Filomat 38:5 (2024), 1793–1806
https://doi.org/10.2298/FIL2405793I

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Improved Jensen's type inequality for (p, h)-convex functions via weak sub-majorization

Mohamed Amine Ighachane[a], **Mohammed Bouchangour**[b,*]

[a] *Sciences and Technologies Team (ESTE), Higher School of Education and Training of El Jadida, Chouaib Doukkali University, El Jadida, Morocco*
[b] *Department of Mathematics, LIABM Laboratory, Faculty of Sciences, Mohammed First University, B. P. 717, 60000 Oujda, Morocco*

Abstract. In this paper, we present several new inequalities for (p, h)-convex functions in a way that complements those known inequalities for (p, h)-convex functions. Further we mainly present multiple term refinements of the well-known Jensen's type inequality for (p, h)-convex functions. Our results improve some celebrated results from the literature.

1. Introduction and preliminaries

Convex functions play a significant role in various fields of mathematics, including analysis, optimization, mathematical physics, functional analysis, and operator theory. We recall that a convex function $f : I \to \mathbb{R}$ is a function that satisfies

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y), \tag{1}$$

for every $x, y \in I$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$ and f is said log-convex function if f is positive and $\log f$ is convex. As a research trend in mathematical inequalities, there is considerable interest in minimizing the difference between the two sides of (1) by introducing specific terms. This inequality has been refined in the literature, with numerous applications presented for both scalars and matrices. We refer the reader to [1, 13, 17–19] for further discussion. Throughout this paper, we denote by I a p-convex subset of $\mathbb{R}$. Recall that a subset $I \subset \mathbb{R}$ is said p-convex if $[\alpha x^p + \beta y^p]^{\frac{1}{p}} \in I$ for all $x, y \in I$ and $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta = 1$. In this paper, we will be interested by (p, h)-convex functions [3] and (p, h)-log-convex functions [8], wish represents a generalization of the known concept of the convexity and the log-convexity as we will see. Let $h : J \to \mathbb{R}$ be a non-negative and non-zero function where J is a subset of $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a function, recall that f is said (p, h)-convex function if for every $x, y \in I$, $p \in \mathbb{R} \setminus \{0\}$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$, we have

$$f\left([\alpha x^p + \beta y^p]^{\frac{1}{p}}\right) \leq h(\alpha) f(x) + h(\beta) f(y). \tag{2}$$

2020 *Mathematics Subject Classification*. 26D07.
Keywords. (p, h)-convex function; Weak sub-majorization; Jensen's inequality.
Received: 29 March 2023; Accepted: 04 September 2023
Communicated by Fuad Kittaneh
* Corresponding author: Mohammed Bouchangour
Email addresses: mohamedamineighachane@gmail.com (Mohamed Amine Ighachane), m.bouchangour@ump.ac.ma (Mohammed Bouchangour)

Using the same notations above, we recall that f is said (p,h)-log-convex function, if it satisfies the following inequality

$$f\left(\left[\alpha x^p+\beta y^p\right]^{\frac{1}{p}}\right)\leq f^{h(\alpha)}(x)f^{h(\beta)}(y).$$

Clearly, if $h = id$ (id stands for the identity function) in (2) then we get the definition of the p-convexity [21], if in addition $p = 1$ then we get the known definition of the convexity (1). We say that $f : I \to \mathbb{R}$ is a (G-A)-h-convex function if f is non-negative and

$$f\left(x^\alpha y^\beta\right)\leq h(\alpha)f(x)+h(\beta)f(y). \tag{3}$$

If the inequality sign in (3) is reversed, then f is said to be a (G-A)-h-concave function.
We say that $f : I \to \mathbb{R}$ is a (G-G)-h-convex function if f is non-negative and

$$f\left(x^\alpha y^\beta\right)\leq f^{h(\alpha)}(x)f^{h(\beta)}(y). \tag{4}$$

If the inequality sign in (4) is reversed, then f is said to be a (G-G)-h-concave function.
In [4], Ighachane and Bouchangour found a result that generalizes another important result due to Sababheh [11], as follows. If f is a positive (p,h)-convex function for a non-negative super-multiplicative and super-additive function h, then we have

$$h^\lambda\left(\frac{\alpha}{\beta}\right)\leq\frac{(h(1-\alpha)f(x)+h(\alpha)f(y))^\lambda-f^\lambda\left[((1-\alpha)x^p+\alpha y^p)^{\frac{1}{p}}\right]}{(h(1-\beta)f(x)+h(\beta)f(y))^\lambda-f^\lambda\left[((1-\beta)x^p+\beta y^p)^{\frac{1}{p}}\right]}\leq h^\lambda\left(\frac{1-\alpha}{1-\beta}\right), \tag{5}$$

for a positive (p,h)-convex function f, when $\lambda\geq 1$, $p\in\mathbb{R}\setminus\{0\}$ and $0\leq\alpha\leq\beta\leq 1$.

The known Jensen inequality extends (1) to n parameters in the following way

$$f\left(\sum_{i=1}^n\alpha_i x_i\right)\leq\sum_{i=1}^n\alpha_i f(x_i), \tag{6}$$

where $f : I \longrightarrow \mathbb{R}$ is a convex function, $\{x_1,\dots,x_n\}\subset I$ and $\{\alpha_1,\dots,\alpha_n\}\subset[0,1]$ be such that $\sum_{i=1}^n\alpha_i = 1$. By applying Jensen's inequality (6) to the function $\log f$ we get the following inequality

$$f\left(\sum_{i=1}^n\alpha x_i\right)\leq\prod_{i=1}^n f^{\alpha_i}(x_i), \tag{7}$$

for the same parameters above, where f is log-convex.

