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Trivial doubly warped products

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Abstract. The aim of the paper is to provide new obstructions to the existence of doubly warped products. We prove that, if the factor manifolds of a doubly warped product are connected and locally product Riemannian manifolds, then, the almost product structure naturally induced on the doubly warped product is parallel if and only if the manifold is a direct product manifold. We also show that there do not exist doubly warped product Kähler manifolds (with respect to the naturally induced almost Hermitian structure) with connected Kähler factors, which are not direct products, neither doubly warped product manifolds which are pointwise slant but not slant submanifolds (with respect to the naturally induced almost Hermitian structure) with pointwise slant factors.

1. Introduction

The goal of the present paper is to determine new conditions under which a doubly warped product manifold is a warped product, or just a direct product. More precisely: for two Riemannian manifolds endowed with a symmetric or skew-symmetric $(1, 1)$-tensor field, we consider their doubly warped product $(\tilde{M}, \tilde{g})$ with some warping functions f_1 and f_2, and the naturally induced $(1, 1)$-tensor field. We prove that, under the connectedness hypothesis, if the factor manifolds are locally product Riemannian manifolds, then, the almost product structure naturally induced on the doubly warped product is parallel if and only if f_1 and f_2 are constant, i.e., if $(\tilde{M}, \tilde{g})$ is a direct product manifold. Also, we show that there do not exist doubly warped product Kähler manifolds (with respect to the naturally induced almost Hermitian structure) with connected Kähler factors, which are not direct products. Finally, we prove that there do not exist doubly warped product manifolds which are pointwise slant but not slant submanifolds (with respect to the naturally induced almost Hermitian structure) with pointwise slant factors.

2. Preliminaries

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds, let ∇^1 and ∇^2 be the Levi-Civita connections on M_1 and M_2, respectively, and let f_1 and f_2 be two positive smooth functions on M_1 and M_2, respectively. We consider the *doubly warped product manifold* ${}_{f_2}M_1 \times_{f_1} M_2 =: (\tilde{M}, \tilde{g})$ defined as [3]:

$$\tilde{M} := M_1 \times M_2, \quad \tilde{g} := \left(\pi_2^*(f_2)\right)^2 \pi_1^*(g_1) + \left(\pi_1^*(f_1)\right)^2 \pi_2^*(g_2)$$

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for $\pi_i : M_1 \times M_2 \to M_i$ the canonical projection, $i = 1, 2$. If only one of f_1 and f_2 is a constant, then $(\tilde{M}, \tilde{g})$ is a *warped product manifold* (see [1]). Moreover, if both f_1 and f_2 are constant, then $(\tilde{M}, \tilde{g})$ is a direct product manifold (and we call it as the *trivial case*).

For the rest of the paper, we shall use the same notation for a function on M_i, $i = 1, 2$, and its pullback on $\tilde{M}$, as well as, for a metric on M_i, $i = 1, 2$, and its pullback on $\tilde{M}$, and also, for a vector field on M_i, $i = 1, 2$, and its lift on $\tilde{M}$. The set of smooth sections of a smooth manifold M will be denoted by $\Gamma(TM)$.

We have the orthogonal decomposition

$$T\tilde{M} = TM_1 \oplus TM_2 ,$$

and for any $\tilde{X} \in \Gamma(T\tilde{M})$, we denote

$$\tilde{X} = P_1\tilde{X} + P_2\tilde{X},$$

where $P_i\tilde{X}$ represents the projection of $\tilde{X}$ on $\Gamma(TM_i)$, $i = 1, 2$.

Let $\epsilon \in \{\pm 1\}$ and let J_i be a $(1, 1)$-tensor field on the Riemannian manifold (M_i, g_i), $i = 1, 2$, satisfying

$$J_i^2 = \epsilon I, \ \ g_i(J_iX, Y) = \epsilon g_i(X, J_iY), \ (\forall)\ X, Y \in \Gamma(TM_i).$$

