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Rupture degree and weak rupture degree of gear graphs

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Abstract. In a communication network, several vulnerability measures are used to determine the resistance of the network to distruption of operation after the failure of certain stations or communication links. If we consider a graph as a model of a communication network, the rupture degree and weak rupture degree of a graph are the measures of graph vulnerability. In this paper, we determine the exact values of the rupture degree of G_n, where $n \geq 4$, and $G_m \square G_n$, where $m, n \geq 4$ and weak rupture degree of G_n, where $n \geq 4$, $\overline{G}_n$, where $n \geq 4$, $L(G_n)$ where $n \geq 4$, $K_2 \square G_n$, where $n \geq 4$ and $G_n \square G_m$, where $n, m \geq 4$.

1. Introduction

A communication network is composed of processors and communication links. Network designers attach importance to reliability and stability of a network. If the network begins losing processors or communication links, then there is a loss in its effectiveness. This event is called the *vulnerability* of the communication network. In a communication network, vulnerability measures the resistance of the network after a breakdown of some of its processors or communication links.

A communication network can be modeled by a graph G whose vertices represent the processors and whose edges represent the lines of communication. Many graph theoretical parameters such as connectivity edge connectivity, toughness,scattering number, integrity, rupture degree, tenacity etc. have been used in the past to describe the stability and reliability of communication networks. Among them, two basic parameters, connectivity and edge connectivity have been extensively used. The higher the connectivity (edge connectivity) of G, the more stable it is considered to be.

In an analysis of the vulnerability of networks to disruption, three important quantities, there may be others, that are considered seriously are (1) the number of elements that are not functioning, (2) the number of remaining connected subnetworks and (3) the size of a largest remaining group within which mutual communication can still occur. Connectivity is the parameter based on quantity (1). Both the toughness and the scattering number are based on quantities (1) and (2). The integrity is based on quantities (1) and (3). The tenacity and the rupture degree take into account of all the three quantities.

In [1], E. Aslan introduced the weak rupture degree of a graph to measure the vulnerability of networks. Geared systems are graph theoretic models that are obtained by using gear graphs are used in dynamic modelling. Similarly the cartesian product of gear graphs, the complement of a gear graph and the line graph of a gear graph can be used to design a gear network. Consequently these considerations motivated us to investigate the weak rupture degree of gear graphs.

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2. Preliminaries

All graphs considered here are finite, simple and undirected.

The *rupture degree* for a connected graph G is defined to be $r(G) = \max\{\omega(G-S) - |S| - m(G-S) : S \subset V(G), \omega(G-S) > 1\}$, where $\omega(G-S)$ is the number of components of $G-S$ and $m(G-S)$ is the order of a largest component of $G-S$. The rupture degree of a complete graph $r(K_n) = 1-n$.

The *weak rupture degree* of a connected graph G is defined to be $R_w(G) = max\{\omega(G-S) - |S| - m_e(G-S) : S \subseteq V(G), \omega(G-S) > 1\}$ where $\omega(G-S)$ is the number of the components of $G-S$ and $m_e(G-S)$ is the number of edges of the largest component of $G-S$. The weak rupture degree of a complete graph $R_w(K_n) = 2-n$.

The *tenacity* of an connected graph G is defined as $T(G) = \min\left\{\frac{|S|-m(G-S)}{\omega(G-S)} : S \subset V(G), \omega(G-S) > 1\right\}$, where $\omega(G-S)$ is the number of components of $G-S$ and $m(G-S)$ is the order of a largest component of $G-S$. The tenacity of a complete graph $T(K_n) = n$.

The *join* of simple graphs G and H, denoted by $G \vee H$, is the graph obtained from the disjoint union $G+H$ by adding the edges $\{xy : x \in V(G), y \in V(H)\}$.

The *wheel graph* W_n is the graph obtained from the join of the complete graph K_1 and the cycle C_{n-1}, which contains n vertices and $2(n-1)$ edges.

