Filomat 38:5 (2024), 1877–1886
https://doi.org/10.2298/FIL2405877B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Fractional Hunkel transform of generalized function

Latifa El Bezdaoui[a,*], Abdelah Taqbibt[a], M'hamed Elomari[a], Lalla Saadia Chadli[a]

[a]*Applied Mathematics and Scientific Computing Laboratory,*
Sultan Moulay Slimane University (SMSU), 23000, Beni Mellal, Morocco

Abstract. In this paper, we introduce the fractional Hankel transform in the space of test functions, and we prove some of its properties. In addition, we define this transform in slowly growing distributions space,i.e., in dual space of test functions space. As application, some examples are given to illustrate the theoretical results.

1. Introduction

In the mathematical literature, a generalization of the Hankel transform, known as the fractional Hankel transform, was introduced a few years ago. This transform has found extensive application in diverse challenges within mathematical physics and applied mathematics. Notable applications span various domains including signal processing, optics, and quantum mechanics, as comprehensively documented in references [1, 4–8]. Kerr [2] introduced a fractional Hankel transform that depends on a parameter α in $L^2(\mathbb{R}^+)$. This transform, which is a generalization of the classical Hankel transform, is defined by

$$h_{\mu,\alpha} f(y) = \int_0^\infty f(x) K_\alpha(x,y)dx,$$

where the Kernel $K_\alpha(x,y)$ is given by

$$K_\alpha(x,y) = A_{\mu,\alpha} e^{-\frac{i}{2}(x^2+y^2)\cot\frac{\alpha}{2}} \left(\frac{xy}{|\sin\frac{\alpha}{2}|}\right)^{1/2} J_\mu\left(\frac{xy}{|\sin\frac{\alpha}{2}|}\right),$$

where J_μ is the Bessel function of the first kind and order μ and $\widehat{\alpha} = sgn\alpha$, $f \in L^2(\mathbb{R}^+)$, $\alpha \in \mathbb{R} \setminus \{2k\pi\}$, $k \in \mathbb{Z}$, $A_{\mu,\alpha} = |\sin\frac{\alpha}{2}|^{-\frac{1}{2}} e^{i(\frac{\pi}{2}\widehat{\alpha}-\frac{\alpha}{2})(\mu+1)}$ and $\mu > 1$. For $\alpha = \pi$, we get the classical Hankel transform

$$h_\mu f(y) = \int_0^\infty \sqrt{xy} f(x) J_\mu(xy)dx$$

Zemanian investigated the Hankel transform on distributions of slow growth in [11] and on distributions of rapid growth in [9], something that apparently had not been done before. To define the Hankel transform on the distribution, Zemanian introduced the functional space $\mathcal{H}_\mu$.

2020 *Mathematics Subject Classification*. 46F12; 44-XX.
Keywords. Hankel transform, Distribution, Fractional Hankel transform, Testing function space.
Received: 19 October 2023; Revised: 09 November 2023; Accepted: 10 November 2023
Communicated by Maria Alessandra Ragusa
* Corresponding author: Latifa El Bezdaoui
Email addresses: latifabezdaoui@gmail.com (Latifa El Bezdaoui), abdellah.taqbibt@usms.ma (Abdelah Taqbibt), m.elomari@usms.ma (M'hamed Elomari), sa.chadli@yahoo.fr (Lalla Saadia Chadli)

Based on the work of Zemanian[11] and from the works cited before, we extend the fractional Hankel transform to function test, i.e. in the space $\mathcal{H}_\mu$. Additionally, we introduce the fractional Hankel transform of generalized functions in $\mathcal{H}'_\mu$ the dual space of $\mathcal{H}_\mu$. Our main achievement lies in the extension of this transform to the space of distribution $\mathcal{H}'_\mu$. Furthermore, we prove some properties of this transform in the framework of these spaces.

This article is structured as follows: section 2, recalls the definitions of the spaces $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$, together with a discussion of their main properties. In section 3, our main objectives are divided into two steps: firstly, to define the concept of fractional Hankel transform and, secondly, to prove that it is an homeomorphism on the space $\mathcal{H}_\mu$ when $\mu \geq -\frac{1}{2}$. In section 4, we introduce the fractional Hankel transform in the context of $\mathcal{H}'_\mu$ and some fundamental properties of this transform are proved.

2. Prelimanaries

In this section, we present the spaces of functions and generalized functions that we consider later.

2.1. The Testing-Function Space $\mathcal{H}_\mu$

Let $I = (0, \infty)$. For each real number μ we define the space $\mathcal{H}_\mu$ as follows

$$\mathcal{H}_\mu = \left\{\phi : I \to \mathbb{C} / \ \phi \in C^\infty(I),\ \forall m, n \in \mathbb{N}, \quad \gamma^\mu_{m,k}(\phi) < \infty\right\},$$

where $\gamma^\mu_{m,k}(\phi) = \sup\limits_{x\in I}\left\{(1+x^2)^m |(\frac{1}{x}D)^k[x^{-\mu-1/2}\phi(x)]|\right\}$.

The space $\mathcal{H}_\mu$ is linear. Moreover, each $\gamma^\mu_{m,k}$ is a seminorm on $\mathcal{H}_\mu$ and since the $\gamma^\mu_{m,0}$ are norms, so the set $\{\gamma^\mu_{m,k}\}_{m,k=0}^\infty$ is a multinorm. The topology on $\mathcal{H}_\mu$ is produced by the set $\{\gamma^\mu_{m,k}\}_{m,k=0}^\infty$.

