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Further additive results on the Drazin inverse

Daochang Zhang[a,*], Yue Zhao[a], Dijana Mosić[b]

[a]*College of Sciences, Northeast Electric Power University, Jilin, P.R. China*
[b]*Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia*

Abstract. In this paper, we provide original representation for the Drazin inverse of $P+Q$ under the conditions $P^2Q = 0$, $Q(PQ)^2 = 0$, $Q^2PQ^2 = 0$, $QPQ^3 = 0$ and $QPQ^2PQ = 0$. Then, we apply our results to derive some new expressions for the Drazin inverse of a 2×2 complex block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{C}^{n\times n}$(where A and D are square matrices but not necessarily of the same size). Finally, several illustrative numerical examples are given to demonstrate our results.

1. Introduction

For $A \in \mathbb{C}^{n\times n}$, where $\mathbb{C}^{n\times n}$ denotes the set of $n \times n$ complex matrices and $rank(A)$ is the rank of A, the smallest non-negative integer k which satisfies $rank(A^{k+1}) = rank(A^k)$ is called the index of A, and marked by $ind(A)$. The Drazin inverse of $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$, denoted by A^d, is the unique matrix satisfying the equations as follows:

$$AA^d = A^dA, \quad A^dAA^d = A^d \text{ and } A^k = A^{k+1}A^d.$$

We denote by $A^e = AA^d$, and by $A^\pi = I - A^e$ the spectral idempotent of A corresponding to $\{0\}$, and define $A^0 = I$, where I is the identity matrix with proper sizes. In the case that $ind(A) = 1$, we called the Drazin inverse of A as group inverse and denoted by $A^\sharp$. The Drazin inverse is useful and its applications are showed in various fields, such as singular linear differential equations and difference equations [17], finite Markov chains [20], iterative methods [21]. And the relevant research about the Drazin inverse was widely developed in [2, 23–28, 30, 31, 35, 36, 38, 39].

According to current papers, it is still an open problem to derive the formula for $(P+Q)^d$ without any side conditions for matrices P and Q. Suppose that $P, Q \in \mathbb{C}^{n\times n}$. In 1958, Drazin [13] gave the explicit representation of $(P+Q)^d$ under the conditions $PQ = QP = 0$. In 2001, Hartwig et al. [16] developed

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* Corresponding author: Daochang Zhang
Email addresses: daochangzhang@126.com (Daochang Zhang), yuezhao0303@163.com (Yue Zhao), dijana@pmf.ni.ac.rs (Dijana Mosić)

an expression of $(P+Q)^d$ when $PQ = 0$. Expressions of the Drazin inverse of the sum of two matrices under the weaker conditions $PQ^2 = PQP = 0$ and $P^2Q = QPQ = 0$ were provided in [33] by Yang and Liu. In 2018, Yousefi and Dana [34] presented a representation for $(P+Q)^d$ when $P^2QP = 0, P^2Q^2 = 0$ and $QPQ = 0$. In 2022, Shakoor et al. [29] gave some results of $(P+Q)^d$ under conditions $P^2QP = PQ^2 = 0$ and $QPQ^2 = P^2Q = 0$.

Likewise, formulae for $(P+Q)^d$ are valuable in computing the representations of a $n\times n$ block matrix:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{1}$$

where A and D are square matrices. In 1979, Campbell and Meyer [4] proposed an open problem to find an explicit representation for the Drazin inverse of M. Until now, there has been no formula for M^d without any side conditions for blocks of matrix M. Here we list some results below:

1. In [8], $ABC = 0$ and $DC = 0$;
2. In [11], $BC = 0, BD = 0$ and $DC = 0$;
3. In [12], $BC = 0, BDC = 0$ and $BD^2 = 0$;
4. In [19], $BD^\pi C = 0, BDD^d = 0, DD^\pi CA = 0$ and $DD^\pi CB = 0$;
5. In [1], $ABD = 0, CBD = 0, BCA = 0, DCA = 0, BCBC = 0$ and $D^\pi CBC = 0$.

Inspired by previous results, we continue to study additive results for the Drazin inverse. Under some weaker conditions, we attain original result for $(P+Q)^d$ in this paper. Applying this result, we investigate the Drazin inverse of arbitrary block matrix. By establishing several original results and combining various facts known in the literature, the article reveals new expressions for the Drazin inverse of the sum and of the block matrix.

We organize the article in five sections. In Sect. 2, we first introduce some lemmas about the results of the Drazin inverse of an anti-triangular matrix. In Sect. 3, we derive a new explicit formula for the Drazin inverse of a sum of two matrices $P, Q \in \mathbb{C}^{n\times n}$ under conditions $P^2Q = 0$, $Q(PQ)^2 = 0$, $Q^2PQ^2 = 0$, $QPQ^3 = 0$ and $QPQ^2PQ = 0$. These results extend the formulae proved in [32] and [33], respectively. In Sect. 4, we apply these formulae for $(P+Q)^d$ to attain the representations for the Drazin inverse of M given by (1) under conditions weaker than those used in some recent papers. In Sect. 5, we demonstrate our results by some numerical examples.

2. Key lemma

To prove the main results, we need the following lemmas. Then, we begin with the well-known Cline's formula.

Lemma 2.1. [6] (Cline's Formula) *For $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$*, $(BA)^d = B(AB)^{2d}A$.

The next representation about the Drazin inverse of the sum of two matrices proved in [16] is valuable for our results.

