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Convergence for $m-$positive currents and $m-$subharmonic functions

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Abstract. In this paper we give sufficient conditions on $m-$subharmonic functions f_k and $m-$positive currents R_k of bidegree (p, p) to ensure the convergence of $f_k.R_k \wedge \gamma^{m-p}$ in the sense of currents. As application of this result, we treat the special case $R_k = (dd^c f_k)^p$ and we show that one of the condition of the main result is necessary in this case.

1. Introduction

In complex analysis and especially in pluripotential theory, the concept of convergence of plurisubharmonic (psh. for short) functions and positive currents represents a fundamental to treat several problems on the Monge-Ampère and the Hessian operator. One of these problems is the product between a psh function and a positive current. It is well-known that such a product cannot be well defined in general without loss of the basic properties for currents (See [8]). For a given domain Ω of $\mathbb{C}^n$ and $m \in \mathbb{N} \cap [1, n]$, Blocki [2] , introduced and studied the set $SH^m(\Omega)$ of $m-$subharmonic functions on Ω. Later on Dhouib and Elkhadhra [6] has defined the notion of $m-$positive currents. So it is natural to extend the problem for the product of $m-$positive current with an $m-$subharmonic function.
In this paper, we consider f an $m-$subharmonic function, R an $m-$positive current defined on Ω and γ the standard K$\ddot{a}$hler form of $\mathbb{C}^n$. We first give sufficient and necessary conditions so that the product $f.R \wedge \gamma^{n-p}$ is well defined. Then we will study the problem of the convergence for the product of an $m-$subharmonic function with an $m-$positive current. In other words "if f_k is a sequence of functions that belong to $SH^m(\Omega)$ and R_k a sequence of $m-$positive currents, with bidegrees less than m, that converge respectively towards a function f and a current R respectively then do we have the convergence of the product $f_k.R_k \wedge (dd^c|z|^2)^{n-m}$ to $f.R \wedge (dd^c|z|^2)^{n-m}$ in the sense of currents? ". We give sufficient conditions that answer the given question. In other words we prove the following theorem.

Theorem :
Let R and R_k be $m-$positive currents defined on Ω with bidegree equal to (p, p) where $(p \leq m \leq n)$. Take f and f_k $m-$subharmonic functions satisfying $f_k \in \mathbb{L}^1_{loc}(R_k \wedge \gamma^{n-p})$ and $f \in \mathbb{L}^1_{loc}(R \wedge \gamma^{n-p})$. Assume that there is $1 \leq r \leq m$ such that

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1. f_k converges to f in Cap_r on every $A \Subset \Omega$,
2. $R_k \wedge \gamma^{n-m} \to R \wedge \gamma^{n-m}$ as currents in Ω,
3. $(R + R_k) \wedge \gamma^{n-p} \ll Cap_r$ on every $A \Subset \Omega$ and the convergence is uniform for all k,
4. $|f|R \wedge \gamma^{n-p} \ll Cap_r$ on every $A \Subset \Omega$ and $|f_k|R_k \wedge \gamma^{n-p} \ll Cap_r$ on every $A \Subset \Omega$ uniformly in k,

then $f_kR_k \wedge \gamma^{n-m} \to fR \wedge \gamma^{n-m}$ and hence $dd^c f_k \wedge R_k \wedge \gamma^{n-m} \to dd^c f \wedge R \wedge \gamma^{n-m}$ as currents in Ω.

The previous theorem generalizes the well-known result of Xing [11] established in the case $m = n$. The second part of this paper will be devoted to give an application of the main result where we will focus on the particular case $R_k = (dd^c f_k)^p$. In this case we prove that condition (1) in the main theorem is necessary in some particular cases.

2. Preliminaries

Throughout this paper we will use the following notations:
$d := \partial + \overline{\partial}$, $d^c := i(\overline{\partial} - \partial)$ and $\gamma := dd^c|z|^2$ the Kähler form defined on $\mathbb{C}^n$. We denote by Ω a bounded domain of $\mathbb{C}^n$. In the following definitions, we recall the definition for $m-$positive forms given by Blocki in [2] and Dhouib and Elkhadhra in [6].