We define $\tilde{J} := J_1P_1 + J_2P_2$. Then, for any $\tilde{X} \in \Gamma(T\tilde{M})$, we have $\tilde{J}\tilde{X} = J_1P_1\tilde{X} + J_2P_2\tilde{X}$, hence

$$\tilde{J}^2\tilde{X} = J_1P_1(J_1P_1\tilde{X}) + J_2P_2(J_2P_2\tilde{X}) = J_1^2(P_1\tilde{X}) + J_2^2(P_2\tilde{X}) = \epsilon P_1\tilde{X} + \epsilon P_2\tilde{X} = \epsilon\tilde{X},$$

$$\begin{aligned}\tilde{g}(\tilde{J}\tilde{X}, \tilde{Y}) &= \tilde{g}(J_1P_1\tilde{X}, P_1\tilde{Y}) + \tilde{g}(J_2P_2\tilde{X}, P_2\tilde{Y}) = f_2^2 g_1(J_1P_1\tilde{X}, P_1\tilde{Y}) + f_1^2 g_2(J_2P_2\tilde{X}, P_2\tilde{Y}) \\ &= f_2^2\epsilon g_1(P_1\tilde{X}, J_1P_1\tilde{Y}) + f_1^2\epsilon g_2(P_2\tilde{X}, J_2P_2\tilde{Y}) = \epsilon\tilde{g}(\tilde{X}, \tilde{J}\tilde{Y})\end{aligned}$$

for any $\tilde{X}, \tilde{Y} \in \Gamma(T\tilde{M})$, and we can state

Lemma 2.1. *Let $\epsilon \in \{\pm 1\}$ and let J_i be a $(1, 1)$-tensor field on the Riemannian manifold (M_i, g_i), $i = 1, 2$, such that*

$$J_i^2 = \epsilon I, \ \ g_i(J_iX, Y) = \epsilon g_i(X, J_iY), \ (\forall)\ X, Y \in \Gamma(TM_i).$$

Then, $\tilde{J} := J_1P_1 + J_2P_2$ satisfies:

$$\tilde{J}^2 = \epsilon I, \ \ \tilde{g}(\tilde{J}\tilde{X}, \tilde{Y}) = \epsilon\tilde{g}(\tilde{X}, \tilde{J}\tilde{Y}), \ (\forall)\ \tilde{X}, \tilde{Y} \in \Gamma(T\tilde{M}).$$

If $\epsilon = 1$, the triple (M_i, g_i, J_i), with $J_i \neq \pm I$, satisfying the two conditions from the previous lemma is called an *almost product Riemannian manifold*, and if $\epsilon = -1$, then it is called an *almost Hermitian manifold*. Also, a $(1, 1)$-tensor field J_i satisfying the second of the two conditions is said to be *g_i-symmetric* if $\epsilon = 1$, and *g_i-skew-symmetric* if $\epsilon = -1$.

Let ∇^i be the Levi-Civita connection of g_i. An almost product Riemannian manifold (M_i, g_i, J_i) is called a *locally product Riemannian manifold* if $\nabla^i J_i = 0$, and an almost Hermitian manifold (M_i, g_i, J_i) is called a *Kähler manifold* if $\nabla^i J_i = 0$.

Lemma 2.2. *Let J_i be a $(1, 1)$-tensor field on a Riemannian manifold (M_i, g_i), $i = 1, 2$. Then, $\tilde{J} := J_1P_1 + J_2P_2$ satisfies:*

$$\begin{cases} (\tilde{\nabla}_X\tilde{J})Y &= (\nabla^i_X J_i)Y - \tilde{g}(X, J_iY)\nabla(\ln f_j) + \tilde{g}(X, Y)\tilde{J}(\nabla(\ln f_j)), \ \ (\forall)\ X, Y \in \Gamma(TM_i), \ \ j \neq i, \\ (\tilde{\nabla}_X\tilde{J})Y &= (J_jY)(\ln f_j)X - Y(\ln f_j)J_iX, \ \ (\forall)\ X \in \Gamma(TM_i), Y \in \Gamma(TM_j), \ \ j \neq i. \end{cases}$$

Proof. For $j \neq i$, we have [3]:

$$\begin{cases} \tilde{\nabla}_X Y = \nabla^i_X Y - \tilde{g}(X, Y)\nabla(\ln f_j), \ (\forall)\ X, Y \in \Gamma(TM_i), \\ \tilde{\nabla}_X Y = X(\ln f_i)Y + Y(\ln f_j)X, \ (\forall)\ X \in \Gamma(TM_i), Y \in \Gamma(TM_j), \end{cases}$$

where ∇f denotes the gradient of a function f on the doubly warped product manifold.