Lemma 2.1. *[10, p:130-131] $\phi \in \mathcal{H}_\mu$ if and only if it satisfies the following three conditions:*

i) *$\phi \in C^\infty(I)$.*

ii) *For each nonnegative integer k,*

$$\phi(x) = x^{\mu+\frac{1}{2}}\left[a_0 + a_2 x^2 + ... + a_{2k}x^{2k} + R_{2k}(x)\right],$$

where
$a_{2k} = \dfrac{1}{k!2^k}\lim\limits_{x\to 0^+}(x^{-1}D)^k x^{-\mu-\frac{1}{2}}\phi(x)$ *and* $(x^{-1}D)^k R_{2k}(x) = O_{x\to 0^+}(1)$.

iii) *For each nonnegative integer k, $D^k\phi(x)$ is of rapid descent as $x \to \infty$ (i.e $D^k\phi(x)$ tends to zero faster than any power of $\frac{1}{x}$ as $x \to \infty$).*

Remark 2.2. *For any fixed $y \in I$, the function $\left(\frac{xy}{|\sin\frac{\alpha}{2}|}\right)^{1/2} J_\mu\left(\frac{xy}{|\sin\frac{\alpha}{2}|}\right)$ satisfies conditions **i)** and **ii)** of Lemma **2.1**. However, it does not satisfy the condition **iii)**. On the other hand, from [3, p:134 and p:147] we get*

$$\left(\frac{xy}{|\sin\frac{\alpha}{2}|}\right)^{1/2} J_\mu\left(\frac{xy}{|\sin\frac{\alpha}{2}|}\right) \sim \sqrt{\frac{2}{\pi}}\cos(\frac{xy}{|\sin\frac{\alpha}{2}|} - \frac{\mu\pi}{2} - \frac{\pi}{4}), \quad x \to \infty.$$

Hence, $\left(\frac{xy}{|\sin\frac{\alpha}{2}|}\right)^{1/2} J_\mu\left(\frac{xy}{|\sin\frac{\alpha}{2}|}\right) \notin \mathcal{H}_\mu$.

Lemma 2.3. *[10, p:131] The space $\mathcal{H}_\mu$ is complete and therefore a Frechet space.*

From the previous discussion, the authors demonstrated that $\mathcal{H}_\mu$ is a space of test functions.

2.2. Distribution space $\mathcal{H}'_\mu$

$\mathcal{H}'_\mu$ denotes the dual of $\mathcal{H}_\mu$. According to [10, Theorem 1.8-3], $\mathcal{H}'_\mu$ is also complete. The members of $\mathcal{H}'_\mu$ are the generalized functions on which our fractional Hankel transform will be defined. Furthermore, $\mathcal{H}_\mu$ can be identified with a subspace of $\mathcal{H}'_\mu$ when $\mu \geq -\frac{1}{2}$.
Now, we list some other properties of $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$, which will be useful for us in the future.

1. It is clear that $\mathcal{D}(I)$ is a subspace of $\mathcal{H}_\mu$ for every choice of μ, and that convergence in $\mathcal{D}(I)$ implies convergence in $\mathcal{H}_\mu$. Consequently, the restriction of any $f \in \mathcal{H}'_\mu$ to $\mathcal{D}(I)$ is a member of $\mathcal{D}'(I)$. However, $\mathcal{D}(I)$is not dense in $\mathcal{H}_\mu$.
2. If q is an even positive integer, then $\mathcal{H}_{\mu+q} \subset \mathcal{H}_\mu$ and the topology of $\mathcal{H}_{\mu+q}$ is stronger than that induced on it by $\mathcal{H}_\mu$.
3. For each μ, $\mathcal{H}_\mu$ is clearly a subspace of $C^\infty(I)$. Moreover, it is dense in $C^\infty(I)$. Moreover, the topology of $\mathcal{H}_\mu$ is stronger than that induced on it by $C^\infty(I)$.

Multipliers in $\mathcal{H}_\mu$: Let O denote the linear space of all smooth functions θ defined on I such that for each nonnegative integer ν, there is an integer n_ν for which $\frac{(x^{-1}D)^\nu \theta(x)}{1+x^{n_\nu}}$ is bounded for all $x \in I$. The product of any two members of O is also in O.
Any $\theta \in O$ is a multiplier for $\mathcal{H}_\mu$ for every μ. Indeed, for $\phi \in \mathcal{H}_\mu$, we have

$$(x^{-1}D)^k x^{-\mu-\frac{1}{2}}\theta\phi = \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{(x^{-1}D)^\nu \theta}{1+x^{n_\nu}} (1+x^{n_\nu})(x^{-1}D)^{k-\nu} x^{-\mu-\frac{1}{2}}\phi.$$

Then,

$$\gamma^{\mu}_{m,k}(\theta\phi) \leq \sum_{\nu=0}^{k} \binom{k}{\nu} B_\nu [\gamma^{\mu}_{m,k-\nu}(\phi) + \gamma^{\mu}_{m+n_\nu,k-\nu}(\phi)],$$

where the B_ν are constants. This prove that the linear operator $\phi \to \theta\phi$ is a continuous mapping of $\mathcal{H}_\mu$ into itself.

Multipliers in $\mathcal{H}'_\mu$: The adjoint operator $f \to \theta f$,which is defined on $\mathcal{H}'_\mu$ by

$$\langle \theta f, \phi \rangle = \langle f, \theta\phi \rangle, \quad f \in \mathcal{H}'_\mu,\ \phi \in \mathcal{H}_\mu, \theta \in O,$$

is a continuous linear mapping of $\mathcal{H}'_\mu$ into itself.

3. Fractional Hankel tranform of generalized functions

In this section, our main objectives are divided into two steps: first, to introduce the concept of the fractional Hankel transform, and second, to prove that it is an homeomorphism on the space $\mathcal{H}_\mu$. To achieve the second objective, we will construct certain transform operations, namely the linear differential operators M_μ and N_μ in the space $\mathcal{H}_\mu$, and the linear operators P_μ and Q_μ within $\mathcal{H}_{\mu+1}$. We will utilize these operations to obtain specific properties of the fractional Hankel transform.