Definition 2.1. *Let ξ be a $(1,1)-$form in Ω and $m \in \mathbb{N} \cap [1, n]$. We say that ξ is $m-$positive if for all $j \in \{1, \cdots, m\}$ one has*

$$\xi^j \wedge \gamma^{n-j} \geq 0$$

for every point of Ω.

Definition 2.2. *[6]*
Let ξ be a real $(p, p) - form$ on Ω and $m \in \mathbb{N} \cap [p, n]$.

1. *The form ξ is said to be $m-$positive if for every point of Ω and for every $(1, 1)$ $m-$positive forms $\varphi_1, \cdots, \varphi_{m-p}$ one has*
$$\xi \wedge \beta^{n-m} \wedge \varphi_1 \wedge \cdots \wedge \varphi_{m-p} \geq 0.$$
2. *The form ξ is called $m-$strongly positive if there exist $m-$positive forms $\varphi_1^k, \cdots, \varphi_p^k$ and c_k are constants such that $c_k \geq 0$ such that*
$$\xi = \sum_{k=1}^{N} c_k \varphi_1^k \wedge \cdots \wedge \varphi_p^k$$

The set of all $m-$strongly positive test forms of bidegree (p, p) on Ω will be denoted by $\mathcal{D}_p(\Omega)$. As in the limit case $m = n$, the notion of $m-$positive current will be defined by duality as follows:

Definition 2.3. *Let R be a current with bidegree equal to (p, p) on Ω and $m \in \mathbb{N} \cap [p, n]$. We say that R is $m-$positive if $\langle R, \gamma^{n-m} \wedge \xi\rangle \geq 0$ for every $\xi \in \mathcal{D}_{m-p}(\Omega)$.*

The set of $m-$positive currents with bidegree equal to (p, p) will be denoted by $\mathcal{D}'_p(\Omega)$.

Remark 2.4. 1. *The previous notions generalize the well-known classical positivity for forms and currents which was given by Lelong [9].*
2. *If $R \in \mathcal{D}'_p(\Omega)$, then the current $R \wedge \gamma^{n-m}$ is positive.*

In [2], the author defined the following notion of $m-$subharmonic functions and developed a pluripotential theory to study several problems related to the complex Hessian operator.

Definition 2.5. *The function $f : \Omega \to \mathbb{R} \cup \{-\infty\}$ is said to be m-subharmonic ($m-$sh for short) if it is subharmonic and*

$$dd^c f \wedge \gamma^{n-m} \wedge \xi_1 \wedge \cdots \wedge \xi_{m-1} \geq 0$$

for all $m-$positive forms $\xi_1, \cdots, \xi_{m-1}$. The set of $m-$sh functions defined on Ω will be denoted by $SH^m(\Omega)$. It is easy to check that the set $SH^n(\Omega)$ is exactly the set of psh functions on Ω.

In [2], [10] and [6], authors developed a detailed theory for the notion of $m-$sh functions.

Example 2.6. 1. *In* $\mathbb{C}^3$ *the* $(1,1)-$*form* $\xi := i(2.dz_1 \wedge d\overline{z}_1 + 2.dz_2 \wedge d\overline{z}_2 - dz_3 \wedge d\overline{z}_3)$ *is* $2-$*positive but it is not positive.*
2. *In* $\mathbb{C}^3$ *the current of integration over the* $\{z_3 = 0\}$ *is a* $2-$*positive current.*
3. *The function* $f(z) := 2|z_1|^2 + 2|z_2|^2 - |z_3|$ *belongs to* $SH^2(\mathbb{C}^3)$ *but it is not plurisubhrmonic.*

In pluripotential theory, the capacity represents an essential tool to solve several problems for the complex operators and polar sets. For every integer r, and $A \subset \Omega$ the $r-$capacity of A denoted by $Cap_r(E,\Omega)$ is defined as follows:

Definition 2.7.

$$Cap_r(E,\Omega) = \sup\{Cap_r(\mathcal{K}), \mathcal{K} \text{ compact subset of } E\}.$$

where

$$Cap_r(\mathcal{K}) := \sup\{\int_{\mathcal{K}} (dd^c f)^r \wedge \gamma^{n-r}, f \in SH^m(\Omega), 0 \le f \le 1\},$$

for $1 \le r \le m$.