Then, for any $X, Y \in \Gamma(TM_i)$ and $j \neq i$, we have

$$\begin{aligned}(\tilde{\nabla}_X \tilde{J})Y &:= \tilde{\nabla}_X \tilde{J}Y - \tilde{J}(\tilde{\nabla}_X \tilde{Y}) \\ &= \tilde{\nabla}_X J_i Y - \tilde{J}(\nabla^i_X Y - \tilde{g}(X, Y)\nabla(\ln f_j)) \\ &= \nabla^i_X J_i Y - \tilde{g}(X, J_i Y)\nabla(\ln f_j) - J_i(\nabla^i_X Y) + \tilde{g}(X, Y)\tilde{J}(\nabla(\ln f_j)) \\ &= (\nabla^i_X J_i)Y - \tilde{g}(X, J_i Y)\nabla(\ln f_j) + \tilde{g}(X, Y)\tilde{J}(\nabla(\ln f_j)),\end{aligned}$$

and, for any $X \in \Gamma(TM_i)$ and $Y \in \Gamma(TM_j)$, $j \neq i$, we have

$$\begin{aligned}(\tilde{\nabla}_X \tilde{J})Y &:= \tilde{\nabla}_X \tilde{J}Y - \tilde{J}(\tilde{\nabla}_X \tilde{Y}) \\ &= \tilde{\nabla}_X J_j Y - \tilde{J}(X(\ln f_i)Y + Y(\ln f_j)X) \\ &= X(\ln f_i)J_j Y + (J_j Y)(\ln f_j)X - X(\ln f_i)J_j Y - Y(\ln f_j)J_i X \\ &= (J_j Y)(\ln f_j)X - Y(\ln f_j)J_i X. \quad \square\end{aligned}$$

We remark that, if we take $f_1 = f_2 = 1$, then, $(g_1 + g_2, \tilde{J})$ is an almost product Riemannian (or, an almost Hermitian) structure on $\tilde{M}$, for (g_i, J_i), $i = 1, 2$, almost product Riemannian (or, almost Hermitian) structures on M_i, $i = 1, 2$.

For J_i, $i = 1, 2$, almost product (or, almost complex) structures on M_i, $i = 1, 2$, we shall further call $\tilde{J} := J_1 P_1 + J_2 P_2$ *the naturally induced almost product* (or, *almost complex*) *structure* on the product manifold $\tilde{M}$. Moreover, for (g_i, J_i), $i = 1, 2$, almost product Riemannian (or, almost Hermitian) structures on M_i, $i = 1, 2$, we shall call $(\tilde{g}, \tilde{J})$ *the naturally induced almost product Riemannian* (or, *almost Hermitian*) *structure* on the doubly warped product manifold $(\tilde{M}, \tilde{g})$.

3. Some triviality conditions for doubly warped products

Let (M, g) be a Riemannian manifold, and let ∇ be the Levi-Civita connection of g. We recall that a g-symmetric $(1, 1)$-tensor field J on (M, g) is called a *Codazzi tensor field* if

$$(\nabla_X J)Y = (\nabla_Y J)X$$

for any $X, Y \in \Gamma(TM)$, and a $(1, 1)$-tensor field J is called *parallel* if

$$(\nabla_X J)Y = 0$$

for any $X, Y \in \Gamma(TM)$.

3.1. Locally product factors

Proposition 3.1. *Let (M_i, g_i, J_i), $i = 1, 2$, be almost product Riemannian manifolds, and let $\tilde{J} := J_1 P_1 + J_2 P_2$. Then, $\tilde{J}$ is a Codazzi tensor field if and only if J_1 and J_2 are Codazzi tensor fields, and*

$$(J_i X)(\ln f_i)Y - X(\ln f_i)J_j Y = (J_j Y)(\ln f_j)X - Y(\ln f_j)J_i X$$

for any $X \in \Gamma(TM_i)$ and $Y \in \Gamma(TM_j)$, $j \neq i$.

Proof. From Lemma 2.2, we obtain

$$(\tilde{\nabla}_X \tilde{J})Y - (\tilde{\nabla}_Y \tilde{J})X = (\nabla^i_X J_i)Y - (\nabla^i_Y J_i)X$$

for any $X, Y \in \Gamma(TM_i)$, and

$$(\tilde{\nabla}_X \tilde{J})Y - (\tilde{\nabla}_Y \tilde{J})X = (J_j Y)(\ln f_j)X - Y(\ln f_j)J_i X - (J_i X)(\ln f_i)Y + X(\ln f_i)J_j Y$$

for any $X \in \Gamma(TM_i)$ and $Y \in \Gamma(TM_j)$, $j \neq i$, hence the conclusion. $\square$

Theorem 3.2. *Let (M_i, g_i, J_i), $i = 1, 2$, be almost product Riemannian manifolds, and let $\tilde{J} := J_1P_1 + J_2P_2$.*

(i) Then, $\tilde{J}$ is a Codazzi tensor field on M_i if and only if J_i is a Codazzi tensor field.