Let $\mu \geq -\frac{1}{2}$, we define the fractional Hankel transform $h_{\mu,\alpha}$ by

$$h_{\mu,\alpha}\phi(y) = \int_0^{+\infty} \phi(x) A_{\mu,\alpha} e^{-\frac{i}{2}(x^2+y^2)\cot\frac{\alpha}{2}} \left(\frac{xy}{|\sin\frac{\alpha}{2}|}\right)^{1/2} J_\mu\left(\frac{xy}{|\sin\frac{\alpha}{2}|}\right) dx,$$

for every $x \in I$ and $\phi \in \mathcal{H}_\mu$.

Now, we define the linear differential operators M_μ, N_μ, P_μ and Q_μ by

$$M_\mu\phi(x) = x^{\mu+\frac{1}{2}}e^{\frac{i}{2}x^2\cot\frac{\alpha}{2}}D_x x^{-\mu-\frac{1}{2}}e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}\phi(x).$$
$$N_\mu\phi(x) = x^{\mu+\frac{1}{2}}e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}D_x x^{-\mu-\frac{1}{2}}e^{\frac{i}{2}x^2\cot\frac{\alpha}{2}}\phi(x),$$
$$P_\mu\phi(x) = x^{-\mu-\frac{1}{2}}e^{\frac{i}{2}x^2\cot\frac{\alpha}{2}}D_x x^{\mu+\frac{1}{2}}e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}\phi(x),$$
$$Q_u\phi(x) = x^{-\mu-\frac{1}{2}}e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}D_x x^{\mu+\frac{1}{2}}e^{\frac{i}{2}x^2\cot\frac{\alpha}{2}}\phi(x).$$

Next, we need the following lemmas

Lemma 3.1. *N_μ and M_μ are a continuous linear mappings of $\mathcal{H}_\mu$ into $\mathcal{H}_{\mu+1}$.*

Proof. for $\phi \in \mathcal{H}_\mu$ and any choice for m and k,

$$\begin{aligned}\gamma^{\mu+1}_{m,k}(N_\mu\phi) &= \sup_{0<x<\infty}|x^m(x^{-1}D)^k[x^{-1}e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}D_x x^{-\mu-\frac{1}{2}}e^{\frac{i}{2}x^2\cot\frac{\alpha}{2}}\phi(x)] \\ &= \sup_{0<x<\infty}|i\cot\frac{\alpha}{2}x^m(x^{-1}D)^k[x^{-\mu-1}\phi(x)] + x^m(x^{-1}D)^{k+1}[x^{-\mu-\frac{1}{2}}\phi(x)] \\ &\le |\cot\frac{\alpha}{2}|\gamma^\mu_{m,k}(\phi(x)) + \gamma^\mu_{m,k+1}(\phi(x))\end{aligned}$$

□

Similarly, we prove that M_μ is a continuous linear mapping of $\mathcal{H}_\mu$ into $\mathcal{H}_{\mu+1}$.

Lemma 3.2. *P_μ and Q_μ are a continuous linear mappings of $\mathcal{H}_{\mu+1}$into $\mathcal{H}_\mu$.*

Proof. for $\phi \in \mathcal{H}_{\mu+1}$ and any choice for m and k, since $\mathcal{H}_{\mu+1} \subset \mathcal{H}_{\mu-1}$, then

$$\begin{aligned}\gamma^{\mu}_{m,k}(P_\mu\phi) &= \sup_{0<x<\infty}|x^m(x^{-1}D)^k[x^{-2\mu-1}e^{\frac{i}{2}x^2\cot\frac{\alpha}{2}}D_x x^{\mu+\frac{1}{2}}e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}\phi]| \\ &= \sup_{0<x<\infty}|(2\mu+2)x^m(x^{-1}D)^k x^{-\mu-\frac{3}{2}}\phi \\ &+ i\cot\frac{\alpha}{2}x^m(x^{-1}D)^k x^2 x^{-\mu-\frac{3}{2}} + x^m(x^{-1}D)^k x^2(x^{-1}D)(x^{-\mu-\frac{3}{2}}\phi)| \\ &\le (2\mu+2)\gamma^{\mu+1}_{m,k}(\phi) + |\cot\frac{\alpha}{2}|\gamma^{\mu+1}_{m,k}(\phi) + \gamma^{\mu+1}_{m+2,k+1}(\phi)\end{aligned}$$

□

Similarly, we prove that Q_μ is a continuous linear mapping of $\mathcal{H}_{\mu+1}$ into $\mathcal{H}_\mu$.

Proposition 3.3. *Let $\phi \in \mathcal{H}_\mu$ such that $\mu \ge -\frac{1}{2}$, then*

i) $N_\mu h_{\mu,\alpha}(\phi) = h_{\mu+1,\alpha}(-\frac{x}{|\sin\frac{\alpha}{2}|}\phi(x))$,

ii) $h_{\mu+1,\alpha}(M_\mu\phi) = -\frac{y}{|\sin\frac{\alpha}{2}|}h_{\mu,\alpha}(\phi(x))$.

If $\phi \in \mathcal{H}_{\mu+1}$, then

iii) $Q_\mu h_{\mu+1,\alpha}(\phi) = h_{\mu,\alpha}(\frac{x}{|\sin\frac{\alpha}{2}|}\phi)$

iv) $h_{\mu,\alpha}(P_\mu\phi) = \frac{y}{|\sin\frac{\alpha}{2}|}h_{\mu+1,\alpha}(\phi)$