Lemma 2.2. [16] *Let $S, R \in \mathbb{C}^{n\times n}$. If $SR = 0$, then*

$$(S+R)^d = \sum_{i=0}^{i_R-1} R^\pi R^i (S^d)^{i+1} + \sum_{i=0}^{i_S-1} (R^d)^{i+1} S^i S^\pi, \tag{2}$$

where $\operatorname{ind}(S) = i_S$ *and* $\operatorname{ind}(R) = i_R$.

We need the next lemma about the Drazin inverse of block triangular matrices.

Lemma 2.3. [15, 22] *Let* $U = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ *and* $V = \begin{bmatrix} D & 0 \\ B & A \end{bmatrix} \in \mathbb{C}^{n\times n}$*, where A and D are square matrices such that* $\mathrm{ind}(A) = i_A$ *and* $\mathrm{ind}(D) = i_D$*. Then*

$$U^d = \begin{bmatrix} A^d & X \\ 0 & D^d \end{bmatrix} \text{ and } V^d = \begin{bmatrix} D^d & 0 \\ X & A^d \end{bmatrix},$$

where

$$X = \sum_{i=0}^{i_D-1} (A^d)^{i+2} B D^i D^\pi + A^\pi \sum_{i=0}^{i_A-1} A^i B (D^d)^{i+2} - A^d B D^d. \tag{3}$$

Furthermore, we provide some results about the Drazin inverse of $\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}$ and $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$, which are extremely useful in Section 4.

Lemma 2.4. [37, Theorem 3.1] *Assume that A and B of a matrix* $\bar{N} = \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}$ *are square matrices of the same size and obey* $AB^2 = 0$, $A^2BA = 0$, $ABA^2 = 0$ *and* $(AB)^2 = 0$*. Then*

$$\bar{N}^d = \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}, \tag{4}$$

where $ind(A) = i_A$ *and* $ind(B) = i_B$*,*

$$\begin{aligned}
E_1 &= -B^d A^d B + \sum_{i=0}^{i_B-1} B^\pi B^i A^{(2i+3)d} B + \sum_{i=0}^{i_B-1} B^\pi B^i A^{(2i+1)d} + \sum_{i=0}^{[\frac{i_A}{2}]-1} B^{(i+2)d} A^{2i+1} A^\pi B \\
&+ \sum_{i=0}^{[\frac{i_A}{2}]-1} B^{(i+1)d} A^{2i+1} A^\pi, \\
E_2 &= \sum_{i=0}^{i_B-1} B^\pi B^i A^{(2i+2)d} B + \sum_{i=0}^{[\frac{i_A}{2}]} B^{(i+1)d} A^{2i} A^\pi B, \\
E_3 &= B^{3d} ABA - B^d A^{2d} B - B^d + \sum_{i=0}^{i_B-1} B^\pi B^i A^{(2i+2)d} + \sum_{i=0}^{[\frac{i_A}{2}]} B^{(i+1)d} A^{2i} A^\pi + \sum_{i=0}^{i_B-1} B^\pi B^i A^{(2i+4)d} B \\
&+ \sum_{i=0}^{[\frac{i_A}{2}]} B^{(i+2)d} A^{2i} A^\pi B, \\
E_4 &= -B^d A^d B + \sum_{i=0}^{i_B-1} B^\pi B^i A^{(2i+3)d} B + \sum_{i=0}^{[\frac{i_A}{2}]} B^{(i+2)d} A^{2i+1} A^\pi B.
\end{aligned}$$

Lemma 2.5. [37, Theorem 3.2] *Assume that A and BC of a matrix* $N = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ *are square matrices of the same size and obey* $A(BC)^2 = 0$, $A^2BCA = 0$, $ABCA^2 = 0$ *and* $(ABC)^2 = 0$*. Then*

$$N^d = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix},$$

where $ind(A) = i_A$ *and* $ind(BC) = i_{BC}$,

$$\begin{aligned}
F_1 &= \sum_{i=0}^{i_{BC}-1}(BC)^{\pi}(BC)^i A^{(2i+1)d} + \sum_{i=0}^{[\frac{i_A}{2}]-1}(BC)^{(i+1)d}A^{2i+1}A^{\pi} + \sum_{i=0}^{i_{BC}-1}(BC)^{\pi}(BC)^i A^{(2i+3)d}BC \\
&+ \sum_{i=0}^{[\frac{i_A}{2}]-1}(BC)^{(i+2)d}A^{2i+1}A^{\pi}BC - (BC)^d A^d BC, \\
F_2 &= \sum_{i=0}^{i_{BC}-1}(BC)^{\pi}(BC)^i A^{(2i+2)d}B + \sum_{i=0}^{i_{BC}-1}(BC)^{\pi}(BC)^i A^{(2i+4)d}BCB + \sum_{i=0}^{[\frac{i_A}{2}]}(BC)^{(i+1)d}A^{2i}A^{\pi}B \\
&+ \sum_{i=0}^{[\frac{i_A}{2}]}(BC)^{(i+2)d}A^{2i}A^{\pi}BCB + (BC)^{3d}ABCAB - (BC)^d A^{2d}BCB - (BC)^d B, \\
F_3 &= C\sum_{i=0}^{[\frac{i_A}{2}]}(BC)^{(i+1)d}A^{2i}A^{\pi} + C\sum_{i=0}^{i_{BC}-1}(BC)^{\pi}(BC)^i A^{(2i+2)d} + C\sum_{i=0}^{[\frac{i_A}{2}]}(BC)^{(i+2)d}A^{2i}A^{\pi}BC \\
&+ C\sum_{i=0}^{i_{BC}-1}(BC)^{\pi}(BC)^i A^{(2i+4)d}BC - C(BC)^d A^{2d}BC - C(BC)^d, \\
F_4 &= C\sum_{i=0}^{[\frac{i_A}{2}]-1}(BC)^{(i+2)d}A^{2i+1}A^{\pi}B + C\sum_{i=0}^{[\frac{i_A}{2}]-1}(BC)^{(i+3)d}A^{2i+1}A^{\pi}BCB + C\sum_{i=0}^{i_{BC}-1}(BC)^{\pi}(BC)^i A^{(2i+3)d}B \\
&+ C\sum_{i=0}^{i_{BC}-1}(BC)^{\pi}(BC)^i A^{(2i+5)d}BCB - C(BC)^d A^d B - C(BC)^d A^{3d}BCB - C(BC)^{2d}A^d BCB.
\end{aligned}$$