The literature has devoted a great deal of attention to improving or reversing (6), and consequently, (7). Chronologically, in [9], the following refinement of (6) was presented and proved.

$$f\left(\sum_{i=1}^n\alpha_i x_i\right)+n\alpha_{\min}\left(\frac{1}{n}\sum_{i=1}^n f(x_i)-f\left(\sum_{i=1}^n\frac{x_i}{n}\right)\right)\leq\sum_{i=1}^n\alpha_i f(x_i), \tag{8}$$

where $\alpha_{\min}=\min\{\alpha_1,\dots,\alpha_n\}$. In the same reference, this inequality was reversed as follows.

$$f\left(\sum_{i=1}^n\alpha_i x_i\right)+n\alpha_{\max}\left(\frac{1}{n}\sum_{i=1}^n f(x_i)-f\left(\sum_{i=1}^n\frac{x_i}{n}\right)\right)\geq\sum_{i=1}^n\alpha_i f(x_i), \tag{9}$$

where $\alpha_{\max}=\max\{\alpha_1,\dots,\alpha_n\}$.
In [13], Sababheh presented new refinements of Jensen's inequality by adding as many refinements as we

wish. Namely, for a convex function $f : I \to \mathbb{R}$, $\left\{x_1^{(1)}, \ldots, x_n^{(1)}\right\} \subset I$ and $\left\{\alpha_1^{(1)}, \ldots, \alpha_n^{(1)}\right\} \subset (0,1)$ be such that $\sum_{i=1}^{n} \alpha_i^{(1)} = 1$. Then for every $N \in \mathbb{N}$, the author proved the following inequality

$$f\left(\sum_{i=1}^{n} \alpha_i^{(1)} x_i^{(1)}\right) + \sum_{k=1}^{N} n\alpha_{min}^{(k)} \left(\frac{1}{n}\sum_{i=1}^{n} f\left(x_i^{(k)}\right) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(k)}\right)\right) \leq \sum_{i=1}^{n} \alpha_i^{(1)} f\left(x_i^{(1)}\right), \tag{10}$$

where the construction of $x_i^{(k)}$, $\alpha_i^{(k)}$ and $\alpha_{\min}^{(k)}$ is given in Section 4.
The known Jensen's type inequality for (p,h)-convexity, where h is a non-negative super-multiplicative function, is as follows

$$f\left(\left[\sum_{i=1}^{n} \alpha_i x_i^p\right]^{\frac{1}{p}}\right) \leq \sum_{i=1}^{n} h(\alpha_i) f(x_i). \tag{11}$$

Applying Jensen's inequality to the (p,h)-convex function $\log f$ yields the following inequality

$$f\left(\left[\sum_{i=1}^{n} \alpha_i x_i^p\right]^{\frac{1}{p}}\right) \leq \prod_{i=1}^{n} f^{h(\alpha_i)}(x_i). \tag{12}$$

In this paper, we aim to extend the inequalities for (p,h)-convex functions, complementing existing results for convex, log-convex, (p,h)-convex, and (p,h)-log-convex functions. For example, we will extend inequality (5) to include n parameters and modify inequality (10) by considering (p,h)-convex functions, allowing for as many refinements as desired.

2. Preliminaries and auxiliary results

The main goal of this section is to prove Theorem 2.2 to help us to prove our main results presented in the next section. First, recall that a function $h : J \to \mathbb{R}$ is said super-multiplicative if for all $x, y \in J$, we have $xy \in J$ and

$$h(x)h(y) \leq h(xy). \tag{13}$$

If the inequality (13) is reversed, then h is said to be a sub-multiplicative function. If the equality holds in (13), then h is said to be a multiplicative function. On the other side, if we have $x + y \in J$ and

$$h(x) + h(y) \leq h(x + y), \tag{14}$$

then h is said to be a super-additive function. If the inequality (14) is reversed, we say that h is a sub-additive function. If the equality (14) holds, we say that h is an additive function.

Example 2.1 ([4]). *Let $h : I \to (0,\infty)$ be given by $h(x) = x^k, x > 0$. Then h is*

(1) *additive if $k = 1$,*
(2) *sub-additive if $k \in (-\infty,-1] \cup [0,1)$,*
(3) *super-additive if $k \in (-1,0) \cup (1,\infty)$.*

Let $h : [1,+\infty) \mapsto \mathbb{R}^+$ be given by $h(x) = x^3 - x^2 + x$. We have

(4) $h(xy) - h(x)h(y) = xy(x + y)(1 - x)(1 - y) \geq 0$
(5) $h(x + y) - h(x) - h(y) = xy(x + y + (x - 1) + (y - 1)) \geq 0$.

Then h is a super-multiplicative and super-additive function.

(6) *Let h be a convex function with $h(0) = 0$. Then h is a super-additive function. In particular the following function $h(x) = \exp(x^k) - 1$ for $k > 0$ is super-additive.*

We would like to emphasize that the following result provides one refining terms for inequality (11), which, in turn, allows us to establish the general form of the first inequality in (5). This general form pertains to inequality (8) for (p, h)-convex functions.

Theorem 2.2. *Let h be a non-negative super-multiplicative and super-additive function on $[0, +\infty)$, f be a positive (p, h)-convex function on $[a, b]$, $\{x_1, \ldots, x_n\} \subset [a, b]$ and $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\} \subset (0, 1)$ be such that $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i = 1$. We have*

$$\sum_{i=1}^{n} h(\alpha_i) f(x_i) \geq f\left(\left[\sum_{i=1}^{n} \alpha_i x_i^p\right]^{\frac{1}{p}}\right) + h\left(\min_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)\left[\sum_{i=1}^{n} h(\beta_i) f(x_i) - f\left(\left[\sum_{i=1}^{n} \beta_i x_i^p\right]^{\frac{1}{p}}\right)\right].$$

Proof. Assume that h is super-multiplicative and super-additive. We have

$$\begin{aligned}
I &:= \sum_{i=1}^{n} h(\alpha_i) f(x_i) - h\left(\min_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)\left[\sum_{i=1}^{n} h(\beta_i) f(x_i) - f\left(\left[\sum_{i=1}^{n} \beta_i x_i^p\right]^{\frac{1}{p}}\right)\right] \\
&= \sum_{i=1}^{n}\left(h(\alpha_i) - h\left(\min_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right) h(\beta_i)\right) f(x_i) + h\left(\min_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right) f\left(\left[\sum_{i=1}^{n} \beta_i x_i^p\right]^{\frac{1}{p}}\right) \\
&\geq \sum_{i=1}^{n}\left(h(\alpha_i) - h\left(\min_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\beta_i\right)\right) f(x_i) + h\left(\min_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right) f\left(\left[\sum_{i=1}^{n} \beta_i x_i^p\right]^{\frac{1}{p}}\right) \\
&\geq \sum_{i=1}^{n}\left[h\left(\alpha_i - \min_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\beta_i\right)\right] f(x_i) + h\left(\min_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right) f\left(\left[\sum_{i=1}^{n} \beta_i x_i^p\right]^{\frac{1}{p}}\right).
\end{aligned}$$