Proof. **i)** From [3, p:154], we can get

$$D_y y^{-\mu} J_\mu(xy) = -xy^{-\mu} J_{\mu+1}(xy)$$

and differentiating under an integral sign as follows

$$\begin{aligned} N_\mu h_{\mu,\alpha}(\phi)(y) &= \int_0^\infty A_{\mu,\alpha}(\frac{-x}{|\sin\frac{\alpha}{2}|})\phi(x)e^{-\frac{i}{2}(x^2+y^2)\cot\frac{\alpha}{2}}(\frac{xy}{|\sin\frac{\alpha}{2}|})^{1/2} J_{\mu+1}(\frac{xy}{|\sin\frac{\alpha}{2}|})dx \\ &= h_{\mu+1,\alpha}(-\frac{x}{|\sin\frac{\alpha}{2}|}\phi(x)) \end{aligned}$$

ii) According to [3, p:154], we obtain

$$D_x x^{\mu+1} J_{\mu+1}(xy) = yx^{\mu+1} J_\mu(xy)$$

and an integration by parts, we can get

$$\begin{aligned} h_{\mu+1,\alpha}(M_\mu\phi)(y) =& \Big[A_{\mu,\alpha} e^{-\frac{i}{2}(x^2+y^2)\cot\frac{\alpha}{2}}\phi(x)(\frac{xy}{|\sin\frac{\alpha}{2}|})^{1/2} J_{\mu+1}(\frac{xy}{|\sin\frac{\alpha}{2}|})\Big]_0^\infty \\ &- \frac{y}{|\sin\frac{\alpha}{2}|}\int_0^\infty A_{\mu,\alpha} e^{-\frac{i}{2}(x^2+y^2)\cot\frac{\alpha}{2}}\phi(x)(\frac{xy}{|\sin\frac{\alpha}{2}|})^{1/2} J_\mu(\frac{xy}{|\sin\frac{\alpha}{2}|})dx. \end{aligned}$$

The limit terms are equal to zero since $\phi(x)$ is of rapid descent as $x \to \infty$ and as $x \to 0^+$(From Lemma **2.1**), $(\frac{xy}{|\sin\frac{\alpha}{2}|})^{1/2} J_{\mu+1}(\frac{xy}{|\sin\frac{\alpha}{2}|}) = O(x)$ and $\phi(x) = O(1)$ when $\mu \geq -\frac{1}{2}$. Then,

$$h_{\mu+1,\alpha}(M_\mu\phi)(y) = -\frac{y}{|\sin\frac{\alpha}{2}|} h_{\mu,\alpha}(\phi(x)).$$

iii) From [3, p:154], we can get

$$D_y y^{\mu+1} J_{\mu+1}(xy) = xy^{\mu+1} J_\mu(xy).$$

Then,

$$\begin{aligned} Q_\mu h_{\mu+1,\alpha}(\phi)(y) &= y^{-\mu-\frac{1}{2}} e^{-\frac{i}{2}y^2\cot\frac{\alpha}{2}} \int_0^\infty A_{\mu,\alpha}(\frac{x}{|\sin\frac{\alpha}{2}|})^{1/2} e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}} D_y[y^{\mu+1} J_{\mu+1}(\frac{xy}{|\sin\frac{\alpha}{2}|})]dx \\ &= \int_0^\infty A_{\mu,\alpha}(\frac{x}{|\sin\frac{\alpha}{2}|})\phi(x) e^{-\frac{i}{2}(x^2+y^2)\cot\frac{\alpha}{2}}(\frac{xy}{|\sin\frac{\alpha}{2}|})^{1/2} J_\mu(\frac{xy}{|\sin\frac{\alpha}{2}|})dx \\ &= h_{\mu,\alpha}(\frac{x}{|\sin\frac{\alpha}{2}|}\phi(x)) \end{aligned}$$

iv) From [3, p:154], we can get

$$D_x x^{-\mu} J_\mu(xy) = -yx^{-\mu} J_{\mu+1}(xy)$$

and an integration by parts ,we obtain

$$\begin{aligned} h_{\mu,\alpha}(P_\mu\phi)(y) &= \int_0^\infty A_{\mu,\alpha} x^{\mu+\frac{1}{2}} e^{-\frac{i}{2}y^2\cot\frac{\alpha}{2}} D_x[x^{\mu+\frac{1}{2}} e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}\phi(x)](\frac{xy}{|\sin\frac{\alpha}{2}|})^{\frac{1}{2}} J_\mu(\frac{xy}{|\sin\frac{\alpha}{2}|})dx \\ &= \Big[A_{\mu,\alpha} e^{-\frac{i}{2}(x^2+y^2)\cot\frac{\alpha}{2}}\phi(x)(\frac{xy}{|\sin\frac{\alpha}{2}|})^{1/2} J_\mu(\frac{xy}{|\sin\frac{\alpha}{2}|})\Big]_0^\infty \\ &+ \frac{y}{|\sin\frac{\alpha}{2}|}\int_0^\infty A_{\mu,\alpha} e^{-\frac{i}{2}(x^2+y^2)\cot\frac{\alpha}{2}}\phi(x)(\frac{xy}{|\sin\frac{\alpha}{2}|})^{1/2} J_{\mu+1}(\frac{xy}{|\sin\frac{\alpha}{2}|})dx. \end{aligned}$$

According to Lemma **2.1**, the limit terms are equal to zero since $\phi(x)$ is of rapid descent as $x \to \infty$ and as $x \to 0^+$, $(\frac{xy}{|\sin\frac{\alpha}{2}|})^{1/2} J_\mu(\frac{xy}{|\sin\frac{\alpha}{2}|}) = O(x)$ and $\phi(x) = O(1)$ when $\mu \geq -\frac{1}{2}$.
Then,

$$h_{\mu+1,\alpha}(P_\mu\phi)(y) = \frac{y}{|\sin\frac{\alpha}{2}|} h_{\mu+1,\alpha}(\phi(x)).$$

□

Theorem 3.4. *For $\mu \geq -\frac{1}{2}$, the fractional Hankel transform $h_{\mu,\alpha} : \mathcal{H}_\mu \longrightarrow \mathcal{H}_\mu$ is an homeomorphism .*