Lemma 2.6. [10, Theorem 3.3] *and* [37, Corollary 3.3] *Let* $N = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$, *where* A *and* BC *are square matrices of the same size. If* $ABC = 0$, *then*

$$N^d = \begin{bmatrix} YA & YB \\ CY & C[YA^d + (BC)^d(YA - A^d)]B \end{bmatrix},$$

where

$$Y = \sum_{i=0}^{[\frac{i_A}{2}]}(BC)^{(i+1)d}A^{2i}A^{\pi} + \sum_{i=0}^{i_{BC}-1}(BC)^{\pi}(BC)^i A^{(2i+2)d} \tag{5}$$

such that $ind(A) = i_A$ *and* $ind(BC) = i_{BC}$.

3. Main results

In this section, we present the explicit formula for $(P+Q)^d$, under the conditions $P^2Q = 0$, $Q(PQ)^2 = 0$, $Q^2PQ^2 = 0$, $QPQ^3 = 0$ and $QPQ^2PQ = 0$, which extends the consequences proved in [32] and [33]. Now, we are in position to state the main result.

Theorem 3.1. *Let $P^2Q = 0$, $Q(PQ)^2 = 0$, $Q^2PQ^2 = 0$, $QPQ^3 = 0$ and $QPQ^2PQ = 0$, where $P, Q \in \mathbb{C}^{n\times n}$ are such that $\mathrm{ind}(Q) = i_Q$ and $\mathrm{ind}(P) = i_P$. Then*

$$\begin{aligned}(P+Q)^d &= \sum_{i=0}^{i_P-1} Q^{(i+1)d}P^iP^\pi + \sum_{i=0}^{i_P-1} Q^{(i+3)d}PQP^iP^\pi + \sum_{i=0}^{i_Q-1} Q^\pi Q^iP^{(i+1)d} + \sum_{i=0}^{i_Q-1} PQ^\pi Q^iPQP^{(i+4)d}\\ &+ \sum_{i=0}^{i_P-1} PQ^{(i+2)d}P^iP^\pi + \sum_{i=0}^{i_P-1} PQ^{(i+4)d}PQP^iP^\pi + \sum_{i=0}^{i_Q-1} PQ^\pi Q^iP^{(i+2)d} + \sum_{i=0}^{i_Q-1} Q^\pi Q^iPQP^{(i+3)d}\\ &+ QPQ^2P^{5d} - PQP^{3d} - Q^dPQP^{2d} - Q^{2d}PQP^d + PQPQ^2P^{6d} - PQ^dPQP^{3d} - P^d\\ &- PQ^{2d}PQP^{2d} - PQ^dP^d - PQ^{3d}PQP^d.\end{aligned}$$

Proof. We denote that $P+Q = \begin{bmatrix} Q & I\end{bmatrix}\begin{bmatrix} I\\ P\end{bmatrix}$. Due to Lemma 2.1, we have

$$(P+Q)^d = \begin{bmatrix} Q & I\end{bmatrix}\begin{bmatrix} Q & I\\ PQ & P\end{bmatrix}^{2d}\begin{bmatrix} I\\ P\end{bmatrix}, \tag{6}$$

The next splitting of $\begin{bmatrix} Q & I\\ PQ & P\end{bmatrix}$ will be used:

$$\begin{bmatrix} Q & I\\ PQ & P\end{bmatrix} = \begin{bmatrix} Q & I\\ PQ & 0\end{bmatrix} + \begin{bmatrix} 0 & 0\\ 0 & P\end{bmatrix} := R + S.$$

Since $P^2Q = 0$, we are able to obtain $SR = 0$. Hence, Lemma 2.2 can be utilized. Now, for $H = \begin{bmatrix} 0 & I\\ I & -Q\end{bmatrix}$ and $H^{-1} = \begin{bmatrix} Q & I\\ I & 0\end{bmatrix}$, we calculate R^d as follows:

$$\begin{aligned}\begin{bmatrix} Q & I\\ PQ & 0\end{bmatrix}^d &= \left(H\begin{bmatrix} Q & PQ\\ I & 0\end{bmatrix}H^{-1}\right)^d = H\begin{bmatrix} Q & PQ\\ I & 0\end{bmatrix}^d H^{-1}\\ &= \begin{bmatrix} E_3Q + E_4 & E_3\\ E_1Q - QE_3Q + E_2 - QE_4 & E_1 - QE_3\end{bmatrix}\\ &= \begin{bmatrix} Q^d + PQ^{2d} + PQ^{4d}PQ + Q^{3d}PQ & Q^{2d} + PQ^{3d} + PQ^{5d}PQ + Q^{4d}PQ\\ PQ^d + PQ^{3d}PQ & PQ^{2d} + PQ^{4d}PQ\end{bmatrix},\end{aligned}$$

where $\begin{bmatrix} Q & PQ\\ I & 0\end{bmatrix}^d$ and E_n, $n = \overline{1,4}$ can be expressed by Lemma 2.4. In addition, we attain the expression for R^π as

$$R^\pi = \begin{bmatrix} Q^\pi - Q^{2d}PQ - PQ^d - PQ^{3d}PQ & -Q^d - Q^{3d}PQ - PQ^{2d} - PQ^{4d}PQ\\ -PQQ^d - PQ^{2d}PQ & I - PQ^d - PQ^{3d}PQ\end{bmatrix}.$$