At this point remark that

$$\alpha_i - \min_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\beta_i \geq \alpha_i - \frac{\alpha_i}{\beta_i}\beta_i = 0,$$

and

$$\sum_{i=1}^{n}\left(\alpha_i - \min_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\beta_i\right) + \min_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\} = 1.$$

Using the definition of the (p, h)-convexity, we get that

$$I \geq f\left(\sum_{i=1}^{n}\left[\left(\alpha_i - \min_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\beta_i\right) x_i^p + \min_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\sum_{i=1}^{n} \beta_i x_i^p\right]^{\frac{1}{p}}\right). \tag{15}$$

It comes that the right side of (15) is exactly equal to

$$f\left(\left[\sum_{i=1}^{n} \alpha_i x_i^p\right]^{\frac{1}{p}}\right).$$

This close the proof. □

As in [4], we have the following version of Theorem 2.2, for (G-A)-h-convex functions.

Corollary 2.3. *Let h be a non-negative super-multiplicative and super-additive function on $[0,+\infty)$, f be a positive (G-A)-h-convex function on $[a,b]$, $\{x_1,\ldots,x_n\}\subset[a,b]$ and $\{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n\}\subset(0,1)$ be such that $\sum_{i=1}^n\alpha_i=\sum_{i=1}^n\beta_i=1$. Then, we have*

$$\sum_{i=1}^n h(\alpha_i)f(x_i)\geq f\left(\prod_{i=1}^n x_i^{\alpha_i}\right)+h\left(\min_{1\leq j\leq n}\left\{\frac{\alpha_i}{\beta_j}\right\}\right)\left[\sum_{i=1}^n h(\beta_i)f(x_i)-f\left(\prod_{i=1}^n x_i^{\beta_i}\right)\right].$$

Now, let us present the reverse of the previous theorem.

Theorem 2.4. *Let h be a non-negative multiplicative and super-additive function on $[0,+\infty)$, f be a positive (p,h)-convex function on $[a,b]$, $\{x_1,\ldots,x_n\}\subset[a,b]$ and $\{\alpha_1,\ldots,\alpha_n\}\subset(0,1)$ and $\{\beta_1,\ldots,\beta_n\}\subset(0,1)$ be such that $\sum_{i=1}^n\alpha_i=\sum_{i=1}^n\beta_i=1$. Then we have*

$$\sum_{i=1}^n h(\alpha_i)f(x_i)\leq f\left(\left[\sum_{i=1}^n\alpha_i x_i^p\right]^{\frac{1}{p}}\right)$$
$$+h\left(\max_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)\left[\sum_{i=1}^n h(\beta_i)f(x_i)-f\left(\left[\sum_{i=1}^n\beta_i x_i^p\right]^{\frac{1}{p}}\right)\right].$$

Proof. Assume that h is multiplicative and supper-additive. We have

$$I:=\sum_{i=1}^n h(\beta_i)f(x_i)-\frac{1}{h\left(\max_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)}\sum_{i=1}^n h(\alpha_i)f(x_i)$$
$$+\frac{1}{h\left(\max_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)}f\left(\left[\sum_{i=1}^n\alpha_i x_i^p\right]^{\frac{1}{p}}\right)$$
$$=\sum_{i=1}^n\left(h(\beta_i)-\frac{h(\alpha_i)}{h\left(\max_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)}\right)f(x_i)+\frac{1}{h\left(\max_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)}f\left(\left[\sum_{i=1}^n\alpha_i x_i^p\right]^{\frac{1}{p}}\right)$$
$$\geq\sum_{i=1}^n h\left(\beta_i-\frac{\alpha_i}{\max_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}}\right)f(x_i)+h\left(\frac{1}{\max_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}}\right)f\left(\left[\sum_{i=1}^n\alpha_i x_i^p\right]^{\frac{1}{p}}\right).$$

Since

$$\sum_{i=1}^n\left(\beta_i-\frac{\alpha_i}{\max_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}}\right)+\frac{1}{\max_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}}=1,$$

it follows from the definition of the (p,h)-convexity that

$$I\geq f\left(\left[\sum_{i=1}^n\beta_i x_i^p\right]^{\frac{1}{p}}\right).$$

Thus complete the proof. □

As a direct consequence of the previous theorem, we get the following result for (G-A)-h-convex functions.

Corollary 2.5. *Let h be a non-negative multiplicative and super-additive function on $[0,+\infty)$, f be a positive (G-A)-h-convex function on $[a,b]$, $\{x_1,\dots,x_n\}\subset[a,b]$, $\{\alpha_1,\dots,\alpha_n\}\subset(0,1)$ and $\{\beta_1,\dots,\beta_n\}\subset(0,1)$ be such that $\sum_{i=1}^n\alpha_i=\sum_{i=1}^n\beta_i=1$. Then we have*

$$\sum_{i=1}^n h(\alpha_i)f(x_i)\le f\left(\prod_{i=1}^n x_i^{\alpha_i}\right)+h\left(\max_{1\le j\le n}\left\{\frac{\alpha_i}{\beta_j}\right\}\right)\left[\sum_{i=1}^n h(\beta_i)f(x_i)-f\left(\prod_{i=1}^n x_i^{\beta_i}\right)\right].$$

3. Further inequalities for (p,h)-convex functions

The purpose of this section is to extend Theorem 2.2 and Theorem 2.4 to the more general setting using the so called weak sub-majorization theory. Throughout this section, we denote by $X^*=\left(X_1^*,\dots,X_n^*\right)$ the vector obtained from the vector $X=(X_1,\dots,X_n)\in\mathbb{R}^n$ by rearranging the components of it in decreasing order. Then, for two vectors $X=(X_1,\dots,X_n)$ and $Y=(Y_1,\dots,Y_n)$ in $\mathbb{R}^n$, Y is said to be weakly sub-majorized by X, written $X\succ_w Y$, if

$$\sum_{i=1}^k X_i^*\ge\sum_{i=1}^k Y_i^*$$

for all $k=1,\dots,n$.