(ii) If M_1 and M_2 are connected, and J_1 and J_2 are parallel, then $\tilde{J}$ is parallel if and only if f_1 and f_2 are constant. In this case, $(\tilde{M}, \tilde{g})$ is a direct product manifold.

Proof. From Lemma 2.2, we get

$$(\tilde{\nabla}_X\tilde{J})Y - (\tilde{\nabla}_Y\tilde{J})X = (\nabla^i_X J_i)Y - (\nabla^i_X J_i)Y$$

for any $X, Y \in \Gamma(TM_i)$, and we get (i).

Again, from Lemma 2.2, we deduce that $\tilde{\nabla}\tilde{J} = 0$ if and only if

$$\begin{cases} (\nabla^i_X J_i)Y &= \tilde{g}(X, J_iY)\nabla(\ln f_j) - \tilde{g}(X, Y)\tilde{J}(\nabla(\ln f_j)), \ (\forall)\ X, Y \in \Gamma(TM_i),\ j \neq i, \\ (J_jY)(\ln f_j)X &= Y(\ln f_j)J_iX, \ (\forall)\ X \in \Gamma(TM_i), Y \in \Gamma(TM_j),\ j \neq i. \end{cases}$$

For (ii), if $\tilde{\nabla}\tilde{J} = 0$, since $\nabla^i J_i = 0$, $i = 1, 2$, from the first relation we get

$$\tilde{g}(X, J_iY)\nabla(\ln f_j) = \tilde{g}(X, Y)\tilde{J}(\nabla(\ln f_j))$$

for any $X, Y \in \Gamma(TM_i)$, $j \neq i$, and, by applying $\tilde{J}$, we infer

$$\tilde{g}(X, J_iY)\tilde{J}(\nabla(\ln f_j)) = \tilde{g}(X, Y)\nabla(\ln f_j)$$

for any $X, Y \in \Gamma(TM_i)$, $j \neq i$, and we obtain

$$\nabla(\ln f_j) = 0.$$

Since M_j is connected, we deduce that f_j, $j = 1, 2$, is constant; hence, $(\tilde{M}, \tilde{g})$ is a direct product manifold. The converse implication is obvious. And we proved (ii). □

Hence, we have

Corollary 3.3. *There do not exist doubly warped products which are locally product Riemannian manifolds (with respect to the naturally induced almost product structure) with connected locally product Riemannian factors, which are not direct products.*

3.2. Kähler factors

Now we shall focus on the Kähler case.

Theorem 3.4. *Let (M_i, g_i, J_i), $i = 1, 2$, be almost Hermitian manifolds, and let $\tilde{J} := J_1P_1 + J_2P_2$.*

(i) If M_j is connected, then $(\tilde{\nabla}_X\tilde{J})Y = 0$ for any $X \in \Gamma(TM_i)$ and $Y \in \Gamma(TM_j)$, $i \neq j$, if and only if f_j is constant. In this case, $(\tilde{M}, \tilde{g})$ is a warped product manifold.

(ii) If M_i is connected and J_j, $j \neq i$, is parallel, then $\tilde{J}$ is parallel on M_j if and only if f_i is constant. In this case, $(\tilde{M}, \tilde{g})$ is a warped product manifold.

(iii) If M_1 and M_2 are connected, and J_1 and J_2 are parallel, then $\tilde{J}$ is parallel if and only if f_1 and f_2 are constant. In this case, $(\tilde{M}, \tilde{g})$ is a direct product manifold.

