Filomat 38:5 (2024), 1531–1545
https://doi.org/10.2298/FIL2405531A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Identities derived from a particular class of generating functions for Frobenius-Euler type Simsek numbers and polynomials

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Dedicated to Professor Yılmaz Simsek on the occassion of his 60th birthday

Abstract. In this paper, by aid of the derivative of a particular class of generating functions for Frobenius-Euler type Simsek numbers and polynomials, we obtain some formulas. Moreover, we derive some Riemann integral and p-adic integral formulas for the Frobenius-Euler type Simsek polynomials mentioned above. We also construct a Szasz-type linear positive operator by using generating function for Frobenius-Euler type Simsek polynomials. Finally, some numerical results of this operator with convergence properties associated with the rate of modulus are presented.

1. Introduction

Throughout this study, the results are given by assuming that $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$, and $\mathbb{C}$ denotes respectively the set of positive integers, the set of integers, the set of non-negative integers, the set of real numbers, and the set of complex numbers.

The Stirling numbers of the first kind, $S_1(n,k)$, are defined by the following generating function:

$$\frac{(\ln(1+t))^k}{k!} = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \quad (k \in \mathbb{N}_0) \tag{1}$$

so that,

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)\, x^k \tag{2}$$

(*cf.* [7, 8, 34, 35]; see also the references cited therein).

The Stirling numbers of the second kind, $S_2(n,k)$, are defined by

$$\frac{(e^t-1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k)\frac{t^n}{n!}, \tag{3}$$

2020 *Mathematics Subject Classification*. 05A10; 05A15; 05A19; 11B37; 11B68; 11B73; 11S80; 11S23; 34A99; 41A25; 41A36.
Keywords. Apostol-type polynomials; Bernoulli numbers; Euler numbers; Generating function; Stirling numbers of the first kind; Derivative formulas; Integral formulas; Korovkin-Bohman type approximation; Modulus of continuity.
Received: 22 July 2023; Accepted: 20 August 2023
Communicated by Dragan S. Djordjević
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and an explicit formula for these numbers are given as follows:

$$S_2(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n \tag{4}$$

which satisfy the following recurrence relation:

$$S_2(n+1,k) = S_2(n,k-1) + kS_2(n,k)$$

with

$$S_2(0,0) = 1,\ S_2(n,k) = 0 \text{ if } k > n,\ S_2(n,0) = 0 \text{ if } n > 0$$

(*cf.* [11], [24], [26], [35]; and the references cited therein).

The Apostol-Bernoulli numbers, denoted by $\mathcal{B}_n(\lambda)$, are defined by the following generating function (*cf.* [1]):

$$F_{\mathcal{B}}(\omega,\lambda) := \frac{t}{\lambda e^t - 1} = \sum_{n=0}^{\infty}\mathcal{B}_n(\lambda)\frac{t^n}{n!}, \tag{5}$$

$$(\lambda \in \mathbb{C};\ |t| < 2\pi \text{ if } \lambda = 1 \text{ and } |t| < |\ln(\lambda)| \text{ if } \lambda \neq 1)$$

which, when $\lambda = 1$, reduces to the following generating function of the Bernoulli numbers of the first kind, denoted by B_n:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \qquad (t < |2\pi|) \tag{6}$$

(*cf.* [1]-[36]).

The Apostol-Euler numbers of the first kind $\mathcal{E}_n(\lambda)$, are defined by the following generating function (*cf.* [35]).

$$F_{\mathcal{E}}(\omega,\lambda) = \frac{2}{\lambda e^t + 1} = \sum_{n=0}^{\infty}\mathcal{E}_n(\lambda)\frac{t^n}{n!}, \tag{7}$$

$$(\lambda \in \mathbb{C};\ |t| < \pi \text{ if } \lambda = 1 \text{ and } |t| < |\ln(-\lambda)| \text{ if } \lambda \neq 1)$$

which, when $\lambda = 1$, reduces to the following generating function of the Euler numbers of the first kind, denoted by E_n:

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \qquad (t < |\pi|) \tag{8}$$

(*cf.* [8]-[36]).

The Frobenius–Euler numbers and polynomials are defined by, respectively:

$$\frac{1-u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u)\frac{t^n}{n!}, \tag{9}$$

and

$$\frac{1-u}{e^t - u}e^{tx} = \sum_{n=0}^{\infty} H_n(x;u)\frac{t^n}{n!},$$

where $u \in \mathbb{C} \setminus \{1\}$ (*cf.* [6, 17, 18, 20, 30–32]; and the references cited therein).

1.1. Frobenius-Euler type Simsek numbers and polynomials

The Frobenius-Euler type Simsek polynomials $\ell_n(x;v)$ are defined by the following generating functions:

$$F_\ell(x;t,v) := \frac{t^v}{\prod_{j=0}^{v-1}(e^t-j)}e^{tx} = \sum_{n=0}^{\infty}\ell_n(x;v)\frac{t^n}{n!} \tag{10}$$

which were recently introduced and investigated by Simsek in [27].

