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Some notes on topology of partially metric spaces

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Abstract. In this paper, we introduce a topology which is weaker than the one introduced by Matthews on partial metric spaces. We present some examples and rolls for our results. Also, we show that the condition $p(x, x) \leq p(x, y)$ is redundant in the initial definition of partial metric.

1. Introduction

After introducing partial metric spaces by Matthews in [10] many papers are written especially in fixed point theory all of them turn on $p(a, a)$ is not zero. In this paper we make a weaker than its topology and we remove the condition $p(x, x) \leq p(x, y)$ in the following main definition of the partial metric. See the more references in [1–9, 11]

Definition 1.1 ([10]). *Let X be a nonempty set and $p : X \times X \to \mathbb{R}^+$ be a self mapping of X such that for all $x, y, z \in X$ the followings are satisfied:*

$p1$ $x = y \iff p(x, x) = p(x, y) = p(y, y)$,

$p2$ $p(x, x) \leq p(x, y)$,

$p3$ $p(x, y) = p(y, x)$,

$p4$ $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Then p is called partial metric on X and the pair (X, p) is called partial metric space (in short PMS).

At first, we show that the condition $p2$ is redundant in Definition 1.1 of partial metric. By $p4$ if we put $y = x$, then

$$p(x, x) \leq p(x, z) + p(z, x) - p(z, z).$$

$$p(x, x) + p(z, z) \leq 2p(x, z).$$

Now we have two cases: $p(x, x) \leq p(z, z)$ or $p(z, z) \leq p(x, x)$. So in the each case

$$2p(x, x) \leq p(x, x) + p(z, z) \leq 2p(x, z) \Rightarrow p(x, x) \leq p(x, z)$$

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or

$$2p(z,z) \leq p(x,x) + p(z,z) \leq 2p(x,z) \Rightarrow p(z,z) \leq p(x,z).$$

So $p(x,x) \leq p(x,y)$, for every $x, y \in X$.

Note also that each partial metric p on X generates a T_0 topology τ_p on X, whose base is a family of open p-balls

$$\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$$

where

$$B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\},$$

for all $x \in X$ and $\varepsilon > 0$.

It's time to introduce new definition of partial metric.

Definition 1.2. *Let X be a nonempty set and $p : X \times X \to \mathbb{R}^+$ be a self mapping of X such that for all $x, y, z \in X$ the followings are satisfied:*

$p1$ $p(x,x) = p(x,y) = p(y,y) \iff x = y,$

$p3$ $p(x,y) = p(y,x),$

$p4$ $p(x,y) \leq p(x,z) + p(z,y) - p(z,z).$

Then p is called partial metric on X and the pair (X, p) is called partial metric space.

Put

$$d(x,y) := p(x,y) - \min\{p(x,x), p(y,y)\} + k|p(x,x) - p(y,y)|, \tag{1}$$

where $k \in (0,1)$.

Proposition 1.3. *d is a metric on X.*

Proof. We see that,

1. If $x = y$, then
$$d(x,x) = p(x,x) - \min\{p(x,x), p(x,x)\} + k|p(x,x) - p(x,x)| = 0.$$
2. And if $d(x,y) = 0$, then
$$p(x,y) - \min\{p(x,x), p(y,y)\} + k|p(x,x) - p(y,y)| = 0.$$
So
$$p(x,y) \leq p(x,y) + k|p(x,x) - p(y,y)| = \min\{p(x,x), p(y,y)\} \leq p(x,y).$$
Thus $p(x,y) = p(x,x)$ or $p(x,y) = p(y,y)$. Hence
$$p(x,y) + k|p(x,x) - p(y,y)| = p(x,y) \Rightarrow p(x,x) = p(y,y).$$
Therefore $p(x,y) = p(x,x) = p(y,y)$ which means $x = y$.
3. Symmetry is obvious.
4. For triangle inequality, by the following inequality
$$\min\{a,c\} + \min\{c,b\} \leq \min\{a,b\} + c \quad \forall a,b,c \in \mathbb{R}^+,$$
we have
$$\begin{aligned} d(x,y) &= p(x,y) - \min\{p(x,x), p(y,y)\} + k|p(x,x) - p(y,y)| \\ &\leq p(x,z) + p(z,y) - p(z,z) \\ &- \min\{p(x,x), p(z,z)\} - \min\{p(z,z), p(y,y)\} + p(z,z) \\ &+ k|p(x,x) - p(z,z)| + k|p(z,z) - p(y,y)| \\ &\leq d(x,z) + d(x,z). \end{aligned}$$