The *gear graph* G_n is a wheel graph with a vertex added between each pair of adjacent vertices of the outer cycle C_{n-1}. The gear graph G_n had $2n-1$ vertices and $3(n-1)$ edges.

The *cartesian product* of the graphs G and H, denoted by $G \,\square\, H$, has the vertex set $V(G \,\square\, H) = V(G) \times V(H)$ and $(u,x)(v,y)$ is an edge of $G \,\square\, H$ if (i) $u = v$ and $xy \in E(H)$ or, (ii) $x = y$ and $uv \in E(G)$.

The *line graph* $L(G)$ of a graph G is a graph such that each vertex of $L(G)$ represents an edge of G and any two vertices of $L(G)$ are adjacent if and only if their edges are adjacent in G, meaning they share a common end vertex in G.

The *complement* of a graph G is a graph $\overline{G}$ on the same vertices such that two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.

A subset S of V is called an *independent set* of G if no two vertices of S are adjacent in G. An independent set S is maximum if G has no independent set S' with $|S'| > |S|$. The independence number of G, $\alpha(G)$ is the number of vertices in a maximum independent set of G.

A subset S of V is called an covering of G if every edge of G has at least one end in S. A covering S is a minimum covering in G if G has no covering S' with $|S'| > |S|$. The covering number, $\beta(G)$, is the number of vertices in a minimum covering of G.

In this paper, the first integer larger than x is denoted by $\lceil x \rceil$, the first integer smaller than x is denoted by $\lfloor x \rfloor$ and the absolute value of x by $|x|$.

Lemma 2.1. [2] Let G be an incomplete connected graph of order n. Then

$$2\alpha(G) - n - 1 \leq r(G) \leq \frac{[\alpha(G)]^2 - \kappa(G)[\alpha(G) - 1] - n}{\alpha(G)}$$

where $r(G)$, $\alpha(G)$ and $\kappa(G)$ respectively, are the rupture degree, independence number and connectivity of G.

Theorem 2.2. [7]Let G is a incomplete connected graph, $\alpha(G)$ is the independence number of G and $\tau(G)$ is the tenacity of G. Then the rupture degree $r(G)$ is given by

$$r(G) \leq \alpha(G)(1 - \tau(G)).$$

Theorem 2.3. [6] Let $m, n \geq 4$ be a positive integers. Then the tenacity of $G_m \square G_n$ is

$$\tau(G_m \square G_n) = 1$$

where G_n is the gear graph of order n.

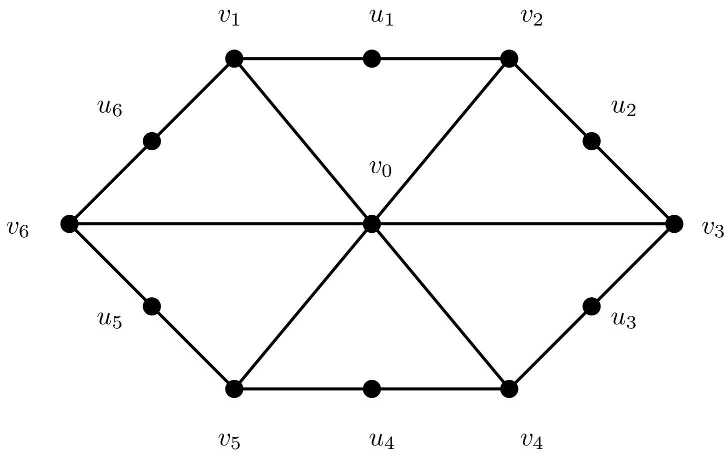


Figure 1: The graph G_7

Lemma 2.4. [6] Let $m, n \geq 4$, then the independence number of $G_m \square G_n$ is

$$\alpha(G_m \square G_n) = 2mn + m + n + 1.$$

Theorem 2.5. [8] If G is a graph of order n with independence number $\alpha(G)$ and covering number $\beta(G)$, then $R_w(G) \geq \alpha(G) - \beta(G)$.