The following proposition proves the quasicontinuity of every locally bounded $m-$subharmonic function. Such property will be used frequently in the rest of this paper.

Proposition 2.8. *Let* $f \in SH^m(\Omega)$. *If* f *is locally bounded then for every* $a > 0$ *there is an open subset* H_a *with* $Cap_r(H_a) < a$ *and so that* $f \in C(\Omega \setminus H_a)$ *(i.e.* f *is continuous on* $\Omega \setminus H_a$ *). Hence the following writing holds*

$$f = g + h$$

where $g \in C(\Omega)$ *and* $h \equiv 0$ *on* $\Omega \setminus H_a$.

Definition 2.9. *Let* $R \in \mathcal{D}'_p(\Omega)$ *and* μ_k *a sequence of positive measures on* Ω.

1. *The trace measure of* $R \wedge \gamma^{n-m}$ *on a subset E denoted by* $|R \wedge \gamma^{n-m}|_E$ *is defined on every subset* $A \subset \Omega$ *as*

$$|R \wedge \gamma^{n-m}|_E(A) = \int_{E\cap A} R \wedge \gamma^{n-p}.$$

2. *We say that the sequence* μ_k *is uniformly absolutely continuous with respect to* Cap_r *(and write* $\mu_k << Cap_r$ *) if* $\forall \varepsilon > 0, \exists \lambda > 0$ *such that for all* $B \subset \Omega$ *one has*

$$Cap_r(B) < \lambda \Rightarrow \mu_k(B) < \varepsilon, \forall k \in \mathbb{N}.$$

3. Convergence of $m-$positive currents

In this section, we consider $R \in \mathcal{D}'_p(\Omega)$ a closed current (i.e. $dR = 0$) and $f \in SH^m(\Omega)$ a negative function. If f is bounded then the product $f.R$ defines an $m-$positive closed current on Ω. However, it was noticed in [8] that, even in the limit case $m = n$, the product $f.R$ can not be well defined without loss of the basic properties for positive currents. We give in this section a sufficient and necessary condition to ensure the definition of $f.R \wedge \gamma^{n-m}$ for some unbounded classes of functions. For that we will establish the following lemma:

Lemma 3.1. *For every* $(f,R) \in SH^m(\Omega) \times \mathcal{D}'_p(\Omega)$ *and* $J \in \mathbb{N}$ *one has*

$$\int_{\{-J\le f<0\}\cap E} (-f) R \wedge \gamma^{n-p} \le \sum_{j=0}^{\infty} |R \wedge \gamma^{n-m}|_E(f < -j).$$

Proof. Using Abel transformation (see [12]), we get

$$\begin{aligned}\sum_{j=0}^{\infty}|R\wedge\gamma^{n-m}|_E(f<-j) &\geq \sum_{j=0}^{J-1}|R\wedge\gamma^{n-m}|_E(-J\leq f<-j)\\ &= \sum_{j=1}^{J} j|R\wedge\gamma^{n-m}|_E(-j\leq f<-j+1)\\ &\geq \sum_{j=1}^{J}\int_{\{-j\leq f<-j+1\}\cap E}(-f)R\wedge\gamma^{p}\\ &= \int_{\{-J\leq f<0\}\cap E}(-f)R\wedge\gamma^{n-p}\end{aligned}$$