Then, we prove, for $n \geq 7$,

$$R^n = \begin{bmatrix} Q^n + Q^{n-2}PQ + PQ^{n-1} + PQ^{n-3}PQ & Q^{n-1} + Q^{n-3}PQ + PQ^{n-2} + PQ^{n-4}PQ\\ PQ^n + PQ^{n-2}PQ & PQ^{n-1} + PQ^{n-3}PQ\end{bmatrix},$$

and for $n \geq 1$,

$$R^{nd} = \begin{bmatrix} Q^{nd} + Q^{(n+2)d}PQ + PQ^{(n+1)d} + PQ^{(n+3)d}PQ & Q^{(n+1)d} + Q^{(n+3)d}PQ + PQ^{(n+2)d} + PQ^{(n+4)d}PQ\\ PQ^{nd} + PQ^{(n+2)d}PQ & PQ^{(n+1)d} + PQ^{(n+3)d}PQ\end{bmatrix}.$$

Consequently, the proof is finished by substituting the above expressions into (2) and (6). □

Now, we have strengthened the conditions of Theorem 3.1 and obtained the following corollaries represented in [32] and [33].

Corollary 3.2. [32, Theorem 3.2] *Let* $P^2Q = 0$, $QPQ^2 = 0$ *and* $(QP)^2 = 0$, *where* $P, Q \in \mathbb{C}^{n\times n}$ *are such that* $\mathrm{ind}(Q) = i_Q$ *and* $\mathrm{ind}(P) = i_P$. *Then*

$$\begin{aligned}(P+Q)^d &= \sum_{i=0}^{i_P-1} Q^{(i+1)d}P^iP^\pi + \sum_{i=0}^{i_Q-1} Q^\pi Q^i P^{(i+1)d} + \sum_{i=0}^{i_P-1} PQ^{(i+2)d}P^iP^\pi + \sum_{i=0}^{i_Q-1} PQ^iQ^\pi P^{(i+2)d}\\ &+ Q^{3d}PQ + PQ^{4d}PQ - P^d - PQ^dP^d.\end{aligned}$$

Corollary 3.3. [33, Theorem 2.2] *Let* $P^2Q = 0$ *and* $QPQ = 0$, *where* $P, Q \in \mathbb{C}^{n\times n}$ *are such that* $\mathrm{ind}(Q) = i_Q$ *and* $\mathrm{ind}(P) = i_P$. *Then*

$$\begin{aligned}(P+Q)^d &= \sum_{i=0}^{i_P-1} Q^{(i+1)d}P^iP^\pi + \sum_{i=0}^{i_Q-1} Q^\pi Q^i P^{(i+1)d} + \sum_{i=0}^{i_P-1} PQ^{(i+2)d}P^iP^\pi + \sum_{i=0}^{i_Q-2} PQ^{i+1}Q^\pi P^{(i+3)d}\\ &- PQ^dP^d - PQQ^dP^{2d}.\end{aligned}$$

We illustrate by the following example that Theorem 3.1 is an extension of Corollary 3.2 and Corollary 3.3.

Example 3.4. *Consider* 4×4 *complex matrices*

$$P = \begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix} \text{ and } Q = \begin{bmatrix} a_1 & a_2 & a_3 & a_4\\ 0 & 0 & b_3 & b_4\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},$$

where $a_1, a_2, a_3, a_4, b_3, b_4 \in \mathbb{C}\backslash\{0\}$. *Since*

$$QPQ = \begin{bmatrix} 0 & 0 & a_1b_3 & a_1b_4\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \neq 0 \quad \text{and} \quad (QP)^2 = \begin{bmatrix} 0 & 0 & 0 & a_1b_3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \neq 0.$$

The assumptions of Corollary 3.2 and Corollary 3.3 do not hold. Then $\mathrm{ind}(P) = 4$, $Q(PQ)^2 = (QP)^2Q = 0$, $P^2Q = 0$ *and* $QPQ^2 = 0$. *So,* $Q^2PQ^2 = 0$, $QPQ^3 = 0$ *and* $QPQ^2PQ = 0$, *that is, the conditions of Theorem 3.1 are satisfied and we obtain*

$$(P+Q)^d = \sum_{i=0}^{3} Q^{(i+1)d}P^i + \sum_{i=0}^{3} Q^{(i+3)d}PQP^i + \sum_{i=0}^{3} PQ^{(i+2)d}P^i + \sum_{i=0}^{3} PQ^{(i+4)d}PQP^i. \tag{7}$$