Theorem 2.6. [8] Let G be a connected graph. Then $R_w(G) \leq r(G) + 1$.

Theorem 2.7. [5] Let G_n be a gear graph. Then $r(K_2 \square G_n) = -1$.

3. Rupture degree of gear graphs

In this section, we calculate the exact values of rupture degree of line graph of the gear graph G_n, where $n \geq 4$ and the rupture degree of the cartesian product $G_m \square G_n$ where $m, n \geq 4$.

Theorem 3.1. Let G_n be a gear graph of order n where $n \geq 4$. Then

$$r(L(G_n)) = -n.$$

Proof. Let S be an arbitrary vertex cut of $L(G_n)$ with $|S| = x$. $|V(L(G_n))| = 3(n-1)$. Then $L(G_n) - S$ has at most $\left\lceil \frac{x}{2} \right\rceil$ components. i.e.,

$$\omega(L(G_n) - S) \leq \left\lceil \frac{x}{2} \right\rceil$$

and

$$m(L(G_n) - S) \geq \left\lceil \frac{3(n-1) - x}{\left\lceil \frac{x}{2} \right\rceil} \right\rceil.$$

Since $\dfrac{3(n-1) - x}{\frac{x}{2}} \geq 1$, then x must be at most $2(n-1)$. Hence we have

$$r(L(G_n) \leq \max \left\{ \left\lceil \frac{x}{2} \right\rceil - x - \left\lceil \frac{3(n-1) - x}{\left\lceil \frac{x}{2} \right\rceil} \right\rceil \right\},$$