Proof. From Lemma 2.2, we deduce that $\tilde{\nabla}\tilde{J} = 0$ if and only if

$$\begin{cases} (\nabla^i_X J_i)Y &= \tilde{g}(X, J_iY)\nabla(\ln f_j) - \tilde{g}(X, Y)\tilde{J}(\nabla(\ln f_j)), \ (\forall)\ X, Y \in \Gamma(TM_i), j \neq i, \\ (J_jY)(\ln f_j)X &= Y(\ln f_j)J_iX, \ (\forall)\ X \in \Gamma(TM_i), Y \in \Gamma(TM_j), j \neq i. \end{cases}$$

If $(\tilde{\nabla}_X\tilde{J})Y = 0$ for any $X \in \Gamma(TM_i)$ and $Y \in \Gamma(TM_j)$, $i \neq j$, from the second relation we get

$$(J_jY)(\ln f_j)X = Y(\ln f_j)J_iX$$

for any $X \in \Gamma(TM_i)$ and $Y \in \Gamma(TM_j)$, $i \neq j$, and by applying J_i, we infer

$$(J_jY)(\ln f_j)J_iX = -Y(\ln f_j)X$$

for any $X \in \Gamma(TM_i)$, $Y \in \Gamma(TM_j)$, $j \neq i$, and we obtain

$$\Big((J_jY)(\ln f_j)\Big)^2 + \Big(Y(\ln f_j)\Big)^2 = 0$$

for any $Y \in \Gamma(TM_j)$. Since M_j is connected, we deduce that f_j is constant; hence, $(\tilde{M}, \tilde{g})$ is a warped product manifold. The converse implication is obvious. And we proved (i).

If $(\tilde{\nabla}_X\tilde{J})Y = 0$ for any $X, Y \in \Gamma(TM_j)$, since $\nabla^jJ_j = 0$, from the first relation we get

$$\tilde{g}(X, J_jY)\nabla(\ln f_i) = \tilde{g}(X, Y)\tilde{J}(\nabla(\ln f_i))$$

for any $X, Y \in \Gamma(TM_j)$, $i \neq j$, and, by applying $\tilde{J}$, we infer

$$\tilde{g}(X, J_jY)\tilde{J}(\nabla(\ln f_i)) = -\tilde{g}(X, Y)\nabla(\ln f_i)$$

for any $X, Y \in \Gamma(TM_j)$, $i \neq j$, and we obtain

$$\nabla(\ln f_i) = 0.$$

Since M_i is connected, we deduce that f_i is constant; hence, $(\tilde{M}, \tilde{g})$ is a warped product manifold. The converse implication is obvious. And we proved (ii). Then, we immediately obtain (iii) by means of (ii). □

Hence, we have

Corollary 3.5. *There do not exist doubly warped products which are Kähler manifolds (with respect to the naturally induced almost complex structure) with connected Kähler factors, which are not direct products.*

3.3. Slant doubly warped products

All the results of this section are valid both in the almost product Riemannian as well as in the almost Hermitian setting. We shall further consider the almost Hermitian case.

Let M_i be a submanifold of an almost Hermitian manifold (M, g, J_i), $i = 1, 2$, defined by an injective immersion. We have the orthogonal decomposition

$$TM = TM_i \oplus T^{\perp}M_i\,,$$

and, for any $X \in \Gamma(TM_i)$, we denote

$$J_iX = T_iX + N_iX,$$

where $T_iX \in \Gamma(TM_i)$ and $N_iX \in \Gamma(T^{\perp}M_i)$ represent the tangential and the normal component of J_iX, respectively.

In view of [2, 4], we call M_i a *pointwise slant submanifold* of (M, g, J_i) if, for any $X \in \Gamma(TM_i) \setminus \{0\}$ and $x \in M_i$ such that $X_x \neq 0$, the angle between J_iX_x and T_xM_i is nonzero and does not depend on the tangent vector X_x but only on the point x of M_i. In this case, denoting by θ_i, $\theta_i(x) \in (0, \frac{\pi}{2}]$ for any $x \in M_i$, the slant function, for any $X \in \Gamma(TM_i) \setminus \{0\}$, we have

$$\|T_iX\|^2 = \cos^2\theta_i \cdot \|X\|^2.$$

Moreover, if $\theta_i(x) \neq \frac{\pi}{2}$ for any $x \in M_i$, then M_i is called a *proper pointwise slant submanifold*.

If the angle between J_iX_x and T_xM_i does not depend on the nonzero tangent vector X_x, neither on the point x of M_i, then M_i is called a *slant submanifold* of (M, g, J_i), with constant slant angle θ_i (in particular, a *proper slant submanifold* if $\theta_i \neq \frac{\pi}{2}$, and an *anti-invariant submanifold* if $\theta_i = \frac{\pi}{2}$).