An important tool in the theory of weak sub-majorization, which will be used to prove our results, is provided by the following lemma.

Lemma 3.1. *[7, pp. 13] Let $X=(X_i)_{i=1}^n$, $Y=(Y_i)_{i=1}^n\in\mathbb{R}^n$ and $J\subset\mathbb{R}$ be an interval containing the components of X and Y. If $X\succ_w Y$ and $\psi:J\to\mathbb{R}$ is a continuous increasing convex function, then*

$$\sum_{i=1}^n\psi(X_i)\ge\sum_{i=1}^n\psi(Y_i).$$

The following lemma will allow us to derive the general form of Theorem 2.2.

Lemma 3.2. *Let h be a non-negative super-multiplicative and super-additive function on $[0,+\infty)$ and let f be a convex function on $[0,1]$. Let $\{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n\}\subset[0,1]$ such that $\sum_{i=1}^n\alpha_i=\sum_{i=1}^n\beta_i=1$ and $\{x_1,\dots,x_n\}\subset(0,1)$. Let $X=(X_1,X_2)$ and $Y=(Y_1,Y_2)$ be two vectors in $\mathbb{R}^2$ with components*

$$X_1=\sum_{i=1}^n h(\alpha_i)f(x_i),\quad X_2=h\left(\min_{1\le j\le n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)f\left(\left[\sum_{i=1}^n\beta_i x_i^p\right]^{\frac{1}{p}}\right)$$

$$Y_1=f\left(\left[\sum_{i=1}^n\alpha_i x_i^p\right]^{\frac{1}{p}}\right)\ \text{and}\ Y_2=h\left(\min_{1\le j\le n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)\sum_{i=1}^n h(\beta_i)f(x_i).$$

Then, we have $X\succ_w Y$, namely, the vectors X^* and Y^* have components satisfying that

$$X_1^*\ \ge\ Y_1^*,\tag{16}$$
$$X_1^*+X_2^*\ \ge\ Y_1^*+Y_2^*.\tag{17}$$

Proof. First of all remark that X_1^* is exactly X_1. Indeed, on one hand we have

$$\begin{aligned} X_1 - Y_2 &= \sum_{i=1}^{n} h(\alpha_i) f(x_i) - h\left(\min_{1\le j\le n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)\sum_{i=1}^{n} h(\beta_i) f(x_i) \\ &= \sum_{i=1}^{n}\left[h(\alpha_i) - h\left(\min_{1\le j\le n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)h(\beta_i)\right] f(x_i) \\ &\ge \sum_{i=1}^{n} h\left(\alpha_i - \min_{1\le j\le n}\left\{\frac{\alpha_j}{\beta_j}\right\}\beta_i\right) f(x_i) \\ &\ge 0. \end{aligned}$$

On the other hand, by the (p, h)-convexity we get that $Y_2 \ge X_2$. This implies that $X_1 \ge X_2$, hence $X_1^* = X_1$. Once again, by the (p, h)-convexity the inequality (16) is established. The second inequality (17) comes directly from Theorem 2.2. □

Theorem 3.3. *Let h be a non-negative super-multiplicative and super-additive function on $[0, +\infty)$, f be a positive (p, h)-convex function on $[a, b]$ and ψ be a strictly increasing convex function defined on an interval J. Let $\{x_1, \ldots, x_n\} \subset [a, b]$, $\{\alpha_1, \ldots, \alpha_n\} \subset (0, 1)$ and $\{\beta_1, \ldots, \beta_n\} \subset (0, 1)$ be such that $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i = 1$. Then we have*

$$\begin{aligned} \psi\left(\sum_{i=1}^{n} h(\alpha_i) f(x_i)\right) \ge{}& \psi \circ f\left(\left[\sum_{i=1}^{n} \alpha_i x_i^p\right]^{\frac{1}{p}}\right) + \psi\left(h\left(\min_{1\le j\le n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)\sum_{i=1}^{n} h(\beta_i) f(x_i)\right) \\ &- \psi\left(h\left(\min_{1\le j\le n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right) f\left(\left[\sum_{i=1}^{n} \beta_i x_i^p\right]^{\frac{1}{p}}\right)\right). \end{aligned}$$

Proof. Let us consider the vectors $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ defined in Lemma 3.2, by the same Lemma we have $X \succ_w Y$. This implies by Lemma 3.1 that

$$\psi(X_1) + \psi(X_2) \ge \psi(Y_1) + \psi(Y_2),$$

becomes

$$\psi(X_1) \ge \psi(Y_1) + \psi(Y_2) - \psi(X_2).$$

□

In order to give the reverse of the previous Theorem, let us show the following helpful lemma.

Lemma 3.4. *Let h be a non-negative multiplicative and supper-additive function on $[0, +\infty)$, f be a convex function on $[0, 1]$, $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\} \subset [0, 1]$ such that $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i = 1$ and $\{x_1, \ldots, x_n\} \subset (0, 1)$. Let $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ be two vectors in $\mathbb{R}^2$ with components*

$$X_1 = h\left(\max_{1\le j\le n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)\sum_{i=1}^{n} h(\beta_i) f(x_i), \quad X_2 = f\left(\left[\sum_{i=1}^{n} \alpha_i x_i^p\right]^{\frac{1}{p}}\right),$$

$$Y_1 = \sum_{i=1}^{n} h(\alpha_i) f(x_i) \text{ and } Y_2 = h\left(\max_{1\le j\le n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right) f\left(\left[\sum_{i=1}^{n} \beta_i x_i^p\right]^{\frac{1}{p}}\right).$$