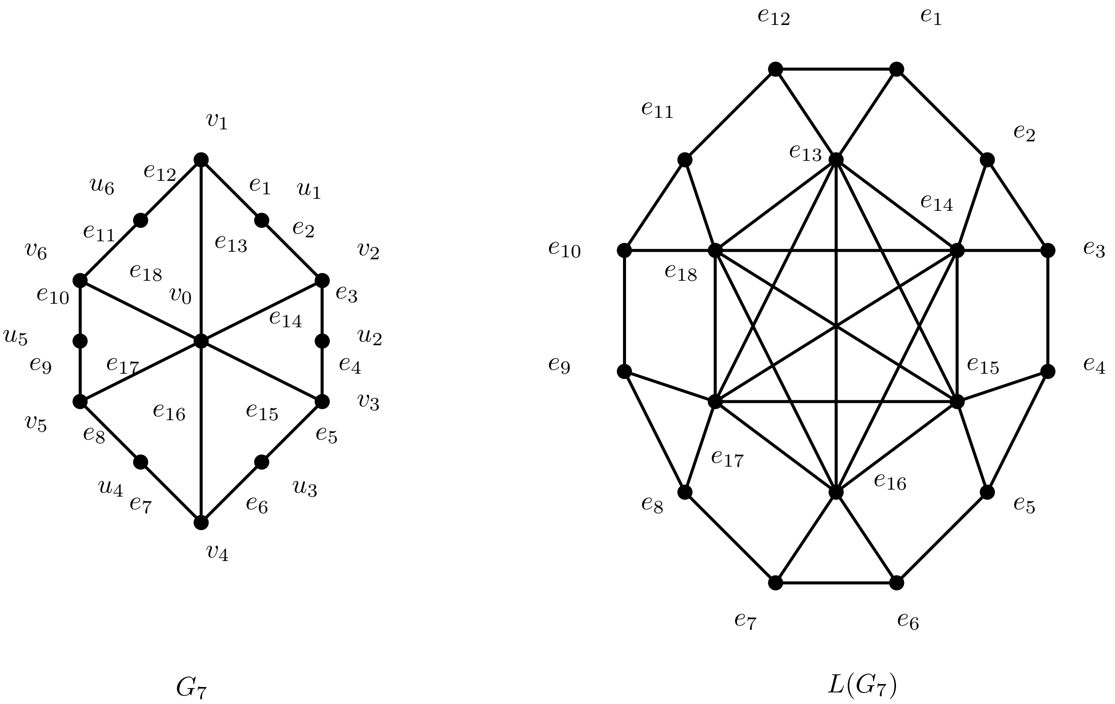


Figure 2: The graph $L(G_7)$

where $x \leq 2(n-1)$.
Consider the function

$$f(x) = \left\{ \frac{x}{2} - x - \frac{3(n-1) - x}{\frac{x}{2}} \right\},$$

Then

$$f(x+1) = \left\{ \frac{x+1}{2} - (x+1) - \frac{3(n-1) - (x+1)}{\frac{x+1}{2}} \right\},$$

and hence

$$f(x+1) - f(x) = \left\{ \frac{6(n-1)}{x} - \frac{6(n-1)}{x+1} - \frac{1}{2} \right\} \geq 0.$$

Hence $f(x)$ is an increasing function and also

$$f^{'}(x) = \left\{ \frac{6(n-1)}{x^2} - \frac{1}{2} \right\},$$

Since $2\sqrt{3(n-1)} \leq x \leq 2(n-1)$ and $f(x)$ is an increasing function, the function

$$f(x) = \left\{ \frac{x}{2} - x - \frac{3(n-1) - \frac{3x}{2}}{\frac{x}{2}} \right\}$$

takes its maximum value at $x = 2(n-1)$ and $f_{max}(x) = -n$. We have

$$r(L(G_n) \leq -n \tag{1}$$

On the other hand, the independence number of $L(G_n)$ is $\alpha(L(G_n)) = (n-1)$. By Lemma 2.1, we have

$$\begin{aligned} r(L(G_n)) &\geq 2\alpha(L(G_n)) - 3(n-1) - 1 \\ &\geq 2(n-1) - 3(n-1) - 1 \\ r(L(G_n)) &\geq -n. \end{aligned} \tag{2}$$

From (1) and (2)

$$r(L(G_n)) = -n.$$

Hence the result follows. □

Theorem 3.2. Let $m, n \geq 4$ be two positive integers. Then

$$r(G_m \Box G_n) = 0.$$

Proof. Let $G = G_m \Box G_n$, where $m, n \geq 4$.
By Theorem 2.3 $\tau(G) = 1$ and by Lemma 2.4, the independence number is $\alpha(G) = 2mn + m + n + 1$.
Now by Theorem 2.2, we have

$$\begin{aligned} r(G) &\leq \alpha(G)(1 - \tau(G)) \\ &\leq (2mn + m + n + 1)(1 - 1) = 0. \end{aligned} \tag{3}$$

On the other hand, we know that $|V(G)| = (2m-1)(2n-1)$ and by Lemma 2.4, $\alpha(G) = 2mn + m + n + 1$. By Lemma 2.1,

$$\begin{aligned} r(G) &\geq 2\alpha(G) - (2m-1)(2n-1) - 1 \\ &= 2(2mn + m + n + 1) - (2m-1)(2n-1) - 1 \\ &= 0. \end{aligned} \tag{4}$$

From (3) and (4)

$$r(G) = r(G_m \Box G_n) = 0.$$

Hence the result follows. □

4. Weak rupture degree of gear graphs

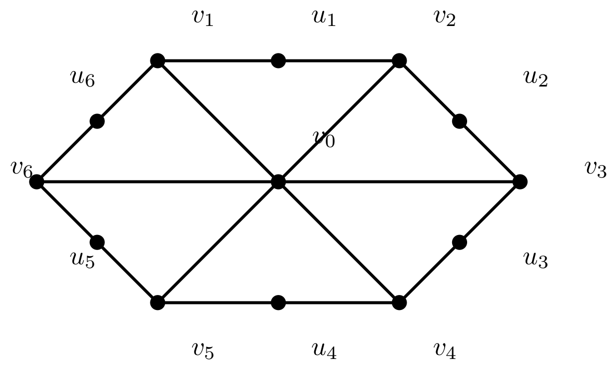
In this section, we calculate the exact values of weak rupture degree of G_n where $n \geq 4$, $\overline{G}_n$ where $n \geq 4$, $L(G_n)$ where $n \geq 4$, $K_2 \Box G_n$ where $n \geq 4$ and $G_n \Box G_m$ where $n, m \geq 4$.
Theorem 4.1. Let G_n be a gear graph of order n, $n \geq 4$. Then $R_w(G_n) = 1$