According to Lemma 2.3, we get

$$Q^d = \begin{bmatrix} a_1^{-1} & a_1^{-2}a_2 & a_1^{-2}a_3 + a_1^{-3}a_2b_3 & a_1^{-2}a_4 + a_1^{-3}a_2b_4\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Substituting the above matrices into (7), *we get*

$$(P+Q)^d = \begin{bmatrix} a_1^{-1} & 0 & 0 & 0\\ a_1^{-2}a_2 + a_1^{-2} & 0 & 0 & 0\\ a_1^{-2}a_3 + a_1^{-3}a_2b_3 + a_1^{-3}a_2 + a_1^{-3} + a_1^{-3}b_3 & 0 & 0 & 0\\ a_1^{-2}a_4 + a_1^{-4}a_2 + a_1^{-4} + a_1^{-3}b_4 + a_1^{-4}b_3 + a_1^{-3}a_2b_4 + a_1^{-3}a_3 + a_1^{-4}a_2b_3 & 0 & 0 & 0\end{bmatrix}^\top.$$

4. Applications to the Drazin inverse of block matrix

In this section, we apply our explicit formulae proved in Section 3 to present the formulae for the Drazin inverse of a block matrix M given by (1). The Drazin inverse of anti-triangular matrices given by lemmas in Section 2 are extremely valuable for us to derive the specific expressions of M^d in this part.

Theorem 4.1. *Let M be defined in* (1)*, if*

$$BCA = 0, \quad DCA = 0, \quad CBCB = 0, \quad DCBD = 0 \quad \text{and} \quad BCBD = 0,$$

then

$$M^d = \begin{bmatrix} \begin{matrix} A^d + A^dXC + XD^dC \\ +(A^{3d}X - A^{3d}BD^{2d} - A^{2d}BD^{3d} + XD^{3d})CBC \end{matrix} & X + (A^{2d}X - A^{2d}BD^{2d} + XD^{2d})CB \\ \begin{matrix} (I + CX)D^{2d}C + (D^{4d} + CA^{4d}X - CA^{4d}BD^{2d} \\ -CA^{3d}BD^{3d} - CA^{2d}BD^{4d} + CXD^{4d})CBC \\ +CA^{2d}(I + XC - BD^{2d}C) \end{matrix} & \begin{matrix} D^d + CA^dX + CXD^d + (D^{3d} + CA^{3d}X \\ -CA^{3d}BD^{2d} - CA^{2d}BD^{3d} + CXD^{3d})CB \end{matrix} \end{bmatrix},$$

where X is given by (3)*.*

Proof. We consider the splitting

$$M = \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} + \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} := K + U,$$

and we can obtain $K^2 = 0$, $K^d = 0$ and $K^\pi = I$. After applying Lemma 2.3, the Drazin inverse of U can be gained as follows:

$$U^d = \begin{bmatrix} A^d & X \\ 0 & D^d \end{bmatrix}.$$

According to Theorem 3.1, we get

$$M^d = (U^d + KU^{2d})(I + U^dK + U^{2d}KU + U^{3d}KUK).$$

We have $K^2U = 0$,

$$U(KU)^2 = \begin{bmatrix} BCBCA & BCBCB \\ DCBCA & DCBCB \end{bmatrix} = 0, \quad UKU^3 = \begin{bmatrix} BCA^3 & BCA^2B + BCABD + BCBD^2 \\ DCA^3 & DCA^2B + DCABD + DCBD^2 \end{bmatrix} = 0,$$

$$U^2KU^2 = \begin{bmatrix} ABCA^2 + BDCA^2 & ABCAB + ABCBD + BDCAB + BDCBD \\ D^2CA^2 & D^2CAB + D^2CBD \end{bmatrix} = 0,$$

$$UKU^2KU = \begin{bmatrix} BCABCA + BCBDCA & BCABCB + BCBDCB \\ DCABCA + DCBDCA & DCABCB + DCBDCB \end{bmatrix} = 0.$$

Obviously, the conditions are hold. After a series of calculations, we get the new formula for M^d. □

On the basis of Theorem 4.1, after strengthening the conditions, we can obtain the following result, which is given in [19, Theorem 3.2].

Corollary 4.2. *Let M be defined in* (1)*, if*

$$BCA = 0, \quad DCA = 0, \quad CBC = 0 \quad \text{and} \quad CBD = 0,$$

then

$$M^d = \begin{bmatrix} A^d + A^dXC + XD^dC & X + (A^{2d}X - A^{2d}BD^{2d} + XD^{2d})CB \\ \begin{matrix} (I + CX)D^{2d}C \\ +CA^{2d}(I + XC - BD^{2d}C) \end{matrix} & \begin{matrix} D^d + CA^dX + CXD^d + (D^{3d} + CA^{3d}X \\ -CA^{3d}BD^{2d} - CA^{2d}BD^{3d} + CXD^{3d})CB \end{matrix} \end{bmatrix},$$

where X is given by (3)*.*

Remark 4.3. *Theorem 4.1 generalizes some known results for M^d under the assumptions:*

1. $CA = 0$ *and* $CB = 0$ *(see* [9, Theorem 2.1]*);*
2. $BD = 0, CA = 0$ *and* $CB = 0$ *(see* [12, Case (b3)]*);*
3. $BCA = 0, BCB = 0, DCA = 0$ *and* $DCB = 0$ *(see* [33, Theorem 3.1]*);*
4. $BC = 0$ *and* $DC = 0$ *(see* [18, Corollary 3.3]*);*
5. $ABD = 0, CBD = 0, BCA = 0, DCA = 0$ *and* $CBC = 0$ *(see* [1, Corollary 3.2]*);*
6. $DCA = 0, BCA = 0, CBD = 0, ABD = 0, CBCB = 0$ *and* $A^{\pi}BCB = 0$ *(see* [1, Theorem 3.3]*).*

Now, another theorem for calculating M^d is proved.