Then, we have $X \succ_w Y$, namely, the vectors X^* and Y^* have components satisfying

$$X_1^* \ge Y_1^*, \tag{18}$$
$$X_1^* + X_2^* \ge Y_1^* + Y_2^*. \tag{19}$$

Proof. In order to prove (18) nothing that $X_1^* = X_1$. Indeed, using the multiplicativity of the function h observe that

$$\begin{aligned} X_1 - Y_1 &= h\left(\max_{1\le j\le n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)\sum_{i=1}^n h(\beta_i)f(x_i) - \sum_{i=1}^n h(\alpha_i)f(x_i) \\ &= \sum_{i=1}^n \left[h\left(\max_{1\le j\le n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)h(\beta_i) - h(\alpha_i)\right]f(x_i) \\ &= \sum_{i=1}^n \left[h\left(\max_{1\le j\le n}\left\{\frac{\alpha_j}{\beta_j}\right\}\beta_i - \alpha_i\right)\right]f(x_i). \end{aligned}$$

At this point remark that

$$\max_{1\le j\le n}\left\{\frac{\alpha_j}{\beta_j}\right\}\beta_i - \alpha_i \ge \frac{\alpha_i}{\beta_i}\beta_i - \alpha_i = 0,$$

for all $i = 1, \dots, n$, hence $X_1 \ge Y_1$. Beside, by the (p, h)-convexity we get that $Y_1 \ge X_2$. This implies that $X_1 \ge X_2$, therefore $X_1^* = X_1$. On the other hand, one more time via the (p, h)-convexity we deduce easily that $X_1 \ge Y_2$. The second inequality (17) follows directly from Theorem 2.4. □

Theorem 3.5. *Let h be a non-negative multiplicative and super-additive function on $[0, +\infty)$, f be a positive (p, h)-convex function on $[a, b]$ and ψ be a strictly increasing convex function defined on an interval J. Let $\{x_1, \dots, x_n\} \subset [a, b]$, $\{\alpha_1, \dots, \alpha_n\} \subset (0, 1)$ and $\{\beta_1, \dots, \beta_n\} \subset (0, 1)$ be such that $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i = 1$. Then we have*

$$\begin{aligned} \psi\left(\sum_{i=1}^n h(\alpha_i)f(x_i)\right) &\le \psi\circ f\left(\left[\sum_{i=1}^n \alpha_i x_i^p\right]^{\frac{1}{p}}\right) + \psi\left(h\left(\max_{1\le j\le n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)\sum_{i=1}^n h(\beta_i)f(x_i)\right) \\ &\quad - \psi\left(h\left(\max_{1\le j\le n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)f\left(\left[\sum_{i=1}^n \beta_i x_i^p\right]^{\frac{1}{p}}\right)\right). \end{aligned}$$

Proof. Let us consider the vectors $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ defined in Lemma 3.4, through the same Lemma we have $X \succ_w Y$. This implies by Lemma 3.1 that

$$\psi(X_1) + \psi(X_2) \ge \psi(Y_1) + \psi(Y_2),$$

becomes

$$\psi(Y_1) \le \psi(X_1) + \psi(X_2) - \psi(Y_2).$$

□

Replacing f by $\log f$, in Theorems 3.3 and 3.5 we state the log-convex version of the previous results as follows.

Theorem 3.6. *Let h be a non-negative function on J, f be a positive (p, h)-*log-*convex function on $[a, b]$ and ψ be a strictly increasing convex function defined on an interval $[0, +\infty)$. Let $\{x_1, \dots, x_n\} \subset [a, b]$, $\{\alpha_1, \dots, \alpha_n\} \subset (0, 1)$ and $\{\beta_1, \dots, \beta_n\} \subset (0, 1)$ be such that $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i = 1$. Then we have*

1. *If h is a super-multiplicative and super-additive function on* $[0, +\infty)$, *then*

$$\psi \circ \log\left(\prod_{i=1}^{n} f^{h(\alpha_i)}(x_i)\right) \geq \psi \circ \log f\left(\left[\sum_{i=1}^{n} \alpha_i x_i^p\right]^{\frac{1}{p}}\right) + \psi\left(\log\left(\prod_{i=1}^{n} f^{h(\beta_i)}(x_i)\right)^{h\left(\min_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)}\right) - \psi\left(\log f\left(\left[\sum_{i=1}^{n} \beta_i x_i^p\right]^{\frac{1}{p}}\right)^{h\left(\min_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)}\right).$$

2. *If h is a multiplicative and super-additive function on* $[0, +\infty)$, *then*

$$\psi \circ \log\left(\prod_{i=1}^{n} f^{h(\alpha_i)}(x_i)\right) \leq \psi \circ \log f\left(\left[\sum_{i=1}^{n} \alpha_i x_i^p\right]^{\frac{1}{p}}\right) + \psi\left(\log\left(\prod_{i=1}^{n} f^{h(\beta_i)}(x_i)\right)^{h\left(\max_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)}\right) - \psi\left(\log f\left(\left[\sum_{i=1}^{n} \beta_i x_i^p\right]^{\frac{1}{p}}\right)^{h\left(\max_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)}\right).$$

Now, by considering $\psi(x) = x^\lambda$ for $\lambda \geq 1$, in Theorems 3.3 and 3.5 we get the following results that extends the inequality (5) to n parameters.

Theorem 3.7. *Let h be a non-negative function on* $[0, +\infty)$, f *be a positive* (p, h)*-convex function on* $[a, b]$ *and* ψ *be a strictly increasing convex function defined on an interval J. Let* $\{x_1, \ldots, x_n\} \subset [a, b]$, $\{\alpha_1, \ldots, \alpha_n\} \subset (0, 1)$ *and* $\{\beta_1, \ldots, \beta_n\} \subset (0, 1)$ *be such that* $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i = 1$. *We have*

1. *If h is a super-multiplicative and super-additive function on* $[0, +\infty)$, *then*

$$\left(\sum_{i=1}^{n} h(\alpha_i) f(x_i)\right)^\lambda \geq f^\lambda\left(\left[\sum_{i=1}^{n} \alpha_i x_i^p\right]^{\frac{1}{p}}\right) + \left(h\left(\min_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)\right)^\lambda \left(\sum_{i=1}^{n} h(\beta_i) f(x_i) - f\left(\left[\sum_{i=1}^{n} \beta_i x_i^p\right]^{\frac{1}{p}}\right)\right)^\lambda.$$

2. *If h is a multiplicative and super-additive function on* $[0, +\infty)$, *then*

$$\left(\sum_{i=1}^{n} h(\alpha_i) f(x_i)\right)^\lambda \leq f^\lambda\left(\left[\sum_{i=1}^{n} \alpha_i x_i^p\right]^{\frac{1}{p}}\right) + \left(h\left(\max_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)\right)^\lambda \left(\sum_{i=1}^{n} h(\beta_i) f(x_i) - f\left(\left[\sum_{i=1}^{n} \beta_i x_i^p\right]^{\frac{1}{p}}\right)\right)^\lambda.$$

By taking $\psi(x) = \exp(x)$, in Theorem 3.6, we get the following results. Which presents one refining term of inequality (12).