Proof. Let v_0 be the center vertex and $v_1, v_2, \ldots, v_{n-1}$ be the vertices of the outer $n-1$-cycle of W_n. We know that a gear graph G_n can be constructed from the wheel graph W_n by adding new vertices between each pair of adjacent vertices of the outer cycle of W_n. Let the new vertices be $u_1, u_2, \ldots, u_{n-1}$.

Let S be a subset of $V(G_n)$ such that $\omega(G-S) - |S| - m_e(G-S) = R_w(G_n)$. Then clearly S must contain all the vertices of $v_1, v_2, \ldots, v_{n-1}$.

It is clear that S is a covering set of G_n and $|S| = n-1$; so $m_e(G_n - S) = 0$ and $\omega(G_n - S) = n$. Consequently $R_w(G_n) = 1$. □

Theorem 4.2. Let G_n be a gear graph of order n, $n \geq 4$. Then $R_w(\overline{G}_n) = 5 - 2n$.



G_7

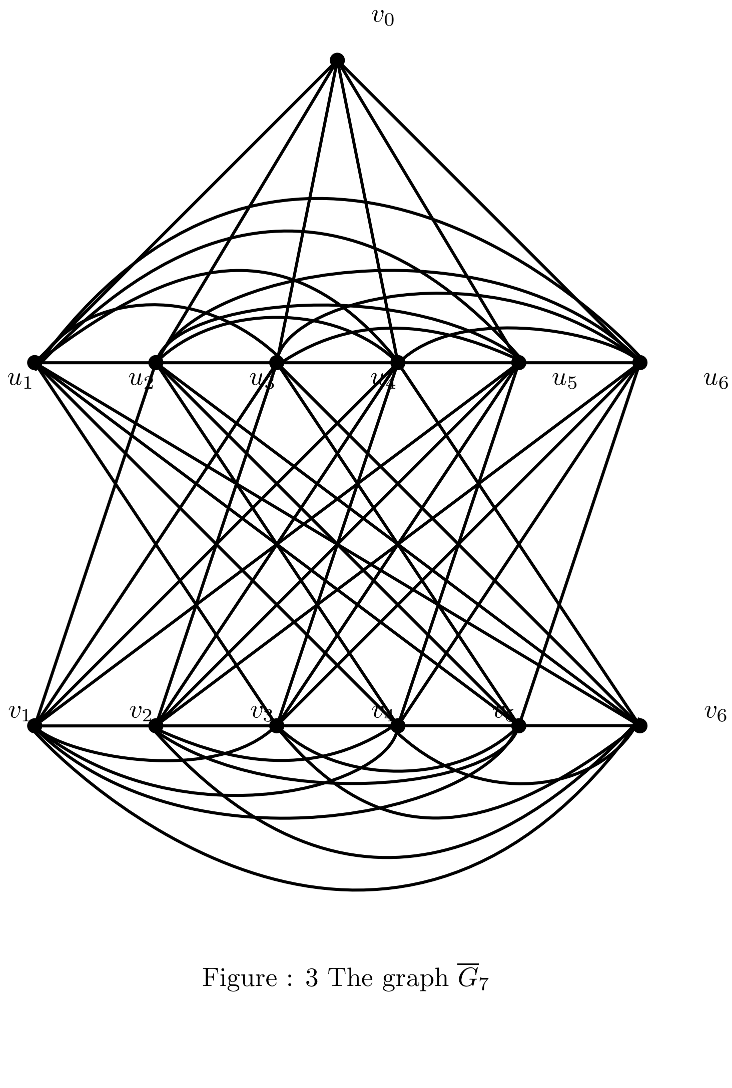


Figure : 3 The graph $\overline{G}_7$

Proof. Let v_0 be the center vertex and $V = \{v_1, v_2, \ldots, v_{n-1}\}$ be the vertices of the outer $n-1$-cycle of W_n. We know that a gear graph G_n can be constructed from the wheel graph W_n by adding new vertices between each pair of adjacent vertices of the outer cycle of W_n. Let the new vertices be $U = \{u_1, u_2, \ldots, u_{n-1}\}$.

Since V is an independent set of G_n, these vertices form a complete graph with order $n - 1 = m$ in $\overline{G}_n$. Similarly, since $U \bigcup \{v_0\}$ is an independent set of G_n, these vertices form a complete graph with order n in $\overline{G}_n$.

Also in the graph $\overline{G}_n$ there are edges joining K_n to K_m. It is obvious that the vertex u_0 in $\overline{G}_n$ is not adjacent to any vertex in K_m.
We distinguish into following three cases.
Case 1.
If $S = V$ with $|S| = n-1$, then $\overline{G}_n - S$ have only one component which is the graph K_n. Then $m_e(\overline{G}_n - S) = \dfrac{n^2 - n}{2}$ and so

$$\omega(\overline{G}_n - S) - |S| - m_e(\overline{G}_n - S) = \frac{4 - n^2 - n}{2}. \tag{5}$$