Substituting $x = 0$ into (10) gives the generating function of the Frobenius-Euler type Simsek numbers $\ell_n(v)$ as follows:

$$F_\ell(t,v) := \frac{t^v}{\prod_{j=0}^{v-1}(e^t-j)} = \sum_{n=0}^{\infty}\ell_n(v)\frac{t^n}{n!} \tag{11}$$

which is equivalent to the special case $f(t) = e^t$, $\overrightarrow{x_v} = (0,1,2,\ldots,v-1)$ and $\overrightarrow{y_v} = (1,1,\ldots,1)$ of the following meromorphic function:

$$F_1\left(t;\overrightarrow{x_v},\overrightarrow{y_v}\right) = \frac{t^v}{h\left(t;\overrightarrow{x_v},\overrightarrow{y_v}\right)}, \tag{12}$$

where

$$h\left(t;\overrightarrow{x_v},\overrightarrow{y_v}\right) = \prod_{j=0}^{v-1}\left(f(t)-x_j\right)^{y_j}, \tag{13}$$

$f(t)$ is an analytic function such that $t \in \mathbb{R}$ (or $\mathbb{C}$); $\overrightarrow{x_v} = (x_0, x_1, \ldots, x_{v-1})$ and $\overrightarrow{y_v} = (y_0, y_1, \ldots, y_{v-1})$ are v-tuples such that $v \in \mathbb{N}$ and $x_j, y_j \in \mathbb{R}$ with $j = 0, 1, \ldots, v-1$. See, for detail, [27].

Observe that

$$\ell_n(v) = \ell_n(0;v) \tag{14}$$

and (10) and (11) gives the following other relation between the Frobenius-Euler type Simsek numbers $\ell_n(v)$ and polynomials $\ell_n(x;v)$:

$$\ell_n(x;v) = \sum_{j=0}^{n}\binom{n}{j}x^j\ell_{n-j}(v) \tag{15}$$

(*cf.* [27]; and see also [3, 28]).

Some special values of the numbers $\ell_n(v)$ are given as follows:

$$\begin{aligned}
&\ell_0(v) = 0; \quad (\forall v \in \mathbb{N}),\\
&\ell_n(1) = (-1)^{n+1}n; \quad (n \in \mathbb{N}_0)\\
&\ell_0(m+1) = \ell_1(m+1) = \cdots = \ell_{m-1}(m+1) = 0, \quad (m \in \mathbb{N})\\
&\ell_m(m+1) \neq 0, \quad (m \in \mathbb{N})
\end{aligned} \tag{16}$$

(*cf.* [27]; and see also [3, 28]).

Some properties of the numbers $\ell_n(v)$ and polynomials $\ell_n(x;v)$ are given below:

The recurrence relation for the numbers $\ell_n(v)$ is given as follows:

If v is an odd positive integer, then we have

$$\sum_{j=0}^{v}(-1)^{j+1}S_1(v,j)\sum_{q=0}^{n}\binom{n}{q}j^{n-q}\ell_q(v) = \begin{cases} 0 & \text{if } n \neq v \\ v! & \text{if } n = v, \end{cases} \tag{17}$$

and, if v is an even positive integer, then we have

$$\sum_{j=0}^{v}(-1)^{j}S_1(v,j)\sum_{q=0}^{n}\binom{n}{q}j^{n-q}\ell_q(v)=\begin{cases}0 & \text{if } n\neq v\\ v! & \text{if } n=v,\end{cases} \tag{18}$$

(*cf.* [27]).

Note that the numbers $\ell_n(v)$ can be computed recursively by the following computation formula, involving the Apostol-Bernoulli numbers:

$$\ell_n(v)=\frac{1}{v-1}\sum_{j=0}^{n}\binom{n}{j}\ell_j(v-1)\mathcal{B}_{n-j}\left(\frac{1}{v-1}\right) \tag{19}$$

where $n\in\mathbb{N}_0$ and $v\in\mathbb{N}\setminus\{1\}$ (*cf.* [27]).

Note that the numbers $\ell_n(v)$ can also be computed recursively by the following computation formula, involving the Apostol-Euler numbers:

$$\ell_n(v)=\sum_{j=0}^{n}\binom{n}{j}\frac{(j-n)\ell_j(v-1)}{2(v-1)}\mathcal{E}_{n-j-1}\left(\frac{1}{1-v}\right)$$

where $n\in\mathbb{N}$ and $v\in\mathbb{N}\setminus\{1\}$ (*cf.* [27]).

Simsek [27] also showed that the generating functions of the numbers $\ell_n(v)$ can be expressed in terms of the Stirling numbers of the first kind, as follows:

$$\sum_{n=0}^{\infty}\ell_n(v)\frac{w^n}{n!}=\begin{cases}\frac{w^v}{\sum_{j=0}^{v}(-1)^{j+1}S_1(v,j)e^{jw}} & \text{if } v \text{ is an odd positive integer },\\ \frac{w^v}{\sum_{j=0}^{v}(-1)^{j}S_1(v,j)e^{jw}} & \text{if } v \text{ is an even positive integer}\end{cases} \tag{20}$$

(*cf.* [27]).