Theorem 3.8. *Let h be a non-negative function on $[0, +\infty)$, f be a positive (p, h)-log-convex function on $[a, b]$ and ψ be a strictly increasing convex function defined on an interval J. Let $\{x_1, \dots, x_n\} \subset [a, b]$, $\{\alpha_1, \dots, \alpha_n\} \subset (0, 1)$ and $\{\beta_1, \dots, \beta_n\} \subset (0, 1)$ be such that $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i = 1$. Then we have*

1. *If h is a super-multiplicative and super-additive function on $[0, +\infty)$, then*

$$\prod_{i=1}^{n} f^{h(\alpha_i)}(x_i) \geq f\left(\left[\sum_{i=1}^{n} \alpha_i x_i^p\right]^{\frac{1}{p}}\right) + \left(\prod_{i=1}^{n} f^{h(\beta_i)}(x_i)\right)^{h\left(\min_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)} - f^{h\left(\min_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)}\left(\left[\sum_{i=1}^{n} \beta_i x_i^p\right]^{\frac{1}{p}}\right).$$

2. *If h is a multiplicative and super-additive function on $[0, +\infty)$, then*

$$\prod_{i=1}^{n} f^{h(\alpha_i)}(x_i) \leq f\left(\left[\sum_{i=1}^{n} \alpha_i x_i^p\right]^{\frac{1}{p}}\right) + \left(\prod_{i=1}^{n} f^{h(\beta_i)}(x_i)\right)^{h\left(\max_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)} - f^{h\left(\max_{1\leq j\leq n}\left\{\frac{\alpha_j}{\beta_j}\right\}\right)}\left(\left[\sum_{i=1}^{n} \beta_i x_i^p\right]^{\frac{1}{p}}\right).$$

4. Refining Jensen's type inequality for (p, h)-convex functions

Throughout this section, we denote by $\alpha^{(1)} = \left\{\alpha_1^{(1)}, \dots, \alpha_n^{(1)}\right\} \subset (0, 1)$ a convex sequence, satisfying $\sum_{i=1}^{n} \alpha_i^{(1)} = 1$. Define

$$J_1 = \left\{i : \alpha_i^{(1)} = \alpha_{\min}^{(1)}\right\},$$

where $\alpha_{\min}^{(1)} = \min\left\{\alpha_i^{(1)} : 1 \leq i \leq n\right\}$. The quantity $|J_1|$ stand for the cardinality of J_1.

For $k \geq 2$ let $\alpha^{(k)}$ be a sequence defined inductively in the following way

$$\alpha_i^{(k)} = \begin{cases} \alpha_i^{(k-1)} - \alpha_{\min}^{(k-1)} & \text{if } \alpha_i^{(k-1)} \neq \alpha_{\min}^{(k-1)} \\ \frac{1}{J_{k-1}} n\alpha_{\min}^{(k-1)} & \text{if } \alpha_i^{(k-1)} = \alpha_{\min}^{(k-1)} \end{cases} \quad \text{where } J_{k-1} = \left\{i : \alpha_i^{(k-1)} = \alpha_{\min}^{(k-1)}\right\}, \tag{20}$$

and for $k \geq 1, \alpha_{\min}^{(k)} = \min\left\{\alpha_1^{(k)}, \dots, \alpha_n^{(k)}\right\}$. Now, let us set $x^{(1)} = \left\{x_1^{(1)}, \dots, x_n^{(1)}\right\} \subset I$, we provide a new sequence $x^{(k)}$ defined by

$$x_i^{(k)} = \begin{cases} x_i^{(k-1)} & \text{if } \alpha_i^{(k-1)} \neq \alpha_{\min}^{(k-1)} \\ \left[\frac{1}{n}\sum_{i=1}^{n} x_i^{p(k-1)}\right]^{\frac{1}{p}} & \text{if } \alpha_i^{(k-1)} = \alpha_{\min}^{(k-1)} \end{cases}, 1 \leq i \leq n \text{ and } p \in \mathbb{R} \setminus \{0\}. \tag{21}$$

We point out that the order of the $\left\{x_i^{(1)}\right\}$ follows the order in which they are associated with the $\left\{\alpha_i^{(1)}\right\}$. That is, $x_1^{(1)}$ is the value multiplied with $\alpha_1^{(1)}$, and so on. Bearing those notations in mind, we present the main result of this section, extending inequality (10) to the notion of (p, h)-convex functions.

Theorem 4.1. *Let h be a non-negative super-multiplicative and super-additive function on $[0,+\infty)$, $f : I \longrightarrow \mathbb{R}$ be (p,h)-convex, $\left\{x_1^{(1)},\dots,x_n^{(1)}\right\} \subset I$ and $\left\{\alpha_1^{(1)},\dots,\alpha_n^{(1)}\right\} \subset (0,1)$ be such that $\sum_{i=1}^n \alpha_i^{(1)} = 1$. Then for every $N \in \mathbb{N}$, we have*

$$\begin{aligned} f\left[\left(\sum_{i=1}^n \alpha_i^{(1)}(x_i^p)^{(1)}\right)^{\frac{1}{p}}\right] \quad &+ \quad \sum_{k=1}^N h\left(n\alpha_{\min}^{(k)}\right)\left(h\left(\frac{1}{n}\right)\sum_{i=1}^n f\left(x_i^{(k)}\right) - f\left[\left(\frac{1}{n}\sum_{i=1}^n (x_i^p)^{(k)}\right)^{\frac{1}{p}}\right]\right) \\ &\leq \quad \sum_{i=1}^n h\left(\alpha_i^{(1)}\right) f\left(x_i^{(1)}\right), \end{aligned} \tag{22}$$

where $\alpha_i^{(k)}$ and $x_i^{(k)}$ are as in (20) *and* (21).