Case 2.
If $S = U$ with $|S| = n - 1$, then there are two components, which are the graphs K_1 and K_m, in $\overline{G}_n - S$. Then $m_e(\overline{G}_n - S) = \dfrac{m^2 - m}{2} = \dfrac{n^2 - 3n + 2}{2}$ and so

$$\omega(\overline{G}_n - S) - |S| - m_e(\overline{G}_n - S) = \frac{4 - n^2 + n}{2}. \tag{6}$$

Case 3.
Let v_j be any vertex in V. Consider $S = (V - \left\{v_j\right\}) \cup U$ in $\overline{G}_n$. Then $|S| = n - 2 + n - 1 = 2n - 3$ and $G_n - S$ contains two components which are isolated vertices with $m_e(\overline{G}_n - S) = 0$. Hence

$$\omega(\overline{G}_n - S) - |S| - m_e(\overline{G}_n - S) = 5 - 2n. \tag{7}$$

From (5), (6) and (7), we have

$$R_w(\overline{G}_n) = \max\left\{\frac{4 - n^2 - n}{2}, \frac{4 - n^2 + n}{2}, 5 - 2n\right\} = 5 - 2n.$$

Hence the result follows. □

Theorem 4.3. Let G_n be a gear graph of order n, $n \geq 4$. Then $R_w(K_2 \square G_n) = 0$.

Proof. Let $G = K_2 \square G_n$, $n \geq 4$, From the structure of the graph G, it is clear that $\alpha(G) = 2n-1$ and $\beta(G) = 2n-1$. By Theorem 2.5 we have,

$$\begin{aligned} R_w(G) &\geq \alpha(G) - \beta(G) \\ &\geq (2n - 1) - (2n - 1) = 0. \end{aligned} \tag{8}$$

Using Theorem 2.7 in Theorem 2.6, we have

$$\begin{aligned} R_w(G) &\leq r(G) + 1 \\ &\leq -1 + 1 = 0. \end{aligned}$$

From (8) and (9),

$$R_w(G) = R_w(K_2 \square G_n) = 0.$$

Hence the result follows. □

Theorem 4.4. Let G_n be a gear graph of order n. Then

$$R_w(L(G_n)) = -(n-1).$$

Proof.
Let S be an arbitrary vertex cut of $L(G_n)$ with $|S| = x$. Then $L(G_n) - S$ has at most $\left\lceil \frac{x}{2} \right\rceil$ components. i.e.,

$$\omega(L(G_n) - S) \leq \left\lceil \frac{x}{2} \right\rceil$$

and

$$m_e(L(G_n) - S) \geq \left\lceil \frac{4(n-1) - \left(3\left\lceil \frac{x}{2} \right\rceil + \left\lfloor \frac{x}{2} \right\rfloor\right)}{\left\lceil \frac{x}{2} \right\rceil} \right\rceil.$$