Proof. Let $\phi \in \mathcal{H}_\mu$, and let m and k be two non-negative integers. By applying **i)** of Proposition **3.3** k times and **ii)** of Proposition **3.3** m times, we obtain

$$(\frac{-y}{|\sin\frac{\alpha}{2}|})^m(N_{\mu+k+m-1}...N_{\mu+m}h_{\mu,\alpha}(\phi))(y) = (\frac{-y}{|\sin\frac{\alpha}{2}|})^m h_{\mu+k+m}(\frac{-x}{|\sin\frac{\alpha}{2}|})^k\phi(x))$$
$$= h_{\mu,\alpha}[M_{\mu+m+k}...M_{\mu+k})((\frac{-x}{|\sin\frac{\alpha}{2}|})^k\phi(x)].$$

Noting that,

$$N_{\mu+k+m-1}...N_{\mu+m+1}N_{\mu+m}x^k\phi(x) = x^kN_{\mu+k-1}...N_{\mu+1}N_\mu\phi(x)$$

and

$$M_{\mu+m+k-1}...M_{\mu+k+1}M_{\mu+k}x^k\phi(x) = x^kM_{\mu+m}...M_{\mu+1}M_\mu\phi(x).$$

Then,

$$(\frac{-y}{|\sin\frac{\alpha}{2}|})^m(N_{\mu+k-1}...N_\mu h_{\mu,\alpha}(\phi))(y) = \int_0^\infty(\frac{-x}{|\sin\frac{\alpha}{2}|})^k[M_{\mu+m}...M_\mu\phi(x)]A_{\mu,\alpha}e^{-\frac{i}{2}(x^2+y^2)\frac{\alpha}{2}}(\frac{xy}{|\sin\frac{\alpha}{2}|})^{\frac{1}{2}}J_{\mu+k+m}(\frac{xy}{|\sin\frac{\alpha}{2}|})dx.$$

Since,

$$N_{\mu+k-1}...N_\mu h_{\mu,\alpha}(\phi))(y) = y^{\mu+k+\frac{1}{2}}e^{-\frac{i}{2}y^2\cot\frac{\alpha}{2}}(y^{-1}D_y)^k[y^{-\mu-\frac{1}{2}}e^{\frac{i}{2}y^2\cot\frac{\alpha}{2}}h_{\mu,\alpha}(\phi)]$$

and

$$M_{\mu+m}...M_\mu\phi(x) = x^{\mu+m+\frac{1}{2}}e^{\frac{i}{2}x^2\cot\frac{\alpha}{2}}(x^{-1}D_x)^m[x^{-\mu-\frac{1}{2}}e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}\phi(x)].$$

Then,

$$(-1)^{m+k}y^m(y^{-1}D_y)^k[y^{-\mu-\frac{1}{2}}e^{\frac{i}{2}y^2\cot\frac{\alpha}{2}}h_{\mu,\alpha}(\phi)] = A_{\mu,\alpha}(\frac{1}{|\sin\frac{\alpha}{2}|})^{\mu+2k-m+\frac{1}{2}}$$
$$\times\int_0^\infty x^{2\mu+2k+m+1}(x^{-1}D_x)^m[x^{-\mu-\frac{1}{2}}e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}\phi(x)]\frac{J_{\mu+k+m}(\frac{xy}{|\sin\frac{\alpha}{2}|})}{(\frac{xy}{|\sin\frac{\alpha}{2}|})^{\mu+k}}dx.$$

Assume that n is an integer no less than $\mu + k + \frac{1}{2}(m+1)$, then

$$x^{2\mu+2k+m+1} < (1+x^2)^n,$$

for $x > 0$. Since $\mu \geq -\frac{1}{2}$, so $\frac{J_{\mu+k+m}(\frac{xy}{|\sin\frac{\alpha}{2}|})}{(\frac{xy}{|\sin\frac{\alpha}{2}|})}$ is bounded on $0 < xy < \infty$ by the constant $C_{k,m}$. Then,

$$\gamma^\mu_{m,k}\left(e^{\frac{i}{2}y^2\cot\frac{\alpha}{2}}h_{\mu,\alpha}(\phi)\right) \leq A_{\mu,\alpha}(\frac{1}{|\sin\frac{\alpha}{2}|})^{\mu+2k-m+\frac{1}{2}}$$
$$\times\int_0^\infty(1+x^2)^{n+1}(x^{-1}D_x)^m[x^{-\mu-\frac{1}{2}}e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}\phi(x)]\frac{C_{k,m}}{1+x^2}dx$$
$$\leq \frac{\pi}{2}A_{\mu,\alpha}C_{k,m}(\frac{1}{|\sin\frac{\alpha}{2}|})^{\mu+2k-m+\frac{1}{2}}\sum_{p=0}^{n+1}\binom{n+1}{p}\gamma^\mu_{2p,m}(e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}}\phi).$$

Since $\phi \in \mathcal{H}_\mu$, the last term is finite. This prove that $e^{\frac{i}{2}y^2\cot\frac{\alpha}{2}}h_{\mu,\alpha}(\phi) \in \mathcal{H}_\mu$.

We multiply by $e^{-\frac{i}{2}y^2\cot\frac{\alpha}{2}}$, we can find that $h_{\mu,\alpha}(\phi) \in \mathcal{H}_\mu$ and this linear mapping is also continuous from $\mathcal{H}_\mu$ into $\mathcal{H}_\mu$.
Furthermore, the classical inversion theorem [2, Theorem 3.2] applies in this case since $\mathcal{H}_\mu \subset L_2(0,\infty)$ when $\mu \geq -\frac{1}{2}$. Also, $h_{\mu,-\alpha} = h^{-1}_{\mu,\alpha}$.
Hence, $h_{\mu,\alpha}$ is an homeomorphism on $\mathcal{H}_\mu$. □

Example 3.5.