Theorem 4.4. *Let M be defined in* (1), *if*

$$A(BC)^2 = 0,\ A^2BCA = 0,\ ABCA^2 = 0,\ (ABC)^2 = 0,\ BDCA = 0,\ BDCB = 0 \text{ and } D^2C = 0,$$

then

$$\begin{aligned} M^d &= \sum_{i=0}^{i_D-1} N^{(i+1)d}\begin{bmatrix}0&0\\0&D\end{bmatrix}^i\begin{bmatrix}I&0\\0&D^{\pi}\end{bmatrix} + N^{3d}\begin{bmatrix}0&0\\DC&0\end{bmatrix} + \sum_{i=0}^{i_N-1} N^{\pi}N^i\begin{bmatrix}0&0\\0&D^{(i+1)d}\end{bmatrix} \\ &+ \sum_{i=0}^{i_D-1}\begin{bmatrix}0&0\\0&D\end{bmatrix}N^{(i+2)d}\begin{bmatrix}0&0\\0&D\end{bmatrix}^i\begin{bmatrix}I&0\\0&D^{\pi}\end{bmatrix} + \begin{bmatrix}0&0\\0&D\end{bmatrix}N^{4d}\begin{bmatrix}0&0\\DC&0\end{bmatrix} \\ &+ \sum_{i=0}^{i_N-1}\begin{bmatrix}0&0\\0&D\end{bmatrix}N^iN^{\pi}\begin{bmatrix}0&0\\0&D^{(i+2)d}\end{bmatrix} - \begin{bmatrix}0&0\\0&D^d+DF_4D^d\end{bmatrix}, \end{aligned}$$

where N^d and F_4 are given by Lemma 2.5 such that $\operatorname{ind}(N) = i_N$ *and* $\operatorname{ind}(D) = i_D$.

Proof. We consider the splitting

$$M = \begin{bmatrix}0&0\\0&D\end{bmatrix} + \begin{bmatrix}A&B\\C&0\end{bmatrix} := P + N.$$

We have $N(PN)^2 = 0$,

$$P^2N = \begin{bmatrix}0&0\\D^2C&0\end{bmatrix} = 0,\quad N^2PN^2 = \begin{bmatrix}ABDCA&ABDCB\\CBDCA&CBDCB\end{bmatrix} = 0,$$

$$NPN^3 = \begin{bmatrix}BDCA^2+BDCBC&BDCAB\\0&0\end{bmatrix} = 0,\quad NPN^2PN = \begin{bmatrix}BDCBDC&0\\0&0\end{bmatrix} = 0.$$

By applying the result in Lemma 2.5, N^d is given and we can prove the following representation

$$N^{\pi} = \begin{bmatrix}I-F_1A-F_2C&-F_1B\\-F_3A-F_4C&I-F_3B\end{bmatrix}.$$

Analogously, using Lemma 2.3, we note that

$$P^d = \begin{bmatrix}0&0\\0&D^d\end{bmatrix} \quad\text{and}\quad P^{\pi} = \begin{bmatrix}I&0\\0&D^{\pi}\end{bmatrix}.$$

After that $ind(P) = i_D$, because, for $i \geq 1$,

$$P^iP^{\pi} = \begin{bmatrix}0&0\\0&D^iD^{\pi}\end{bmatrix}.$$

Consequently, applying Theorem 3.1, we finish the proof. □

In order to facilitate the application, we give the following deduction, and the conditions about D are strengthened on the basis of Theorem 4.4.

Corollary 4.5. *Let M be defined in* (1), *if*

$$A(BC)^2 = 0, \quad A^2BCA = 0, \quad ABCA^2 = 0, \quad (ABC)^2 = 0, \quad BDC = 0 \quad \text{and} \quad D^2C = 0,$$

then

$$\begin{aligned} M^d &= \sum_{i=0}^{i_D-1} N^{(i+1)d} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} + \sum_{i=0}^{i_N-1} N^\pi N^i \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+1)d} \end{bmatrix} \\ &+ \sum_{i=0}^{i_D-1} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} N^{(i+2)d} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} \\ &+ \sum_{i=0}^{i_N-1} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} N^i N^\pi \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+2)d} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & D^d + DF_4D^d \end{bmatrix}, \end{aligned}$$

where N^d and F_4 are given by Lemma 2.5 such that $\operatorname{ind}(N) = i_N$ *and* $\operatorname{ind}(D) = i_D$.

After strengthening the conditions about the Drazin inverse of anti-triangular matrix N of Theorem 4.4, we can obtain the result represented in Corollary 4.6.

Corollary 4.6. *Let M be defined in* (1), *if*

$$ABC = 0, \quad BDCA = 0, \quad BDCB = 0 \quad \text{and} \quad D^2C = 0,$$

then

$$\begin{aligned} M^d &= \sum_{i=0}^{i_D-1} N^{(i+1)d} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} + Q^{3d} \begin{bmatrix} 0 & 0 \\ DC & 0 \end{bmatrix} + \sum_{i=0}^{i_N-1} N^\pi N^i \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+1)d} \end{bmatrix} \\ &+ \sum_{i=0}^{i_D-1} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} N^{(i+2)d} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} + \sum_{i=0}^{i_N-1} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} N^i N^\pi \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+2)d} \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} Q^{4d} \begin{bmatrix} 0 & 0 \\ DC & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & D^d + DC[YA^d + (BC)^d(YA - A^d)]BD^d \end{bmatrix}, \end{aligned}$$

where N^d and Y are given by Lemma 2.6 such that $\operatorname{ind}(N) = i_N$ *and* $\operatorname{ind}(D) = i_D$.