Since $\dfrac{4(n-1) - 2x}{\frac{x}{2}} \geq 0$, then x must be at most $2(n-1)$. Hence we have that

$$R_w(L(G_n) \leq \max\left\{\left\lceil \frac{x}{2} \right\rceil - x - \frac{4(n-1) - \left(3\left\lceil \frac{x}{2} \right\rceil + \left\lfloor \frac{x}{2} \right\rfloor\right)}{\left\lceil \frac{x}{2} \right\rceil}\right\rceil,$$

where $x \leq 2(n-1)$.
Consider the function

$$f(x) = \left\{\frac{x}{2} - x - \frac{4(n-1) - 2x}{\frac{x}{2}}\right\},$$

Then

$$f(x+1) = \left\{\frac{x+1}{2} - (x+1) - \frac{4(n-1) - 2(x+1)}{\frac{x+1}{2}}\right\},$$

and hence

$$f(x+1) - f(x) = \left\{\frac{8(n-1)}{x} - \frac{8(n-1)}{x+1} - \frac{1}{2}\right\} \geq 0.$$

Thus $f(x)$ is an increasing function. Also,

$$f'(x) = \left\{\frac{8(n-1)}{x^2} - \frac{1}{2}\right\},$$

Since $4\sqrt{n-1} \leq x \leq 2(n-1)$ and $f(x)$ is an increasing function, the function

$$f(x) = \left\{\frac{x}{2} - x - \frac{4(n-1) - 2x}{\frac{x}{2}}\right\},$$

takes its maximum value at $x = 2(n-1)$ and $f_{max}(x) = -(n-1)$. Hence we have

$$R_w(L(G_n) \leq -(n-1) \quad (9)$$

On the other hand, the independence number is $\alpha(L(G_n)) = n-1$ and the covering number is $\beta(L(G_n)) = 2(n-1)$. By Theorem 2.5

$$\begin{aligned} R_w(L(G_n)) &\geq \alpha(L(G_n)) - \beta(L(G_n)) \\ &\geq (n-1) - 2(n-1) \\ &\geq -(n-1). \end{aligned} \quad (10)$$

From (9) and (10)

$$R_w(L(G_n)) = -(n-1).$$

Hence the result follows. □

Theorem 4.5. Let $m, n \geq 3$ be two positive intergers. Then

$$R_w(G_m \square G_n) = 1.$$

Proof. From Theorem 3.2, we have $r(G_m \square G_n) = 0$. By Theorem 2.6, we have

$$\begin{aligned} R_w(G_m \square G_n) &\leq r(G_m \square G_n) + 1 \\ &\leq 0 + 1 \\ R_w(G_m \square G_n) &\leq 1. \end{aligned} \quad (11)$$

On the other hand, we see that independence number is $\alpha(G_m \square G_n) = 2mn + m + n + 1$ and covering number is $\beta(G_m \square G_n) = 2mn + m + n$. Hence by Theorem 2.5,

$$\begin{aligned} R_w(G_m \square G_n) &\geq \alpha(G_m \square G_n) - \beta(G_m \square G_n) \\ &\geq (2mn + m + n + 1) - (2mn + m + n) \\ R_w(G_m \square G_n) &\geq 1. \end{aligned} \quad (12)$$

Using (11) and (12), we have

$$R_w(G_m \square G_n) = 1.$$

Hence the result follows. □

5. Conclusion

Geared system can be used for dynamic analysis. To examine the vulnerability of a geared system, gear graphs can be used for exploring the vulnerability of the network. In this study, the rupture degree and weak rupture degree of gear graphs are obtained.

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