1}^{p} \sum_{r,j=1}^{q} g\left(h\left(Je_i, e_j^*\right), FTe_r^*\right)^2 \\ &+ 2\csc^2\theta\sec^4\theta \sum_{i=1}^{p} \sum_{r,j=1}^{q} g\left(h\left(e_i, Te_j^*\right), FTe_r^*\right)^2 + 2\csc^2\theta\sec^4\theta \sum_{i=1}^{p} \sum_{r,j=1}^{q} g\left(h\left(Je_i, Te_j^*\right), FTe_r^*\right)^2 . \end{aligned} \tag{35}$$

Thus, by Lemma 3.3 (ii) we obtain

$$||h||^2 \geq 2q\csc^2\theta \sum_{i=1}^{2p} (e_i \ln f)^2 + \frac{2q}{9}\cot^2\theta \sum_{i=1}^{2p} (e_i \ln f)^2 = \frac{2q}{9}\left\{10\csc^2\theta - 1\right\} ||\nabla^T(\ln f)||^2,$$

which is the inequality (32). If the equality holds in (32), then by Leaving term and Lemma 3.3, we get

$$h(D^T, D^T) = 0, h\left(D^\theta, D^\theta\right) = 0. \tag{36}$$

Then, N_T is totally geodesic and N_θ is totally umbilical submanifold of $\bar{M}$ by using the fact that N_T is totally geodesic and N_θ is totally umbilical in M [4, 8] with equality holding case of (32) in (36). Furthermore, M is a minimal submanifold, which proves the theorem completely. □

For the special cases of Theorem 4.1, we have the following remarks.

Remark 4.2. *If the slant function θ is globally constant on M then warped product pointwise semi-slant submanifolds reduce warped product semi-slant submanifolds those are studied in [3] and hence the main Theorem* 4.1 *of [3] is a special case of Theorem 4.1.*

Remark 4.3. *If we assume $\theta = \frac{\pi}{2}$ in Theorem 4.1, then warped product becomes $M = N_T \times_f N_\perp$, where N_T and $N_\perp$ are holomorphic and totally real submanifolds of $\bar{M}$, respectively. In this case, M is a CR-warped product submanifold studied in [20]. In this sense, Theorem* 4.2 *of [20] is a special case of Theorem 4.1.*

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