Proof. We prove this by induction on N. For $N = 1$, the result follows from Theorem 2.2. Now assume that (22) holds for some $N \in \mathbb{N}$. We point out here that this means, given any convex sequence $\left\{\beta_i^{(1)} : 1 \le i \le n\right\}$ and any elements $\left\{y_i^{(1)} : 1 \le i \le n\right\} \subset I$, we have the inductive step

$$\begin{aligned} f\left[\left(\sum_{i=1}^n \beta_i^{(1)}(y_i^p)^{(1)}\right)^{\frac{1}{p}}\right] \quad &+ \quad \sum_{k=1}^N h\left(nq_{\min}^{(k)}\right)\left(h\left(\frac{1}{n}\right)\sum_{i=1}^n f\left(y_i^{(k)}\right) - f\left[\left(\frac{1}{n}\sum_{i=1}^n (y_i^p)^{(k)}\right)^{\frac{1}{p}}\right]\right) \\ &\leq \quad \sum_{i=1}^n h\left(\beta_i^{(1)}\right) f\left(y_i^{(1)}\right). \end{aligned} \tag{23}$$

Then

$$\begin{aligned} A \quad :=& \quad \sum_{i=1}^n h\left(\alpha_i^{(1)}\right) f\left(x_i^{(1)}\right) - h\left(n\alpha_{\min}^{(1)}\right)\left(h\left(\frac{1}{n}\right)\sum_{i=1}^n f\left(x_i^{(1)}\right) - f\left[\left(\frac{1}{n}\sum_{i=1}^n (x_i^p)^{(1)}\right)^{\frac{1}{p}}\right]\right) \\ =& \quad \sum_{i=1}^n h\left(\alpha_i^{(1)}\right) f\left(x_i^{(1)}\right) - h\left(n\alpha_{\min}^{(1)}\right) h\left(\frac{1}{n}\right)\sum_{i=1}^n f\left(x_i^{(1)}\right) \\ +& \quad h\left(\frac{|J_1|}{|J_1|}n\alpha_{\min}^{(1)}\right) f\left[\left(\frac{1}{n}\sum_{i=1}^n (x_i^p)^{(1)}\right)^{\frac{1}{p}}\right] \qquad (24) \\ \geq& \quad \sum_{\substack{i=1 \\ \alpha_i^{(1)} \neq \alpha_{\min}^{(1)}}}^n h\left(\alpha_i^{(1)} - \alpha_{\min}^{(1)}\right) f\left(x_i^{(1)}\right) + |J_1|\left(h\left(\frac{1}{|J_1|}n\alpha_{\min}^{(1)}\right) f\left[\left(\frac{1}{n}\sum_{i=1}^n (x_i^p)^{(1)}\right)^{\frac{1}{p}}\right]\right) \\ =& \quad \sum_{\substack{i=1 \\ \alpha_i^{(1)} \neq \alpha_{\min}^{(1)}}}^n h\left(\alpha_i^{(1)} - \alpha_{\min}^{(1)}\right) f\left(x_i^{(1)}\right) + \sum_{\alpha_i^{(1)} = \alpha_{\min}^{(1)}}\left(h\left(\frac{1}{|J_1|}n\alpha_{\min}^{(1)}\right) f\left[\left(\frac{1}{n}\sum_{i=1}^n (x_i^p)^{(1)}\right)^{\frac{1}{p}}\right]\right) \\ =& \quad \sum_{i=1}^n h\left(\alpha_i^{(2)}\right) f\left(x_i^{(2)}\right), \end{aligned} \tag{25}$$

where the last line comes from the definitions of $\left(\alpha_i^{(k)}\right)$ and $\left(x_i^{(k)}\right)$ in (20) and (21). For convenience, we denote

$\alpha_i^{(2)}$ by $\beta_i^{(1)}$ and $x_i^{(2)}$ by $y_i^{(1)}$. Nothing that

$$\begin{aligned}\sum_{i=1}^{n}\beta_i^{(1)} &= \sum_{i=1}^{n}\alpha_i^{(2)} \\ &= \sum_{i\notin J_1}\left(\alpha_i^{(1)}-\alpha_{\min}^{(1)}\right)+\sum_{i\in|J_1|}\frac{n\alpha_{\min}^{(1)}}{|J_1|} \\ &= \sum_{i=1}^{n}\alpha_i^{(1)}-\sum_{i\in J_1}\alpha_i^{(1)}-\sum_{i\notin J_1}\alpha_{\min}^{(1)}+n\alpha_{\min}^{(1)} \\ &= 1-|J_1|\,\alpha_{\min}^{(1)}-(n-|J_1|)\,\alpha_{\min}^{(1)}+n\alpha_{\min}^{(1)} \\ &= 1.\end{aligned}$$

Consequently, we may apply the inductive step (25) on (23) to get

$$\begin{aligned} I &= \sum_{i=1}^{n} h\left(\beta_i^{(1)}\right) f\left(y_i^{(1)}\right) \\ &\geq f\left(\left[\sum_{i=1}^{n}\beta_i^{(1)}(y_i^p)^{(1)}\right]^{\frac{1}{p}}\right)+\sum_{k=1}^{N} h\left(nq_{\min}^{(k)}\right)\left(h\left(\frac{1}{n}\right)\sum_{i=1}^{n} f\left(y_i^{(k)}\right)-f\left[\left(\frac{1}{n}\sum_{i=1}^{n}(y_i^p)^{(k)}\right)^{\frac{1}{p}}\right]\right). \end{aligned} \tag{26}$$