Remark 4.7. *A list of results extended by Theorem 4.4 is given below:*

1. *In* [5, Theorem 2.1], $A = 0$ and $D = 0$;
2. In [11, Theorem 5.3], $BC = 0, BD = 0$ and $DC = 0$;
3. In [14, Lemma 2.2], $BC = 0, DC = 0$ and D is nilpotent;
4. In [3, Theorem 2.2], $ABC = 0$ and $DC = 0$;
5. In [7, Theorem 1], $ABC = 0, BD = 0$ and $DC = 0$;
6. In [7, Theorem 2, Theorem 3], $ABC = 0, DC = 0$ and BC is nilpotent (or D is nilpotent).

5. Numerical examples

To illustrate our results, we present numerical examples in this section.

Firstly, we describe that Theorem 4.1 generalizes results listed in Remark 4.3.

Example 5.1. *Let*

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad and \quad B = C = D = \begin{bmatrix} 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $b, c, d \in \mathbb{C}\backslash\{0\}$. *Then*

$$CB = BC = DC = \begin{bmatrix} 0 & 0 & bc & 0 \\ 0 & 0 & 0 & cd \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq 0, \quad BCB = CBC = CBD = \begin{bmatrix} 0 & 0 & 0 & bcd \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq 0.$$

Hence, the assumptions of [9, Theorem 2.1], [12, Case (b3)], [33, Theorem 3.1], [18, Corollary 3.3], [1, Corollary 3.2] *and* [1, Theorem 3.3] *are not satisfied. Notice that* $BCA = DCA = 0$ *and* $CBCB = DCBD = BCBD = 0$*, i.e, the conditions of Theorem 4.1 hold. Using* $A^2 = A = A^\#$ *and* $D^d = 0$*, by Theorem 4.1, we obtain*

$$M^d = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & d & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^d = \begin{bmatrix} 1 & 1 & bc & cd+bcd & 0 & b & c+bc & cd+2bcd \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We now verify that Theorem 4.4 extends results given in Remark 4.7.

Example 5.2. *Consider matrices A, B and C presented in Example 5.1 and, for* $a, e \in \mathbb{C}\backslash\{0\}$,

$$D = \begin{bmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

By

$$DC = \begin{bmatrix} 0 & 0 & 0 & ad \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq 0,$$

we observe that the conditions of [5, Theorem 2.1], [11, Theorem 5.3], [14, Lemma 2.2], [3, Theorem 2.2], [7, Theorem 1] *and* [7, Theorem 2,Theorem 3] *are not met. Since* $(BC)^2 = 0$*, we have* $A(BC)^2 = 0$ *and* $(BC)^d = 0$*. The equalities* $BCA = 0$*,* $DCA = 0$ *and* $D^2 = 0$ *imply that the assumptions of Theorem 4.4 are satisfied. Also, we see that* $D^d = 0$*. Applying Lemma 2.5 and Theorem 4.4, we get*

$$N^d = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix},$$

where

$$F_1 = A + ABC = \begin{bmatrix} 1 & 1 & bc & cd \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$F_2 = A^2B + A^4BCB = AB + ABCB = \begin{bmatrix} 0 & b & c & bcd \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$F_3 = CA + CABC = 0_{4\times 4},$$

$$F_4 = CAB + CABCB = 0_{4\times 4},$$

and

$$M^d = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 & e \\ 0 & 0 & 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^d = \begin{bmatrix} 1 & 1 & bc & cd & 0 & b & c & be+bcd \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The conflict of interest statement

The authors declare that there is no conflict of interest.

References

[1] M.S. Abdolyousefi, *The representations of the g-Drazin inverse in a Banach algebra*, Hacet. J. Math. Stat. **50** (2021), 659–667.

[2] R. Behera, A.K. Nandi, J.K. Sahoo, *Further results on the Drazin inverse of even-order tensors*, Numerical Linear Algebra with Applications **27** (2020), e2317.

[3] C. Bu, K. Zhang, *The explicit representations of the Drazin inverses of a class of block matrices*, Electron. J. Linear Algebra **20** (2010), 406–418.

[4] S.L. Campbell, C.D. Meyer, *Generalized inverse of linear transformations*, Pitman, London, 1979.

[5] M. Catral, D.D. Olesky, P. Van Den Driessche, *Block representations of the Drazin inverse of a bipartite matrix*, Electron. J. Linear Algebra **18** (2009), 98–107.

[6] R.E. Cline, *An application of representation for the generalized inverse of a matrix*, MRC Technical Report **592**, 1965.

[7] A.S. Cvetković, G.V. Milovanović, *On Drazin inverse of operator matrices*, J. Math. Anal. Appl. **375** (2011), 331–335.

[8] D.S. Cvetković-Ilić, *New additive results on Drazin inverse and its applications*, Appl. Math. Comput. **218** (2011), 3019–3024.

[9] D.S. Cvetković-Ilić, J. Chen, Z. Xu, *Explicit representations of the Drazin inverse of block matrix and modified matrix*, Linear Multilinear Algebra **14** (2008), 1–10.