Let M_i be a pointwise slant submanifold of the almost Hermitian manifold (M, g, J_i), with the slant function θ_i, $i = 1, 2$, and let $\tilde{J} := J_1P_1 + J_2P_2$ be the naturally induced $(g+g)$-skew-symmetric almost complex structure on the direct product manifold $(M \times M, g + g)$. We prove the following result.

Proposition 3.6. *$M_1 \times M_2$ is a pointwise slant submanifold of the almost Hermitian manifold $(M \times M, g + g, \tilde{J})$, with a slant function θ, if and only if θ_1 and θ_2 are constant, equal to the same value. In this case, θ is also constant, has the same value like them, and the submanifolds $M_1 \times M_2$, M_1, and M_2 are slant.*

Proof. Let $X_i \in \Gamma(TM_i)$, $i = 1, 2$. We denote by $T_i X_i$ the tangential component of $J_i X_i$, $T_i X_i \in \Gamma(TM_i)$. For any $(x_1, x_2) \in M_1 \times M_2$, we identify $T_{(x_1,x_2)}(M_1 \times M_2)$ with $T_{x_1}M_1 \oplus T_{x_2}M_2$ and further,

$$T(M_1 \times M_2) \cong \pi_1^*(TM_1) \oplus \pi_2^*(TM_2).$$

Let $\tilde{X} \in \Gamma(T(M_1 \times M_2))$. Then, $\tilde{X} = X_1 + X_2$, $X_i \in \Gamma(TM_i)$, $i = 1, 2$. We denote by $\tilde{T}\tilde{X}$ the tangential component of $\tilde{J}\tilde{X}$, $\tilde{T}\tilde{X} \in \Gamma(T(M_1 \times M_2))$. Then, $\tilde{T}\tilde{X} = T_1X_1 + T_2X_2$ and we get

$$\begin{aligned} \|\tilde{T}\tilde{X}\|_{g+g}^2 &= \|T_1X_1\|_g^2 + \|T_2X_2\|_g^2 \\ &= \cos^2\theta_1 \cdot \|X_1\|_g^2 + \cos^2\theta_2 \cdot \|X_2\|_g^2, \\ \|\tilde{X}\|_{g+g}^2 &= \|X_1\|_g^2 + \|X_2\|_g^2; \end{aligned}$$

hence, $M_1 \times M_2$ is a pointwise slant submanifold of the almost Hermitian manifold $(M \times M, g + g, \tilde{J})$ with the slant function θ if and only if

$$\cos^2\theta(\|X_1\|_g^2 + \|X_2\|_g^2) = \cos^2\theta_1 \cdot \|X_1\|_g^2 + \cos^2\theta_2 \cdot \|X_2\|_g^2$$

for any $X_1 \in \Gamma(TM_1)$ and $X_2 \in \Gamma(TM_2)$, equivalent to

$$(\cos^2\theta - \cos^2\theta_1)\|X_1\|_g^2 = -(\cos^2\theta - \cos^2\theta_2)\|X_2\|_g^2$$

for any $X_1 \in \Gamma(TM_1)$ and $X_2 \in \Gamma(TM_2)$, and we get the conclusion. □

Remark 3.7. *Obviously, for f_1 and f_2 two positive smooth functions on M_1 and M_2, respectively, if we consider the metric $\tilde{g} := f_2^2 g_1 + f_1^2 g_2$ on $M_1 \times M_2$, then the doubly warped product ${}_{f_2}M_1 \times_{f_1} M_2$ is a pointwise slant submanifold of the almost Hermitian manifold $(M \times M, \tilde{g}, \tilde{J})$, with a slant function θ, if and only if θ_1 and θ_2 are constant, equal to the same value. In this case, θ is also constant, has the same value like them, and the submanifolds $M_1 \times M_2$, M_1, and M_2 are slant. Indeed, we just notice that, for any $\tilde{X} \in \Gamma(T(M_1 \times M_2))$, $\tilde{X} = X_1 + X_2$, $X_i \in \Gamma(TM_i)$, $i = 1, 2$, we have:*

$$\begin{aligned} \|\tilde{T}\tilde{X}\|_{\tilde{g}}^2 &= f_2^2\|T_1X_1\|_g^2 + f_1^2\|T_2X_2\|_g^2 \\ &= f_2^2\cos^2\theta_1 \cdot \|X_1\|_g^2 + f_1^2\cos^2\theta_2 \cdot \|X_2\|_g^2, \\ \|\tilde{X}\|_{\tilde{g}}^2 &= f_2^2\|X_1\|_g^2 + f_1^2\|X_2\|_g^2; \end{aligned}$$