□

2. Main results

We define weak topology τ_d by the balls

$$B_d^k(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\},$$

for every $k \in (0, 1)$.

$$\forall x(x \neq y) \quad \text{put} \quad \varepsilon := p(x, y) - \min\{\rho(x, x), \rho(y, y)\} + k|\rho(x, x) - \rho(y, y)|,$$

then $y \notin B_d^k(x, \varepsilon)$, which means τ_d is T_0.

Theorem 2.1. *Balls $B_d^k(x, \varepsilon)$ for every $x \in X$ and $\varepsilon > 0$ makes a base for topology τ_d.*

Proof. Let

$$y \in B_d^k(x, \varepsilon) \Rightarrow p(x, y) - \min\{p(x, x), p(y, y)\} + k|p(x, x) - \rho(y, y)| < \varepsilon.$$

Our claim is

$$\exists \delta > 0 \quad B_d^k(y, \delta) \subseteq B_d^k(x, \varepsilon).$$

It's enough that, we put

$$\delta := \varepsilon - (p(x, y) - \min\{p(x, x), p(y, y)\} + k|p(x, x) - p(y, y)|).$$

$$z \in B_d^k(y, \delta) \Rightarrow p(z, y) - \min\{p(z, z), p(y, y)\} + k|\rho(z, z) - \rho(y, y)| < \delta,$$

thus

$$\begin{aligned} & p(z, y) - \min\{p(z, z), p(y, y)\} + k|p(z, z) - p(y, y)| \\ < \quad & \varepsilon - (p(x, y) - \min\{p(x, x), p(y, y)\} + k|p(x, x) - p(y, y)|), \end{aligned}$$

therefore

$$\begin{aligned} & p(z, y) - \min\{p(z, z), p(y, y)\} + k|p(z, z) - p(y, y)| \\ + \quad & p(x, y) - \min\{p(x, x), p(y, y)\} + k|p(x, x) - p(y, y)| < \varepsilon \end{aligned} \tag{2}$$

so we obtain

$$\begin{aligned} & p(x, z) - \min\{p(x, x), p(z, z)\} + k|p(x, x) - p(z, z)| \\ \leq & p(x, y) + p(y, z) - p(y, y) - \min\{p(x, x), p(z, z)\} + k|p(x, x) - p(z, z)| \\ \leq & p(x, y) - \min\{p(x, x), p(y, y)\} + k|p(x, x) - p(y, y)| \\ + & p(y, z) - \min\{p(y, y), p(z, z)\} + k|p(y, y) - p(z, z)| < \varepsilon \end{aligned}$$

therefore by (2)

$$p(x, z) - \min\{p(x, x), p(z, z)\} + k|p(x, x) - p(z, z)| \leq \varepsilon \Rightarrow z \in B_d^k(x, \varepsilon).$$