Remark 1.1. *For the other relationships of the numbers $\ell_n(v)$ with the Bernoulli numbers, the Apostol-Bernoulli numbers, the Apostol-Euler numbers, the Apostol-Genocchi numbers and polynomials, the Fubini numbers, and the others, the readers may consult the recent paper* [27] *of Simsek.*

The remainder of this article will be succinctly summarized. In section 2, by using the generating functions for the special numbers and polynomials, we derive some identities. In section 3, we derive an identity for special numbers. In section 4, we obtain some theorems between Frobenious-Euler type Simsek polynomials and special numbers by using integral operators. The last section of this article (Section 5), we construct Szász-type linear positive operators involving the generating functions of Frobenius-Euler type Simsek polynomials for a special value. We also investigate convergence properties of these operators.

2. Formulas arising from the derivative of the generating function for the polynomials $\ell_n(x;v)$ and the numbers $\ell_n(v)$

In this section, we obtain some formulas by aid of the derivative of the generating function for the polynomials $\ell_n(x;v)$ and the numbers $\ell_n(v)$.

Taking derivative of (10) with respect to x, we obtain the following ordinary differential equation:

$$\frac{d}{dx}\{F_\ell(x;t,v)\}=tF_\ell(x;t,v). \tag{21}$$

Using (10), we get

$$\sum_{n=0}^{\infty}\frac{d}{dx}\{\ell_n(x;v)\}\frac{t^n}{n!}=\sum_{n=0}^{\infty}n\ell_{n-1}(x;v)\frac{t^n}{n!} \tag{22}$$

Equating coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we have

Theorem 2.1. *Let $n\in\mathbb{N}$. Then we have*

$$\frac{d}{dx}\{\ell_n(x;v)\}=n\ell_{n-1}(x;v). \tag{23}$$

Remark 2.2. *Theorem* 2.1 *shows that the polynomials $\ell_n(x;v)$ forms an Appell sequence. See, for details,* [24, Theorem 2.5.6, p.27].

Recall from [22, Eq. (9)] and [19] that

$$\frac{d}{dx}\left\{x_{(v)}\right\}=\begin{cases}x_{(v)}\sum\limits_{j=0}^{v-1}\frac{1}{x-j} & \text{if} \quad v\in\mathbb{N}\\ 0 & \text{if} \quad v=0,\end{cases} \tag{24}$$

where $x_{(v)}$ denotes the falling factorial defined by

$$x_{(v)}=x(x-1)(x-2)\dots(x-v+1), \tag{25}$$

where $x\in\mathbb{C}$, $n\in\mathbb{N}_0$ with $x_{(0)}=1$ (*cf.* [1]-[36]).

It is also known from [22, Eq. (11)] that

$$\frac{d}{dx}\left\{x_{(k)}\right\}=(x)_k\,H_k\,(x)\,;\qquad (k\in\mathbb{N}) \tag{26}$$

where

$$H_k\,(x)=\sum_{j=0}^{k-1}\frac{1}{x-j}$$

which is associated with the harmonic numbers.

Let $v\in\mathbb{N}$. Taking derivative of (11) with respect to t and also using (24), we obtain

$$\frac{d}{dt}\{F_\ell\,(t,v)\}=\frac{vt^{v-1}\prod\limits_{j=0}^{v-1}(e^t-j)-t^v\left(\prod\limits_{j=0}^{v-1}(e^t-j)\right)\left(\sum\limits_{j=0}^{v-1}\frac{1}{e^t-j}\right)}{\left(\prod\limits_{j=0}^{v-1}(e^t-j)\right)^2}$$

which gives the following ordinary differential equation:

$$\frac{d}{dt}\{F_\ell\,(t,v)\}=\left(\frac{v}{t}-\sum_{j=0}^{v-1}\frac{1}{e^t-j}\right)F_\ell\,(t,v)\,. \tag{27}$$

By combining the above equation with (11), we have

$$\sum_{n=0}^{\infty}n\ell_n(v)\frac{t^{n-1}}{n!}=\left(\frac{v}{t}-\sum_{j=0}^{v-1}\frac{1}{e^t-j}\right)\sum_{n=0}^{\infty}\ell_n(v)\frac{t^n}{n!}.$$

Thus, by (5), we get

$$\sum_{n=0}^{\infty} n\ell_n(v)\frac{t^{n-1}}{n!} = \left(\frac{v}{t} - \left(\frac{1}{e^t} + \frac{1}{t}\sum_{j=1}^{v-1}\frac{1}{j}\sum_{n=0}^{\infty}\mathcal{B}_n\left(\frac{1}{j}\right)\frac{t^n}{n!}\right)\right)\sum_{n=0}^{\infty}\ell_n(v)\frac{t^n}{n!}.$$