hence, $M_1 \times M_2$ is a pointwise slant submanifold of $(M \times M, \tilde{g}, \tilde{J})$ with the slant function θ if and only if

$$\cos^2\theta(f_2^2\|X_1\|_g^2 + f_1^2\|X_2\|_g^2) = f_2^2\cos^2\theta_1 \cdot \|X_1\|_g^2 + f_1^2\cos^2\theta_2 \cdot \|X_2\|_g^2$$

for any $X_1 \in \Gamma(TM_1)$ and $X_2 \in \Gamma(TM_2)$, equivalent to

$$f_2^2(\cos^2\theta - \cos^2\theta_1)\|X_1\|_g^2 = -f_1^2(\cos^2\theta - \cos^2\theta_2)\|X_2\|_g^2$$

for any $X_1 \in \Gamma(TM_1)$ and $X_2 \in \Gamma(TM_2)$, and we get the conclusion.

Hence, we have

Corollary 3.8. *(i) Any doubly warped product manifold which is a pointwise slant submanifold (with respect to the naturally induced almost Hermitian structure) is slant, its factors are also slant, and all have the same slant angle.*

(ii) There do not exist doubly warped product manifolds which are pointwise slant but not slant submanifolds (with respect to the naturally induced almost Hermitian structure), with pointwise slant factors.

We shall further underline a case when a doubly warped product has to be a direct product.

Example 1. *If M_1 and M_2 are proper pointwise slant submanifolds (i.e., $\theta_i - \frac{\pi}{2}$ is nowhere zero on M_i, $i = 1, 2$), and if the warping functions are $f_i(x_i) = \cos\theta_i(x_i)$, $x_i \in M_i$, $i = 1, 2$, so $\tilde{g} = (\cos^2\theta_2)g_1 + (\cos^2\theta_1)g_2$, then we deduce:*

(i) ${}_{(\cos\theta_2)}M_1 \times_{(\cos\theta_1)} M_2$ is a pointwise slant submanifold of $(M \times M, \tilde{g}, \tilde{J})$ if and only if θ_1 and θ_2 are constant, i.e., if the manifold is a direct product.

(ii) There do not exist slant (or pointwise slant) doubly warped product submanifolds ${}_{(\cos\theta_2)}M_1 \times_{(\cos\theta_1)} M_2$ of $(M \times M, \tilde{g}, \tilde{J})$ with proper slant (or pointwise slant) submanifold factors having the slant angle (or slant functions) θ_i, $i = 1, 2$, which are not direct products.

The result from Proposition 3.6 can be extended as follows.

Proposition 3.9. *Let M_i be a pointwise slant submanifold of an almost Hermitian manifold $(\bar{M}_i, g_i, J_i)$, with the slant function θ_i, $i = 1, 2$, and let $\tilde{J} := J_1P_1 + J_2P_2$. Then, $M_1 \times M_2$ is a pointwise slant submanifold of the almost Hermitian manifold $(\bar{M}_1 \times \bar{M}_2, g_1 + g_2, \tilde{J})$, with a slant function θ, if and only if θ_1 and θ_2 are constant, equal to the same value. In this case, θ is also constant, has the same value like them, and the submanifolds $M_1 \times M_2$, M_1, and M_2 are slant.*

Proof. It follows by repeating the steps from the proof of Proposition 3.6. □

As a generalization, we have

Proposition 3.10. *Let M_i be a pointwise slant submanifold of an almost Hermitian manifold $(\bar{M}_i, g_i, J_i)$, with the slant function θ_i, $i = \overline{1,k}$, and let $\tilde{J} := \sum_{i=1}^{k} J_iP_i$. Then, $M_1 \times \cdots \times M_k$ is a pointwise slant submanifold of the almost Hermitian manifold $(\bar{M}_1 \times \cdots \times \bar{M}_k, g_1 + \cdots + g_k, \tilde{J})$, with a slant function θ, if and only if θ_i, $i = \overline{1,k}$, are constant, equal to the same value. In this case, θ is also constant, has the same value like them, and the submanifolds $M_1 \times \cdots \times M_k$, $M_1, \ldots, M_k$ are slant.*

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