□

Theorem 2.2. *Topology τ_d is weaker than topology τ_p.*

Proof. Put $y \in B_d^k(x, \varepsilon)$. Hence

$$p(x, y) - \min\{p(x, x), p(y, y)\} + k|p(x, x) - p(y, y)| < \varepsilon$$

thus

$$p(x, y) - p(x, x) \leq \rho(x, y) - \min\{\rho(x, x), \rho(y, y)\} + k|\rho(x, x) - \rho(y, y)| < \varepsilon$$

$$p(x, y) - p(x, x) < \varepsilon \Rightarrow y \in B_p(x, \varepsilon)$$

which means $B_d^k(x, \varepsilon) \subseteq B_p(x, \varepsilon)$. □

3. Second weak topology

If we put

$$D(x,y) := p(x,y) - \min\{p(x,x), p(y,y)\} \tag{3}$$

and

$$B_D(x,\varepsilon) = \{y \in X : D(x,y) < \varepsilon\},$$

then

$$\bigcap_{k\in(0,1)} B_d^k(x,\varepsilon) = B_D(x,\varepsilon).$$

Also, we know that

$$p(x,y) - p(x,x) \leq D(x,y) := p(x,y) - \min\{p(x,x), p(y,y)\}.$$

We define weak topology τ_D which is T_0, by the balls

$$B_D(x,\varepsilon) = \{y \in X : D(x,y) < \varepsilon\}.$$

Remark 3.1. *Dis not a metric. Put $X := \{1,2\}$ and define p as follows:*

$$p(1,1) = 1,\ p(2,2) = 2,\ p(1,2) = p(2,1) = 3,$$

So p is a partial metric and $D(2,2) = p(2,2) - \min\{p(1,1), p(2,2)\} = 2 - 1 = 1$.

Theorem 3.2. *Balls $B_D(x,\varepsilon)$ for every $x \in X$ and $\varepsilon > 0$ makes a base for topology τ_D.*

Proof. It's similar to proof Theorem 2.1. □

Theorem 3.3. *Topology τ_d is weaker than topology τ_D and topology τ_D is weaker than topology τ_p.*

Proof. Put $y \in B_d^k(x,\varepsilon)$. Hence

$$p(x,y) - \min\{p(x,x), p(y,y)\} + k|p(x,x) - p(y,y)| < \varepsilon$$

thus

$$p(x,y) - p(x,x) \leq p(x,y) - \min\{p(x,x), p(y,y)\} + k|p(x,x) - p(y,y)| < \varepsilon$$

$$p(x,y) - p(x,x) \leq D(x,y) \leq d(x,y) < \varepsilon \Rightarrow y \in B_D(x,\varepsilon) \subseteq B_p(x,\varepsilon).$$

which means $B_d^k(x,\varepsilon) \subseteq B_D(x,\varepsilon) \subseteq B_p(x,\varepsilon)$. □

Definition 3.4. *Let (X,p) be a partial metric space. Then*

- *a sequence $\{a_n\}$ in (X,p) is said to be convergent to a point $a \in X$ if and only if*

$$\lim_{n\to\infty} d(a_n,a) = 0 \iff a_n \xrightarrow{\tau_d} a.$$

$$(\lim_{n\to\infty} D(a_n,a) = 0 \iff a_n \xrightarrow{\tau_D} a).$$

- *a sequence $\{a_n\}$ is called a Cauchy sequence if*

$$\lim_{m,n\to\infty} d(a_m, a_n) \quad (\lim_{m,n\to\infty} D(a_m, a_n))$$

exists and finite;

- *(X, p) is said to be complete if every Cauchy sequence $\{a_n\}$ in X converges to a point $a \in X$ with respect to τ_d. Furthermore,*

$$\lim_{m,n\to\infty} d(a_m, a_n) = \lim_{n\to\infty} d(a, a_n) = 0$$

- *A mapping $f : X \to X$ is said to be continuous at $a_0 \in X$ if for*

$$\forall \varepsilon > 0\ \exists \delta > 0 \quad f(B^k_d(a_0, \delta)) \subseteq B^k_d(f(a_0), \varepsilon).$$

$$(\forall \varepsilon > 0\ \exists \delta > 0 \quad f(B_D(a_0, \delta)) \subseteq B_D(f(a_0), \varepsilon)).$$