Therefore, we have

$$\begin{aligned}\sum_{n=0}^{\infty} n\ell_n(v)\frac{t^{n-1}}{n!} &= v\sum_{n=0}^{\infty}\ell_n(v)\frac{t^{n-1}}{n!} - \sum_{n=0}^{\infty}(-1)^n\frac{t^n}{n!}\sum_{n=0}^{\infty}\ell_n(v)\frac{t^n}{n!} \\ &\quad -\frac{1}{t}\sum_{j=1}^{v-1}\frac{1}{j}\sum_{n=0}^{\infty}\mathcal{B}_n\left(\frac{1}{j}\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}\ell_n(v)\frac{t^n}{n!}\end{aligned}$$

which, when the Cauchy product is applied, gives

$$\begin{aligned}\sum_{n=0}^{\infty} n\ell_n(v)\frac{t^{n-1}}{n!} &= v\sum_{n=0}^{\infty}\ell_n(v)\frac{t^{n-1}}{n!} - \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\ell_{n-k}(v)\frac{t^n}{n!} \\ &\quad -\frac{1}{t}\sum_{j=1}^{v-1}\frac{1}{j}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k\left(\frac{1}{j}\right)\ell_{n-k}(v)\frac{t^n}{n!}.\end{aligned}$$

Equating coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we have

$$n\ell_n(v) = v\ell_n(v) - n\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\ell_{n-k-1}(v) - \sum_{j=1}^{v-1}\frac{1}{j}\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k\left(\frac{1}{j}\right)\ell_{n-k}(v).$$

After some arrangement at above equation, we arrive at the following theorem:

Theorem 2.3. *Let $n, v \in \mathbb{N}$ such that $n \neq v$. Then we have*

$$\ell_n(v) = \frac{1}{v-n}\left(n\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\ell_{n-k-1}(v) + \sum_{j=1}^{v-1}\frac{1}{j}\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k\left(\frac{1}{j}\right)\ell_{n-k}(v)\right).$$

On the other hand, if we rearrange the equation (27), we also get

$$\frac{d}{dt}\{F_\ell(t,v)\} = \frac{1}{t}F_\ell(t,v-1)F_{\mathcal{B}}\left(t,\frac{1}{v-1}\right) - F_\ell(t,v)\left(e^{-t} + \frac{1}{t}\sum_{j=1}^{v-1}\frac{1}{j}F_{\mathcal{B}}\left(t,\frac{1}{j}\right)\right).$$

By multiplying with w both sides of above equation, we also have

$$t\frac{d}{dt}\{F_\ell(t,v)\} = F_\ell(t,v-1)F_{\mathcal{B}}\left(t,\frac{1}{v-1}\right) - F_\ell(t,v)\left(te^{-t} + \sum_{j=1}^{v-1}\frac{1}{j}F_{\mathcal{B}}\left(t,\frac{1}{j}\right)\right).$$

By using (11) and (5), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\ell_n(v)\frac{t^{n+1}}{n!} &= \sum_{n=0}^{\infty}\ell_n(v-1)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{B}_n\left(\frac{1}{v-1}\right)\frac{t^n}{n!} \\ &\quad -\left(\sum_{n=0}^{\infty}(-1)^n\frac{t^{n+1}}{n!} + \sum_{j=1}^{v-1}\frac{1}{j}\sum_{n=0}^{\infty}\mathcal{B}_n\left(\frac{1}{j}\right)\frac{t^n}{n!}\right)\sum_{n=0}^{\infty}\ell_n(v)\frac{t^n}{n!}.\end{aligned}$$

which, when the Cauchy product is applied, gives

$$\begin{aligned}\sum_{n=0}^{\infty}\ell_n(v)\frac{t^{n+1}}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\ell_k(v-1)\mathcal{B}_{n-k}\left(\frac{1}{v-1}\right)\frac{t^n}{n!}-\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\ell_{n-k}(v)\frac{t^{n+1}}{n!}\\ &\quad+\sum_{j=1}^{v-1}\frac{1}{j}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k\left(\frac{1}{j}\right)\ell_{n-k}(v)\frac{t^n}{n!}.\end{aligned}$$

Equating coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we have

$$\begin{aligned}\ell_{n-1}(v) &= \sum_{k=0}^{n}\binom{n}{k}\ell_k(v-1)\mathcal{B}_{n-k}\left(\frac{1}{v-1}\right)-\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\ell_{n-k-1}(v)\\ &\quad+\sum_{j=1}^{v-1}\frac{1}{j}\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k\left(\frac{1}{j}\right)\ell_{n-k}(v),\end{aligned}$$

where $n\in\mathbb{N}$ and $v\in\mathbb{N}\setminus\{1\}$.

Using (19) in the above equation, we arrive at the following theorem:

Theorem 2.4. *Let $n\in\mathbb{N}$ and $v\in\mathbb{N}\setminus\{1\}$. Then we have*

$$\ell_{n-1}(v)=(v-1)\,\ell_n(v)-\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\ell_{n-k-1}(v)+\sum_{j=1}^{v-1}\frac{1}{j}\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k\left(\frac{1}{j}\right)\ell_{n-k}(v).$$

By combining (11) and (28), we also arrive at the following corollary:

Corollary 2.5. *Let $n,v\in\mathbb{N}\setminus\{1\}$. Then we have*

$$\sum_{j=1}^{v-1}\frac{1}{j}\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k\left(\frac{1}{j}\right)\ell_{n-k}(v)=\frac{(n-1)\,\ell_n(v)-\ell_{n-1}(v)}{1-n}.$$