[10] C. Deng, Y. Wei, *A note on the Drazin inverse of an anti-triangular matrix*, Linear Algebra Appl. **431** (2009), 1910–1922.

[11] D.S. Djordjević, P.S. Stanimirović, *On the generalized Drazin inverse and generalized resolvent*, Czechoslovak Math. J. **51** (2001), 617–634.

[12] E. Dopazo, M.F. Martínez-Serrano, *Further results on the representation of the Drazin inverse of a* 2×2 *block matrix*, Linear Algebra Appl. **432** (2010), 1896–1904.

[13] M.P. Drazin, *Pseudoinverses in associative rings and semigroups*. Am. Math. **65** (1958), 506–514.

[14] R.E. Hartwig, X. Li, Y. Wei, *Representations for the Drazin inverse of a* 2×2 *block matrix*, SIAM J. Matrix Anal. Appl. **27** (2006), 757–771.

[15] R.E. Hartwig, J.M. Shoaf, *Group inverses and Drazin inverses of bidiagonal and triangular Toeplitz matrices*, J. Aust. Math. Soc. **24** (1977), 10–34.

[16] R.E. Hartwig, G. Wang, Y. Wei, *Some additive results on Drazin inverse*, Linear Algebra Appl. **322** (2001), 207–217.

[17] I. Kyrchei, *Determinantal representations of the Drazin inverse over the quaternion skew field with applications to some matrix equations*, Appl. Math. Comput. **238** (2014), 193–207.

[18] X. Li, Y. Wei, *A note on the representations for the Drazin inverse of* 2×2 *block matrices*, Linear Algebra Appl. **423** (2007), 332–338.

[19] J. Ljubisavljević, D.S. Cvetković-Ilić, *Additive results for the Drazin inverse of block matrices and applications*, J. Comput. Appl. Math. **235** (2011), 3683–3690.

[20] C.D. Meyer, *The role of the group generalized inverse in the theory of finite Markov chains*, SIAM Review. **17** (1975), 443–464.

[21] C.D. Meyer, R.J. Plemmons, *Convergent powers of a matrix with applications to iterative methods for singular systems of linear systems*, SIAM J. Numer. Anal. **14** (1977), 699–705.

[22] C.D. Meyer, N.J. Rose, *The index and the Drazin inverse of block triangular matrices*, SIAM J. Appl. Math. **33** (1977), 1–7.

[23] D. Mosić, D.S. Djordjević, *Block representations of the generalized Drazin inverse*, Appl. Math. Comput. **331** (2018), 200–209.

[24] D. Mosić, D.S. Djordjević, *Formulae for the generalized Drazin inverse of a block matrix in terms of Banachiewicz–Schur forms*, J. Math. Anal. Appl. **413** (2014), 114–120.

[25] D. Mosić, D.S. Djordjević, *Representation for the generalized Drazin inverse of block matrices in Banach algebras*, Appl. Math. Comput. **218** (2012), 12001–12007.
[26] D. Mosić, D.S. Djordjević, *Weighted pre-orders involving the generalized Drazin inverse*, Appl. Math. Comput. **270** (2015), 496–504.
[27] J. Robles, M.F. Martínez-Serrano, E. Dopazo, *On the generalized Drazin inverse in Banach algebras in terms of the generalized Schur complement*, Appl. Math. Comput. **284** (2016), 162–168.
[28] J.R. Sendra, J. Sendra, *Symbolic computation of Drazin inverses by specializations*, J. Comput. Anal. Appl. **301** (2016), 201–212.
[29] A. Shakoor, I. Ali, S. Wali, A. Rehman, *Some formulas on the Drazin inverse for the sum of two matrices and block matrices*, B. Iran. Math. Soc. **48** (2022), 351–366.
[30] M. Sohrabi, *Relationship between Cauchy dual and Drazin inverse of conditional type operators*, Bull. Sci. Math. **176** (2022), 103119.
[31] P.S. Stanimirović, M.D. Petković, D. Gerontitis, *Gradient neural network with nonlinear activation for computing inner inverses and the Drazin inverse*, Neural Process Lett **48** (2018), 109–133.
[32] L. Sun, B. Zheng, S. Bai, C. Bu, *Formulas for the Drazin Inverse of Matrices over Skew Fields*, Filomat **30** (2016), 3377–3388.
[33] H. Yang, X. Liu, *The Drazin inverse of the sum of two matrices and its applications*, J. Comput. Appl. Math. **235** (2011), 1412–1417.
[34] R. Yousefi, M. Dana, *Generalizations of some conditions for Drazin inverses of the sum of two matrices*, Filomat **32** (2018), 6417–6430.
[35] D. Zhang, Y. Jin, D. Mosić, *The Drazin inverse of anti-triangular block matrices*, J. Appl. Math. Comput. **68** (2022), 2699–2716.
[36] D. Zhang, D. Mosić, L. Guo, *The Drazin inverse of the sum of four matrices and its applications*, Linear Multilinear Algebra **68** (2020), 133–151.
[37] D. Zhang, D. Mosić, Y. Jin, *Explicit formulae for the Drazin inverse of anti-triangular block matrices*, Filomat **36** (2022), 6215–6229.
[38] D. Zhang, D. Mosić, T. Tam, *On the existence of group inverses of Peirce corner matrices*, Linear Algebra Appl. **582** (2019), 482–498.
[39] D. Zhang, Y. Zhao, D. Mosić, V.N. Katsikis, *Exact expressions for the Drazin inverse of anti-triangular matrices*, J. Comput. Appl. Math. **428** (2023), 115187.