$$\begin{aligned}
h_{\mu,\alpha}(y^{\mu+\frac{1}{2}}) &= \int_0^\infty x^{\mu+\frac{1}{2}} A_{\mu,\alpha} e^{-\frac{i}{2}(x^2+y^2)\cot\frac{\alpha}{2}} \left(\frac{xy}{|\sin\frac{\alpha}{2}|}\right)^{1/2} J_\mu\left(\frac{xy}{|\sin\frac{\alpha}{2}|}\right)dx \\
&= A_{\mu,\alpha} e^{-\frac{i}{2}y^2\cot\frac{\alpha}{2}} \left(\frac{y}{|\sin\frac{\alpha}{2}|}\right)^{1/2} \int_0^\infty e^{-\frac{i}{2}x^2\cot\frac{\alpha}{2}} x^{\mu+\frac{1}{2}} J_\mu\left(\frac{xy}{|\sin\frac{\alpha}{2}|}\right)dx \\
&= A_{\mu,\alpha} e^{-\frac{i}{2}y^2\cot\frac{\alpha}{2}} \left(\frac{y}{|\sin\frac{\alpha}{2}|}\right)^{1/2} \frac{y^\mu}{(i\cot\frac{\alpha}{2})^{\mu+1}} e^{\frac{-y^2}{2i\cot\frac{\alpha}{2}}} \\
&= A_{\mu,\alpha} e^{-\frac{i}{2}y^2\cot\frac{\alpha}{2}} \frac{(\frac{y}{|\sin\frac{\alpha}{2}|})^{\mu+\frac{1}{2}}}{(i\cot\frac{\alpha}{2})^{\mu+1}} e^{\frac{i}{2}(\frac{y}{\sin\frac{\alpha}{2}})^2\cot\frac{\alpha}{2}}
\end{aligned}$$

4. A distributional fractional Hankel transform

In this section, we define the fractional Hankel transform for some slowly growing distributions. Let μ be a real number such that $-\frac{1}{2} \le \mu < \infty$, the fractional Hankel transform $h^*_{\mu,\alpha}$ on $\mathcal{H}'_\mu$ as the adjoint of $h_{\mu,\alpha}$ on $\mathcal{H}_\mu$ is defined by

$$\langle h^*_{\mu,\alpha}(f), h_{\mu,\alpha}(\phi)\rangle = \langle f, \phi\rangle, \quad \text{for all } \phi \in \mathcal{H}_\mu,\ f \in \mathcal{H}'_\mu. \tag{1}$$

From this equality we immediately obtain the uniquness of $h^*_{\mu,\alpha}$.

Lemma 4.1. *Let $f, g \in \mathcal{H}'_\mu$, such that $\langle h^*_{\mu,\alpha}(f), h_{\mu,\alpha}(\phi)\rangle = \langle h^*_{\mu,\alpha}(g), h_{\mu,\alpha}(\phi)\rangle$ for all $\phi \in \mathcal{H}_\mu$. Then, $f = g$ in the sense of distributions.*

Theorem 4.2. *The fractional Hankel transform $h^*_{\mu,\alpha} : \mathcal{H}'_\mu \longrightarrow \mathcal{H}'_\mu$ is an homeomorphism.*

Proof. Since $h_{\mu,-\alpha} = h^{-1}_{\mu,\alpha}$ on $\mathcal{H}_\mu$ and from [10, Theorem 1.10-2.p29], we obtain that $(h^*_{\mu,\alpha})^{-1} = h^*_{\mu,-\alpha}$ □

The fractional Hankel transform $h_{\mu,\alpha}$ of order $\mu \ge -\frac{1}{2}$, when acting on a function $f \in L_2(\mathbb{R}_+)$, is a special case of our generalized fractional Hankel transform. Indeed, let

$$F_c(y) = h_{\mu,\alpha} f = \int_0^{+\infty} f(x) A_{\mu,\alpha} e^{-\frac{i}{2}(x^2+y^2)\cot\frac{\alpha}{2}} \left(\frac{xy}{|\sin\frac{\alpha}{2}|}\right)^{1/2} J_\mu\left(\frac{xy}{|\sin\frac{\alpha}{2}|}\right)dx,$$

where $\mu \ge -\frac{1}{2}$ and $f \in L_2(\mathbb{R}_+)$. It is clear that F_c is both continuous and bounded on I.
According to [10, Note V. p133], F_c generates a regular generalized function in $\mathcal{H}'_\mu$. Morover, let $F = h^*_{\mu,\alpha} f$ and $\Phi = h_{\mu,\alpha}\phi = h^{-1}_{\mu,-\alpha}\phi$, by the formula **(1)** we can get

$$\langle F, \Phi\rangle = \langle f, \phi\rangle = \int_0^\infty f(x)\phi(x)dx, \quad \text{for all } \phi \in \mathcal{H}_\mu.$$

Since $\Phi \in \mathcal{H}_\mu \subset L_2(\mathbb{R}_+)$ when $\mu \ge -\frac{1}{2}$, we may invoke Parseval's equation [10, Eq(3). p127] to write

$$\int_0^\infty f(x)\phi(x)dx = \int_0^\infty F_c(y)\Phi(y)dy.$$

Therfore,

$$\langle F, \Phi\rangle = \int_0^\infty F_c(y)\Phi(y)dy.$$

Then, we can identify $F = h^*_{\mu,\alpha}$ with $F_c = h_{\mu,\alpha}$ under suitable conditions on f when $\mu \ge -\frac{1}{2}$. Thus, the definition **(1)** of the generalized fractional hankel transform becomes

$$\langle h_{\mu,\alpha}(f), h_{\mu,\alpha}(\phi)\rangle = \langle f, \phi\rangle, \quad \text{for all } \phi \in \mathcal{H}_\mu,\ f \in \mathcal{H}'_\mu.$$

4.1. Some operations on transform formulas:

In $\mathcal{H}'_\mu$ we introduce the differential operators descried in section 3, when they are acting on certain generalized functions. In order to establish certain properties cited in Proposition **3.3** for the generalized fractional Hankel transform in $\mathcal{H}'_\mu$.