3. A further identity on the numbers $\ell_n(v)$

On the reciprocal of (25), the Cauchy product of the geometric series, for $|x|<1$, gives

$$\begin{aligned}\frac{1}{x_{(v)}} &= \frac{1}{x(x-1)(x-2)\dots(x-v+1)}\\ &= \frac{(-1)^{v-1}}{(v-1)!x}\sum_{m=0}^{\infty}x^m\sum_{m=0}^{\infty}\left(\frac{x}{2}\right)^m\sum_{m=0}^{\infty}\left(\frac{x}{3}\right)^m\cdots\sum_{m=0}^{\infty}\left(\frac{x}{v-1}\right)^m\\ &= \frac{(-1)^{v-1}}{(v-1)!x}\sum_{m=0}^{\infty}\left(\sum_{n_1+n_2+\ldots+n_{v-2}=m}\frac{1}{(v-1)^{n_{v-2}}}\frac{1}{(v-2)^{n_{v-3}}}\cdots\frac{1}{3^{n_2}}\frac{1}{1^{n_1}}\frac{1}{2^{m-n_1-n_2-\ldots-n_{v-2}}}\right)x^m,\end{aligned}$$

where

$$\sum_{n_1+n_2+\ldots+n_{v-2}=m}=\sum_{n_{v-2}=0}^{m}\sum_{n_{v-3}=0}^{m-n_{v-2}}\cdots\sum_{n_1=0}^{m-n_1-n_2-\ldots-n_{v-2}}.$$

Setting

$$\alpha(m,v)=\sum_{n_1+n_2+\ldots+n_{v-2}=m}\frac{1}{(v-1)^{n_{v-2}}}\frac{1}{(v-2)^{n_{v-3}}}\cdots\frac{1}{3^{n_2}}\frac{1}{1^{n_1}}\frac{1}{2^{m-n_1-n_2-\ldots-n_{v-2}}},\tag{28}$$

we have

$$\frac{x}{x_{(v)}} = \frac{(-1)^{v-1}}{(v-1)!}\sum_{m=0}^{\infty}\alpha(m,v)x^m.$$

Substituting $x = e^t$ into the above equation, we have

$$\frac{e^t}{(e^t)_{(v)}} = \frac{(-1)^{v-1}}{(v-1)!}\sum_{m=0}^{\infty}\alpha(m,v)\left(e^t\right)^m. \tag{29}$$

By combining the above equation with (11), we have

$$\begin{aligned}\frac{e^t}{t^v}\sum_{n=0}^{\infty}\ell_n(v)\frac{t^n}{n!} &= \frac{(-1)^{v-1}}{(v-1)!}\sum_{m=0}^{\infty}\alpha(m,v)\left(e^t\right)^m \\ &= \frac{(-1)^{v-1}}{(v-1)!}\sum_{m=0}^{\infty}\alpha(m,v)\left(e^t-1+1\right)^m \\ &= \frac{(-1)^{v-1}}{(v-1)!}\sum_{m=0}^{\infty}\alpha(m,v)\sum_{k=0}^{m}m_{(k)}\frac{\left(e^t-1\right)^k}{k!}.\end{aligned}$$

Thus, by (3), we have

$$\begin{aligned}\frac{e^t}{t^v}\sum_{n=0}^{\infty}\ell_n(v)\frac{t^n}{n!} &= \frac{(-1)^{v-1}}{(v-1)!}\sum_{m=0}^{\infty}\alpha(m,v)\sum_{k=0}^{m}m_{(k)}\sum_{n=0}^{\infty}S_2(n,k)\frac{t^n}{n!} \\ &= \frac{(-1)^{v-1}}{(v-1)!}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\alpha(m,v)m_{(k)}S_2(n,k)\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, by the Cauchy product, we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\ell_k(v)\frac{t^n}{n!} = \frac{(-1)^{v-1}}{(v-1)!}\sum_{n=0}^{\infty}\left(n_{(v)}\sum_{m=0}^{n-v}\sum_{k=0}^{m}\alpha(m,v)m_{(k)}S_2(n-v,k)\right)\frac{t^n}{n!}.$$

Equating coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 3.1. *Let $n, v \in \mathbb{N}$. Then we have*

$$\sum_{k=0}^{n}\binom{n}{k}\ell_k(v) = \frac{(-1)^{v-1}\,n_{(v)}}{(v-1)!}\sum_{m=0}^{n-v}\sum_{k=0}^{m}\alpha(m,v)m_{(k)}S_2(n-v,k),$$

where $\alpha(m,v)$ is as defined in the equation (28).

4. Formulas arising from the integrals of the polynomials $\ell_n(x;v)$

4.1. Riemann integral formulas

By applying the Riemann integral to both-sides of (15) with respect to x from 0 to 1, we get

$$\int_0^1 \ell_n(x;v)dx = \sum_{j=0}^{n}\binom{n}{j}\ell_{n-j}(v)\int_0^1 x^j dx \tag{30}$$

which gives a formula for the definite integral of the polynomials $\ell_n(x;v)$, given by the following theorem:

Theorem 4.1. *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_0^1 \ell_n(x;v)dx = \sum_{j=0}^{n} \binom{n}{j} \frac{\ell_{n-j}(v)}{j+1}. \tag{31}$$

It is known from [28] that the polynomials $\ell_n(x;v)$ satisfy the following equality, for $n, v \in \mathbb{N}$:

$$n_{(v)} x^{n-v} = \sum_{k=0}^{v} \sum_{j=0}^{n} \binom{n}{j} k^{n-j} S_1(v,k) \ell_j(x;v), \tag{32}$$

(*cf.* [28]).