Now,

$$\begin{aligned}\left[\sum_{i=1}^{n}\beta_i^{(1)}(y_i^p)^{(1)}\right]^{\frac{1}{p}} &= \left[\sum_{i=1}^{n}\alpha_i^{(2)}(x_i^p)^{(2)}\right]^{\frac{1}{p}} \\ &= \left[\sum_{\substack{i=1\\ i\notin J_1}}^{n}\left(\alpha_i^{(1)}-\alpha_{\min}^{(1)}\right)(x_i^p)^{(1)}+\sum_{j\in J_1}\left(\frac{n\alpha_{\min}^{(1)}}{|J_1|}\sum_{i=1}^{n}\frac{(x_i^p)^{(1)}}{n}\right)\right]^{\frac{1}{p}} \\ &= \left[\sum_{\substack{i=1\\ i\notin J_1}}^{n}\alpha_i^{(1)}(x_i^p)^{(1)}-\sum_{\substack{i=1\\ i\notin J_1}}^{n}\alpha_{\min}^{(1)}(x_i^p)^{(1)}+\sum_{i=1}^{n}\alpha_{\min}^{(1)}(x_i^p)^{(1)}\right]^{\frac{1}{p}} \\ &= \left[\sum_{\substack{i=1\\ i\notin J_1}}^{n}\alpha_i^{(1)}(x_i^p)^{(1)}+\sum_{\substack{i=1\\ i\in J_1}}^{n}\alpha_{\min}^{(1)}(x_i^p)^{(1)}\right]^{\frac{1}{p}} \\ &= \left[\sum_{i=1}^{n}\alpha_i^{(1)}(x_i^p)^{(1)}\right]^{\frac{1}{p}}. \end{aligned} \tag{27}$$

Moreover, since $\beta_i^{(1)}=\alpha_i^{(2)}$ and $y_i^{(1)}=x_i^{(2)}$, we have $\beta_i^{(k)}=\alpha_i^{(k+1)}$ and $y_i^{(k)}=x_i^{(k+1)}$ for $k\geq 1$. Therefore, invoking

(27) in (26), we get

$$\begin{aligned}
A &= \sum_{i=1}^{n} h\left(\alpha_i^{(1)}\right) f\left(x_i^{(1)}\right) - h\left(n\alpha_{\min}^{(1)}\right)\left(h\left(\frac{1}{n}\right)\sum_{i=1}^{n} f\left(x_i^{(1)}\right) - f\left(\left[\frac{1}{n}\sum_{i=1}^{n}(x_i^p)^{(1)}\right]^{\frac{1}{p}}\right)\right)\\
&\geq f\left[\left(\sum_{i=1}^{n}\alpha_i^{(1)}(x_i^p)^{(1)}\right)^{\frac{1}{p}}\right]\\
&+ \sum_{k=1}^{N} h\left(n\alpha_{\min}^{(k+1)}\right)\left(h\left(\frac{1}{n}\right)\sum_{i=1}^{n} f\left(x_i^{(k+1)}\right) - f\left(\left[\frac{1}{n}\sum_{i=1}^{n}(x_i^p)^{(k+1)}\right]^{\frac{1}{p}}\right)\right)\\
&= f\left[\left(\sum_{i=1}^{n}\alpha_i^{(1)}(x_i^p)^{(1)}\right)^{\frac{1}{p}}\right]\\
&+ \sum_{k=2}^{N+1} h\left(n\alpha_{\min}^{(k)}\right)\left(h\left(\frac{1}{n}\right)\sum_{i=1}^{n} f\left(x_i^{(k)}\right) - f\left(\left[\frac{1}{n}\sum_{i=1}^{n}(x_i^p)^{(k)}\right]^{\frac{1}{p}}\right)\right).
\end{aligned}$$

Thus

$$\begin{aligned}
f\left[\left(\sum_{i=1}^{n}\alpha_i^{(1)}(x_i^p)^{(1)}\right)^{\frac{1}{p}}\right] &+ \sum_{k=1}^{N+1} h\left(n\alpha_{\min}^{(k)}\right)\left(h\left(\frac{1}{n}\right)\sum_{i=1}^{n} f\left(x_i^{(k)}\right) - f\left(\left[\frac{1}{n}\sum_{i=1}^{n}(x_i^p)^{(k)}\right]^{\frac{1}{p}}\right)\right)\\
&\leq \sum_{i=1}^{n} h\left(\alpha_i^{(1)}\right) f\left(x_i^{(1)}\right), \qquad (28)
\end{aligned}$$

completing the proof. □

Remark 4.2. *Notice that when $n = 2$ in the previous theorem, we obtain a version similar to [5, Theorem 2.1].*

Replacing f by $\log f$, in Theorem 4.1 we get the following refinement of (p, h)-log-convex functions.

Corollary 4.3. *Let h be a non-negative super-multiplicative and super-additive function on $[0, +\infty)$ and $f : I \to \mathbb{R}^+$ be (p, h)-log-convex, $\left\{x_1^{(1)}, \ldots, x_n^{(1)}\right\} \subset I$ and $\left\{\alpha_1^{(1)}, \ldots, \alpha_n^{(1)}\right\} \subset (0, 1)$ be such that $\sum_{i=1}^{n}\alpha_i^{(1)} = 1$. Then for every $N \in \mathbb{N}$, we have*

$$\prod_{k=1}^{N}\left(\frac{\prod_{i=1}^{n} f^{h\left(\frac{1}{n}\right)}\left(x_i^{(k)}\right)}{f\left[\left(\frac{1}{n}\sum_{i=1}^{n}(x_i^p)^{(k)}\right)^{\frac{1}{p}}\right]}\right)^{h\left(n\alpha_{\min}^{(k)}\right)} \leq \frac{\prod_{i=1}^{n} f_i^{h\left(\alpha_i^{(1)}\right)}\left(x_i^{(1)}\right)}{f\left[\left(\sum_{i=1}^{n}\alpha_i^{(1)}(x_i^p)^{(1)}\right)^{\frac{1}{p}}\right]}.$$

5. Acknowledgments

The authors express their gratitude to the referee for their valuable feedback and suggestions, which have significantly improved the quality of our paper.

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