Let $\phi \in \mathcal{H}_{\mu+1}$, then $P^*_\mu\phi, Q^*_\mu\phi \in \mathcal{H}_\mu$. The operators N^*_μ and M^*_μ are defined on $\mathcal{H}'_\mu$ as follows

$$\langle N^*_\mu f, \phi\rangle = \langle f, -P^*_\mu\phi\rangle, \quad \text{for all } f \in \mathcal{H}'_\mu.$$

$$\langle M^*_\mu f, \phi\rangle = \langle f, -Q^*_\mu\phi\rangle, \quad \text{for all } f \in \mathcal{H}'_\mu.$$

By Lemma **3**, $f \to N^*_\mu f$ and $f \to M^*_\mu f$ are a continuous linear mapping of $\mathcal{H}'_\mu$ into $\mathcal{H}'_{\mu+1}$.
Similary,for $\phi \in \mathcal{H}_\mu$ we define Q^*_μ and P^*_μ on $\mathcal{H}'_{\mu+1}$ by

$$\langle Q^*_\mu f, \phi\rangle = \langle f, -M^*_\mu\phi\rangle, \quad \text{for all } f \in \mathcal{H}'_{\mu+1}.$$

$$\langle P^*_\mu f, \phi\rangle = \langle f, -N^*_\mu\phi\rangle, \quad \text{for all } f \in \mathcal{H}'_{\mu+1}.$$

It is clear that $M^*_\mu\phi, N^*_\mu\phi \in \mathcal{H}_{\mu+1}$ and by Lemma **3.2**, $f \to Q^*_\mu f$ and $f \to P^*_\mu f$ are a continous linear mapping of $\mathcal{H}'_{\mu+1}$ into $\mathcal{H}'_\mu$.

From [10, Note V,p133], we have $\mathcal{H}_\mu \subset \mathcal{H}'_\mu$ when $\mu \geq \frac{-1}{2}$; in this case, the generalized fractional Hankel transformation $h^*_{\mu,\alpha}$, the generalized operators M^*_μ and N^*_μ when acting on $\mathcal{H}_\mu$, can be identified with $h_{\mu,\alpha}$, M_μ and $N_\mu$respectively. Similary for the generalized operators P^*_μ and Q^*_μ when acting on $\mathcal{H}_{\mu+1}$.

$$\begin{aligned} h^*_{\mu,\alpha}\phi &= h_{\mu,\alpha}\phi, \quad \forall\phi \in \mathcal{H}_\mu. \\ M^*_\mu\phi &= M_\mu\phi, \quad \forall\phi \in \mathcal{H}_\mu. \\ N^*_\mu\phi &= N_\mu\phi, \quad \forall\phi \in \mathcal{H}_\mu. \\ P^*_\mu\phi &= P_\mu\phi, \quad \forall\phi \in \mathcal{H}_{\mu+1}. \\ Q^*_\mu\phi &= Q_\mu\phi, \quad \forall\phi \in \mathcal{H}_{\mu+1}. \end{aligned}$$

We use these equalities and the properties of Proposition **3.3** to prove the following theorem

Theorem 4.3. *Let $\mu \geq -\frac{1}{2}$ and $f \in \mathcal{H}'_\mu$,then*

i) $h^*_{\mu+1,\alpha}(N^*_\mu f) = -\frac{y}{|\sin\frac{\alpha}{2}|}h^*_{\mu,\alpha}(f)$

ii) $N^*_\mu h^*_{\mu,\alpha}(f) = h^*_{\mu+1,\alpha}(-\frac{x}{|\sin\frac{\alpha}{2}|}f)$

If $f \in \mathcal{H}'_{\mu+1}$, then

iii) $h^*_{\mu,\alpha}(Q^*_\mu f) = \frac{y}{|\sin\frac{\alpha}{2}|}h^*_{\mu+1,\alpha}(f)$

iv) $Q^*_\mu h^*_{\mu+1,\alpha}(f) = h^*_{\mu,\alpha}(\frac{x}{|\sin\frac{\alpha}{2}|}f)$

Proof. **i)** Let $\phi \in \mathcal{H}_{\mu+1}$ and $\Phi = h_{\mu+1,\alpha}(\phi)$. Since $\mathcal{H}'_{\mu-1} \subset \mathcal{H}'_{\mu+1}$. Then,

$$\begin{aligned} \langle h^*_{\mu+1,\alpha}(N^*_\mu f), \Phi\rangle &= \langle N^*_\mu f, \phi\rangle \\ &= \langle f, -P^*_\mu\phi\rangle \\ &= \langle h^*_{\mu,\alpha}(f), \frac{-y}{|\sin\frac{\alpha}{2}|}\Phi\rangle \\ &= \langle \frac{-y}{|\sin\frac{\alpha}{2}|}h^*_{\mu,\alpha}(f), \Phi\rangle. \end{aligned}$$

ii) Let $F = h^*_{\mu,-\alpha}(f)$ and note that $h^*_{\mu,\alpha}h^*_{\mu,-\alpha}$ is the identity operator. Then,

$$\begin{aligned} N^*_\mu h^*_{\mu,\alpha}F &= h^*_{\mu+1,\alpha}h^*_{\mu+1,-\alpha}N^*_\mu f \\ &= h^*_{\mu+1,\alpha}(-\frac{y}{|\sin\frac{\alpha}{2}|)}h^*_{\mu,-\alpha}(f) \\ &= h^*_{\mu+1,\alpha}(-\frac{y}{|\sin\frac{\alpha}{2}|}F). \end{aligned}$$

So we get the result, just take F=f and y=x.

iii) Let $\phi \in \mathcal{H}_{\mu+1}$ and $h_{\mu,\alpha}(\phi) = \Phi$. Then,

$$\begin{aligned} \langle h^*_{\mu,\alpha}(Q^*_\mu f), \Phi\rangle &= \langle Q^*_\mu f, \phi\rangle \\ &= \langle f, -M^*_\mu \phi\rangle \\ &= \langle h^*_{\mu+1,\alpha}(f), \frac{y}{|\sin\frac{\alpha}{2}|)}\Phi\rangle \\ &= \langle \frac{y}{|\sin\frac{\alpha}{2}|)}h^*_{\mu+1,\alpha}(f), \Phi\rangle. \end{aligned}$$

iv) Let $F = h^*_{\mu+1,-\alpha}f$. Then,

$$\begin{aligned} Q^*_\mu h^*_{\mu+1,\alpha}(F) &= h^*_{\mu,\alpha}h^*_{\mu,-\alpha}Q^*_\mu f \\ &= h^*_{\mu,\alpha}(\frac{y}{|\sin\frac{\alpha}{2}|}h^*_{\mu+1,-\alpha}(f)) \\ &= h^*_{\mu,\alpha}(\frac{y}{|\sin\frac{\alpha}{2}|}F). \end{aligned}$$

By replacing F by f and y by x, we obtain the result .