By applying the Riemann integral to both-sides of (32) with respect to x from 0 to 1, we get

$$n_{(v)} \int_0^1 x^{n-v} dx = \sum_{k=0}^{v} \sum_{j=0}^{n} \binom{n}{j} k^{n-j} S_1(v,k) \int_0^1 \ell_j(x;v)dx \tag{33}$$

where $n, v \in \mathbb{N}$

Using (31) in the above equation, we arrive at the following theorem:

Theorem 4.2. *Let $n, v \in \mathbb{N}$. Then we have*

$$\frac{n_{(v)}}{n-v+1} = \sum_{k=0}^{v} \sum_{j=0}^{n} \sum_{r=0}^{j} \binom{n}{j} k^{n-j} S_1(v,k) \binom{j}{r} \frac{\ell_{j-r}(v)}{r+1}. \tag{34}$$

4.2. p-adic bosonic and fermionic integral formulas

Let $\mathbb{Z}_p$ denote a set of p-adic integers. Let $f(x)$ be a uniformly differentiable function on $\mathbb{Z}_p$. The Volkenborn integral (or p-adic bosonic integral) of the function $f(x)$ is given by:

$$\int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{35}$$

where,

$$\mu_1(x) = \mu_1(x + p^N \mathbb{Z}_p) = \frac{1}{p^N}$$

(*cf.* [25]; see also [14, 15]).

The p-adic fermionic integral of the function $f(x)$ is given by (*cf.* [15]):

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} (-1)^x f(x) \tag{36}$$

where $p \neq 2$ and,

$$\mu_{-1}(x) = \mu_{-1}\left(x + p^N \mathbb{Z}_p\right) = (-1)^x$$

(*cf.* [13, 15]).

It is known that the p-adic bosonic integral of the function $f(x) = x^n$ gives the Bernoulli numbers as follows (*cf.* [14, 25]):

$$B_n = \int_{\mathbb{Z}_p} x^n d\mu_1(x). \tag{37}$$

The p-adic fermionic integral of the function $f(x) = x^n$ gives the Euler numbers as follows (*cf.* [14]):

$$E_n = \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x). \tag{38}$$

For more p-adic integral formulas and their obtaining techniques, the reader consult the papers [13–15, 25, 29].

In [28], by applying the p-adic integrals to (32), Simsek obtained some formulas involving the numbers $\ell_n(v)$, the Bernoulli numbers, the Euler numbers and the Stirling numbers of the first kind. For details, see [28].

By applying p-adic bosonic integral at (15), we obtain

$$\int_{\mathbb{Z}_p} \ell_n(x;v) d\mu_1(x) = \int_{\mathbb{Z}_p} \sum_{j=0}^{n} \binom{n}{j} x^j \ell_{n-j}(v) d\mu_1(x). \tag{39}$$

By using (36) in the above equation, we arrive at the following theorem:

Theorem 4.3. *Let $n, v \in \mathbb{N}$. Then we have*

$$\int_{\mathbb{Z}_p} \ell_n(x;v) d\mu_1(x) = \sum_{j=0}^{n} \binom{n}{j} \ell_{n-j}(v) B_j. \tag{40}$$

By applying the p-adic fermionic integral to (15), we obtain

$$\int_{\mathbb{Z}_p} \ell_n(x;v) d\mu_{-1}(x) = \int_{\mathbb{Z}_p} \sum_{j=0}^{n} \binom{n}{j} x^j \ell_{n-j}(v) d\mu_{-1}(x) \tag{41}$$

By using (37) in the above equation, we arrive at the following theorem:

Theorem 4.4. *Let $n, v \in \mathbb{N}$. Then we have*

$$\int_{\mathbb{Z}_p} \ell_n(x;v) d\mu_{-1}(x) = \sum_{j=0}^{n} \binom{n}{j} \ell_{n-j}(v) E_j. \tag{42}$$

5. Convergence Properties of a Szász-Type Operator Including Generating Function of the polynomials $\ell_n(x;2)$

In this section, we give Szász-type linear positive operators which involve the generating function of the polynomials $\ell_n(x;2)$. The methods to be applied in this section are the Korovkin-Bohman theorem, which shows the uniformly convergence of the operator, and the concept of the modulus of continuity, which estimates the rate of convergence of the operator, respectively.

Theorem 5.1. *(cf. [23], P. 8, Theorem 1.2.2) Let a sequence of linear positive operators* $(L_n)_n$, $L_n : V \to \mathcal{F}[a,b]$ *where* $\mathcal{F}[a,b]$ *is space of all real-valued functions in the interval [a,b] and V is a linear subspace of* $\mathcal{F}[a,b]$*. Suppose that* $\varphi_0, \varphi_1, \varphi_2 \in V \cap C[a,b]$ *forms a Chebychev system on the interval* $[a,b]$*, if we have*

$$lim_{n\to\infty} L_n(\varphi_j) = \varphi_j,$$

uniformly for $j = 0, 1, 2$*, then*

$$lim_{n\to\infty} L_n(f) = f,$$

uniformly, for any $f \in V \cap C[a,b]$.