Example 3.5. *Let $X := \{1, 2, 3\}$, $x_n := 1$ and $x = 3$. Hence $x_n \to x$ in τ_p but $x_n \not\to x$ in τ_d, when $p(x, y) = \max\{x, y\}$.*

Example 3.6. *Let $X := \{\frac{n+1}{n} : n \in \mathbb{N}\} \cup \{1\}$, $x_n := \frac{n+1}{n}$ and $x = 1$. Hence $x_n \to x$ in τ_d, so $x_n \to x$ in τ_p, when $p(x, y) = \max\{x, y\}$.*

Lemma 3.7. *Let (X, p) be a partial metric space. If $\{a_n\}$ be a sequence in (X, p) such that $p(a_n, a_{n+1}) \to 0$ as $n \to \infty$. Then $d(a_n, a_{n+1}) \to 0$ as $n \to \infty$.*

Proof. By $p(a_n, a_n) \leq p(a_n, a_{n+1})$ so $p(a_n, a_n) \to 0$ as $n \to \infty$ with respect τ_p. Therefore $d(a_n, a_{n+1}) \to 0$ as $n \to \infty$. □

The next lemma states that converse convergent conditions in τ_d and τ_p topologies.

Lemma 3.8. *Let (X, p) be a partial metric space. If $a_n \xrightarrow{\tau_p} a$ and $\lim_{n\to\infty} p(a_n, a_n)$ exists. Then*

$$\lim_{n\to\infty} d(a_n, a) = \lim_{n\to\infty} D(a_n, a) = (k + 1)(p(a, a) - \lim_{n\to\infty} p(a_n, a_n)).$$

Further more $\lim_{n\to\infty} p(a_n, a_n) = p(a, a)$, then

$$\lim_{n\to\infty} d(a_n, a) = 0, \quad \text{and} \quad \lim_{n\to\infty} D(a_n, a) = 0,$$

or

$$a_n \xrightarrow{\tau_d} a, \quad \text{and} \quad a_n \xrightarrow{\tau_D} a.$$

Proof. According to

$$d(a_n, a) = p(a_n, a) - \min\{p(a, a), p(a_n, a_n)\} + k|p(a, a) - p(a_n, a_n)|$$

and

$$p(a_n, a_n) \leq p(a_n, a) + p(a, a_n) - p(a, a)$$

assertion is clear. □

About the condition $\lim_{n\to\infty} p(a_n, a_n) = p(a, a)$, in Lemma 3.8, look at Examples 3.5 and 3.6.

The next theorem is an application in fixed point theory as base on Banach's theorem.

Theorem 3.9. *Let (X, p) be a complete partial metric space. T a self mapping on X and*

$$p(Tx, Ty) - \min\{p(Tx, Tx), p(Ty, Ty)\} + k|p(Tx, Tx) - p(Ty, Ty)| \\ \leq l(p(x, y) - \min\{p(x, x), p(y, y)\} + k|p(x, x) - p(y, y)|),$$

for some $l \in [0, 1)$ and for every $x, y \in X$. Then T has a unique fixed point on X.

Proof. By Proposition 1.3, d is a metric and $d(Tx, Ty) \leq ld(x, y)$. □

By the new topology and metric d, many complicated contractions could be verified in the same way.

Corollary 3.10. *Let (X, p) be a complete partial metric space. T a self mapping on X and*

$$p(Tx, Ty) - \min\{p(Tx, Tx), p(Ty, Ty)\} \leq l(p(x, y) - \min\{p(x, x), p(y, y)\}),$$

for some $l \in [0, 1)$ and for every $x, y \in X$. Then T has a unique fixed point on X.

Proof. By Definition 3, $D(Tx, Ty) \leq lD(x, y)$. □

Conclusion

We introduce a weak topology for partial metric spaces with applying to fixed point theorem. Some illustrated examples are included. Also, we showed that the condition $p(x, x) \leq p(x, y)$ is redundant in the initial definition of partial metric.

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