□

Example 4.4. *Let τ be a real positive number and ϑ a nonnegative integer. The generalized function δ^ϑ is defined on $\mathcal{H}_\mu$ by*

$$\langle \delta^\vartheta(x-\tau), \phi(x)\rangle = (-1)^\vartheta \phi^\vartheta(\tau), \quad \phi \in \mathcal{H}_\mu$$

Clearly, $\delta^\vartheta(x-\tau) \in \mathcal{H}'_\mu$ for $\mu \geq -\frac{1}{2}$ and

$$h^*_{\mu,\alpha}\delta^\vartheta(x-\tau) = (-1)^\vartheta D^\vartheta_\tau[K_\alpha(x,\tau)].$$

where D^ϑ_τ is a conventional differentiation of ϑ th order. Indeed, for $\phi \in \mathcal{H}_\mu$ and $\Phi = h_{\mu,\alpha}\phi$,

$$\langle \delta^\vartheta(y-\tau), \Phi(y)\rangle = (-1)^\vartheta D^\vartheta_\tau[h^*_{\mu,\alpha}\phi(\tau)] = \langle (-1)^\vartheta D^\vartheta_\tau[K_\alpha(x,\tau)], \phi(x)\rangle.$$

Thus, we have

$$h^*_{\mu,\alpha}D^\vartheta_\tau[K_\alpha(x,\tau)] = (-1)^\vartheta\delta^\vartheta(y-\tau).$$

Since $h^*_{\mu,\alpha} = (h^*_{\mu,\alpha})^{-1}$, then

$$h^*_{\mu,\alpha}\delta^\vartheta(x-\tau) = (-1)^\vartheta D^\vartheta_\tau[K_\alpha(y,\tau)].$$

For $\alpha = \pi$, we obtain

$$h^*_{\mu,\pi}\delta^\vartheta(x-\tau) = (-1)^\vartheta D^\vartheta_\tau[\sqrt{\tau y}J_\mu(\tau y)],$$

Example 4.5. *The function f is defined on $\mathcal{H}_\mu$ by*

$$\langle f, g\rangle = \sum_{n=1}^{+\infty} \int_0^\infty \phi(x) K_\alpha(x, nT) dx,$$

where $\mu \geq -\frac{1}{2}$, $\phi \in \mathcal{H}_\mu$ and T is a real positive number. We prove that $f \in \mathcal{H}'_\mu$. For $\phi \in \mathcal{H}_\mu$ and $\Phi = h_{\mu,\alpha}\phi$, we have

$$\begin{aligned}\langle f, g\rangle &= \sum_{n=1}^{+\infty} \langle K_\alpha(x, nT), \phi(x)\rangle \\ &= \sum_{n=1}^{+\infty} h_{\mu,\alpha}\phi(nT) \\ &= \sum_{n=1}^{+\infty} \Phi(nT) \\ &= \langle \sum_{n=1}^{+\infty} \delta(y - nT), \Phi(y)\rangle.\end{aligned}$$

Then, $f \in \mathcal{H}'_\mu$ and $h^*_{\mu,\alpha} f(x) = \sum_{n=1}^{+\infty} \delta(y - nT)$.

References

[1] F. Ge, D. Zhao, S. Wang, *Fractional Hankel transform and the diffraction of misaligned optical systems*, Journal of Modern Optics. **52**(1) (2005), 61–71.

[2] H. K. Fiona, *A Fractional power theory for Hankel transforms in $L^2(\mathbb{R}^+)$*, Int. J. of Mathematical Analysis and Application. **158** (1991), 114–123.

[3] E. Jahnke, F. Emde, F. Losch, *Tables of Higher Functions* , (3rd edition), McGraw-Hill, New York, 1986.

[4] A.Kassymov, MA. Ragusa, M. Ruzhansky, D.Suragan, *Stein-Weiss-Adams inequality on Morrey spaces*, Journal of Functional Analysis, **285**(2023).

[5] V. Namias, *The fractional order Fourier transform and its application to quantum mechanics*, IMA Journal of Applied Mathematics. **25**(3)(1980) , 241–265.

[6] H.Ozaktas, Z. Zalevsky, M. A. Kutay, *The Fractional Fourier Transform with Applications in Optics and Signal Processing*, Jhon Wiley and Sons. New York, (2001).

[7] A. Selvam, S. Sabarinathan, S. Noeiaghdam, V. Govindan, *Fractional Fourier transform and Ulam stability of fractional differential equation with fractional Caputo-type derivative*, Journal of Function Spaces, **2022**, art.n. 3777566, (2022).

[8] H. M. Srivastava, K. K. Mishran, S. K. Upadhyay, Abelian theorems involving the fractional wavelet transforms, Filomat, **37**(2023), 9453–9468.

[9] A. H. Zemanian, *The Hankel transformation of certain distributions of rapid growth*, SIAM Journal on Applied Mathematics. **14**(4)(1966), 678–690.

[10] A. H. Zemanian, *Generalized Integral Transformations*, Interscience Publishers. New York, (1968).

[11] A. H. Zemanian, *A distributional Hankel transformation*, SIAM Journal on Applied Mathematics. **14**(3)(1966), 561–576.