The theorem of Bohman is the particular version of above theorem when $\varphi_j = e_j$, $j = 0, 1, 2$. The monomial functions denoted by e_j are defined to be as:

$$e_j(x) = x^j$$

where $x \in [a,b]$ and $j \in \mathbb{N} \cup 0$. $e_j(x)$ functions are also called moment functions. Furthermore the j- order central moment function of the operator L_n is defined as follows:

$$L_n((e_1 - e_0 x)^j),$$

(*cf.* [12]).

The concept of modulus of continuity is the primary tool in positive linear operators' approximation theory. This concept is effective for producing quantifiable estimates.

Definition 5.2. *(cf. [10], P. 40) Let f be uniformly continuous function on* $[0,\infty)$ *and* $\delta > 0$*. The modulus of continuity,* $\omega(f,x)$*, of function of f is defined to be*

$$\omega(f,\delta) := sup|f(x) - f(y)|, \tag{43}$$

where $x, y \in [0,\infty)$ *and* $|x - y| < \delta$.

Then for any $\delta > 0$, and each $x \in [0,\infty)$ the following relation holds

$$|f(x) - f(y)| \leq \omega(f,\delta)\left(\frac{|x-y|}{\delta} + 1\right) \tag{44}$$

(*cf.* [2, 10]).

By means of generating function method, $\ell_n(x;2)$ are given as follow:

$$\frac{t^2}{e^t - 1} e^{t(x-1)} = \sum_{n=0}^{\infty} \ell_n(x;2) \frac{t^n}{n!}. \tag{45}$$

The Taylor expansion of the generating function of $\ell_n(x;2)$ is defined by the following expression:

$$\frac{t}{e^t-1} + \frac{t^2(x-1)}{e^t-1} + \frac{t^3(x-1)^2}{2(e^t-1)} + \frac{t^4(x-1)^3}{6(e^t-1)} + \frac{t^5(x-1)^4}{24(e^t-1)} + \frac{t^6(x-1)^5}{120(e^t-1)} + O((x-1)^6). \tag{46}$$

This series converges everywhere and is positive at $x \in (1,\infty)$.

The Szász-type linear positive operators which involving the generating function of $\ell_n(x;2)$ are defined to be as:

$$\mathfrak{L}_n(f,x) = (e^2 - e)e^{-nx} \sum_{k=0}^{\infty} \frac{\ell_k(x;2)}{k!} f\left(\frac{k}{n}\right) \tag{47}$$

where $f \in C[0,\infty)$.

Let $f(x) = e_0(x) = 1$, and be $x \to nx$ at Eq. (44). We have,

$$\mathfrak{L}_n(1,x) = (e^2-e)e^{-nx}\sum_{k=0}^{\infty}\frac{\ell_k(nx;2)}{k!} = (e^2-e)e^{-nx}\frac{1}{(e^2-e)}e^{nx} = 1$$

By taking the first derivative of Eq. (42), we give

$$\sum_{k=1}^{\infty}\ell_k(x;2)\frac{kt^{k-1}}{k!} = \frac{te^{t(x-1)}(t(-x)+e^t(t(x-2)+2)+t-2)}{(e^t-1)^2}. \tag{48}$$

Let $f(x) = e_1(x) = x$, and be $x \to nx$ at Eq. (44). We get,

$$\mathfrak{L}_n(x,x) = (e^2-e)\frac{e^{-nx}}{n}\frac{e^{(nx-1)}(-nx+e(nx-1))}{(e-1)^2} = x - \frac{e}{(e-1)n}$$

where $t = 1$ and $x \to nx$.

By taking the second derivative of Eq. (45), we obtain

$$\sum_{k=1}^{\infty}\ell_k(x;2)\frac{k(k-1)t^{k-2}}{k!} = \frac{e^{t(x-1)}(-e^t(t^2(2x^2-6x+3)+4t(2x-3)+4)+t^2(x-1)^2+e^{2t}(t^2(x-2)^2+4t(x-2)+2)+4t(x-1)+2)}{(e^t-1)^3} \tag{49}$$

Similarly, If the mathematical operations applied for $f(x) = 1$ and $f(x) = x$ are applied for $f(x) = x^2$, the following equation is obtained:

$$\mathfrak{L}_n(x^2,x) = x^2 + (2-2e)\frac{x}{n} + \frac{4e-2e^2}{(e-1)^2}\frac{1}{n^2}.$$

With the help of the above equations, the moment functions for $\mathfrak{L}_n(f,x)$ are given in the following theorem (*cf.* [12]):

Theorem 5.3. *For all $x \in (1,\infty)$ and $n \in \mathbb{N}$ the operators $\mathfrak{L}$ satisfy the following:*

$$\mathfrak{L}_n(e_0(x),x) = 1, \tag{50}$$

$$\mathfrak{L}_n(e_1(x),x) = x - \frac{e}{(e-1)n}, \tag{51}$$

and

$$\mathfrak{L}_n(e_2(x),x) = x^2 + (2-2e)\frac{x}{n} + \frac{4e-2e^2}{(e-1)^2}\frac{1}{n^2}. \tag{52}$$

By using (50), (51), and (52) we give uniformly convergence of $\mathfrak{L}$ with the aid of the Korovkin-Bohman theorem.

We have evidence that,

$$lim_{n\to\infty}\mathfrak{L}_n(e_i(x);x) = x^i$$

for $i = 0, 1, 2$ at Theorem 5.1 and then we can use the Korovkin-Bohman theorem to obtain at the following theorem (*cf.* [4, 23]):

Theorem 5.4. *Let $f \in C[0,\infty)$. Then*

$$lim_{n\to\infty}\mathfrak{L}_n(f;x) = f(x) \tag{53}$$

uniformly on each compact subset of $[0,\infty)$.

By using Theorem 5.3 and linearity property of $\mathfrak{L}_n$, we obtain second-order central moment function for $\mathfrak{L}$ as follows:

$$\mathfrak{L}_n((e_1 - e_0x)^2;x) = \frac{(-2e^2+5e-2)}{(e-1)}\frac{x}{n} + \frac{2e(2-e)}{n^2(e-1)^2}. \tag{54}$$

According to Theorem 5.3 and monotonicity property of operators $\mathfrak{L}$, we have

$$|\mathfrak{L}_n(f;x) - f(x)| \leq \mathfrak{L}_n(|f(x) - f(y);x)|. \tag{55}$$

Applying (44), we get the following from (55)

$$|\mathfrak{L}_n(f;x) - f(x)| \leq \omega(f,\delta)\left(1 + \frac{1}{\delta}\mathfrak{L}_n(|x-y|;x)\right). \tag{56}$$

Applying the Cauchy–Schwarz inequality to the right side of (56), we get

$$|\mathfrak{L}_n(f;x) - f(x)| \leq \omega(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{\mathfrak{L}_n((x-y)^2,x)}\right). \tag{57}$$

Assuming that $\delta := \delta_n(x) = \mathfrak{L}_n((x-y)^2,x)$ in (57), the following theorem is given as follows:

Theorem 5.5. *Let $f \in C_B[0,\infty) \cap E$. The following inequality holds:*

$$|\mathfrak{L}_n(f;x) - f(x)| \leq 2\omega(f,\delta_n), \tag{58}$$

where $E = \{f(x) : \lim_{x\to\infty}\frac{f(x)}{1+x^2}$ exists and is finite and $0 \leq x < \infty$ \} and $\delta_n(x) = \mathfrak{L}_n((x-y)^2,x)$.

We construct error estimation tables analyzing the convergence of the operator $\mathfrak{L}_n(f;x)$ to a few example functions. The following examples contain functions from the exponential, trigonometric and fractional function families, respectively. The numerical values of $\mathfrak{L}_n$'s approximation errors to these functions are as follows:

Example 1 In Table 1, we demonstrate the numerical results of the approximation of $\mathfrak{L}_n(f;x)$ to the function $f(x) = x^2e^{-2x}$.

n	**Estimation by** $\omega(f,\delta)$
10	0.2706705664
10^2	0.04649728780
10^3	0.004682218110
10^4	0.0004683039334
10^5	0.00004683092196
10^6	0.000004683125408
10^7	0.0000004682160322

Table 1: Error of approximation of the operators $\mathfrak{L}_n(f;x)$ to $f(x) = x^2e^{-2x}$

Example 2 In Table 2, we show the numerical results of the approximation of $\mathfrak{L}_n(f;x)$ to the function $f(x) = sin(\pi x)$.

n	**Estimation by** $\omega(f,\delta)$
10	3.999974312
10^2	0.6345392848
10^3	0.06379521134
10^4	0.006380536830
10^5	0.0006380614300
10^6	0.00006380621810
10^7	0.000006380622558

Table 2: Error of approximation of the operators $\mathfrak{L}_n(f;x)$ to $f(x) = sin(\pi x))$ for $n = 1..7$

Example 3 In Table 3, we show the numerical results of the approximation of $\mathfrak{L}_n(f,x)$ to the function $f(x) = \frac{x}{\sqrt{1+x^2}}$.

n	**Estimation by** $\omega(f,\delta)$
10	1.415824119
10^2	0.2018018226
10^3	0.02030646068
10^4	0.002030987768
10^5	0.0002031012610
10^6	0.00002031015002
10^7	0.000002031015240

Table 3: Error of approximation of the operators $\mathfrak{L}_n(f;x)$ to $f(x) = \frac{x}{\sqrt{1+x^2}}$ for $n = 1..7$

In these examples, we numerically find the approximation of $\mathfrak{L}_n(f;x)$ to function $f(x) = x^2e^{-2x}$, $f(x) = sin(\pi x)$, and $f(x) = \frac{x}{\sqrt{1+x^2}}$,respectively, by using the modulus of continuity. When we examine the tables in three examples, we notice that the approximation errors of the operators $\mathfrak{L}_n(f;x)$ decrease as n increases.

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