The result follows. □

If we repeat the same argument as in the proof of the above lemma we can observe that for every $J\in\mathbb{N}$ one has

$$\sum_{j=1}^{J-1}|R\wedge\gamma^{n-m}|_E(-J\leq f<-j)\leq\int_E(-f)R\wedge\gamma^{p}. \quad (*)$$

Indeed we have

$$\begin{aligned}\int_{\{-J\leq f<0\}\cap E}(-f)R\wedge\gamma^{n-p} &= \sum_{j=1}^{J}\int_{\{-j\leq f<-j+1\}\cap E}(-f)R\wedge\gamma^{p}\\ &\geq \sum_{j=1}^{J}(j-1)|R\wedge\gamma^{n-m}|_E(-j\leq f<-j+1)\\ &= \sum_{j=0}^{J-1}|R\wedge\gamma^{n-m}|_E(-J\leq f<-j)\\ &- \sum_{j=1}^{J}|R\wedge\gamma^{n-m}|_E(-j\leq f<-j+1)\\ &= \sum_{j=1}^{J-1}|R\wedge\gamma^{n-m}|_E(-J\leq f<-j).\end{aligned}$$

Now we will give necessary and sufficient conditions to guarantee that the product $f.R\wedge\gamma^{n-m}$ is well defined.

Theorem 3.2. *If $f\in SH^m(\Omega)$ and $R\in\mathcal{D}_p'(\Omega)$ then the statement that follow are equivalent*

1. *For any $E\Subset\Omega$ one has* $\sum_{j=0}^{\infty}|R\wedge\gamma^{n-m}|_E(f<-j)<\infty$.
2. $f\in\mathbb{L}^1_{loc}(R\wedge\gamma^{n-p})$.
3. *There exists $E_0\Subset\Omega$ satisfying $f\in\mathbb{L}^1_{loc}(R\wedge\gamma^{n-p})$ in $\Omega\setminus E_0$.*

Proof. Firstly we will prove that (1) ⇔ (2). Using Lemma 3.1, we get that

$$\int_{\{-J\leq f<0\}\cap E}(-f)R\wedge\gamma^{n-p}\leq\sum_{j=0}^{\infty}|R\wedge\gamma^{n-m}|_E(f<-j).$$

By tending J to $+\infty$, we obtain (1) ⇒ (2). For the converse sense, we combine (∗) and the assumption that $f \in \mathbb{L}^1_{loc}(R \wedge \gamma^{n-p})$ to get that

$$\sum_{j=0}^{\infty} |R \wedge \gamma^{n-m}|_E(f < -j) < \infty.$$

Hence we obtain the assertion (1).

Now we will prove that (3) ⇒ (2). Let $V_2 \Subset V_1 \Subset \Omega$ and f_j a sequence of negative smooth $m-$sh functions which is decreasing towards f in V_1. Using the hypothesis (3), we get that for all $j \in \mathbb{N}$, $|f_j R \wedge \gamma^{n-m}|_\Omega$ have a uniformly bounded mass on $V_1 \backslash V_2$. If φ is a positive test function with support relatively compact in V_1 and $\varphi \equiv 1$ near V_2 then using the integration by parts we get

$$\begin{aligned} \int_{V_2} dd^c f_j \wedge R \wedge \gamma^{n-p-1} &\leq \int_{V_1} \varphi dd^c f_j \wedge R \wedge \gamma^{n-p-1} \\ &= \int_{V_1 \backslash V_2} f_j dd^c \varphi \wedge R \wedge \gamma^{n-p-1} \end{aligned}$$

which are uniformly bounded for all j. Now if we take $V_3 \Subset V_2$ and ψ a positive test function with support relatively compact in V_2 such that, for a neighborhood of V_3, the function ψ is equal to $|z|^2$ then

$$\begin{aligned} \int_{V_3} |f_j| \wedge R \wedge \gamma^{n-p} &= \int_{V_2} (-f_j) R \wedge dd^c \psi \wedge \gamma^{n-p-1} + \int_{V_2 \backslash V_3} f_j R \wedge dd^c \psi \wedge \gamma^{n-p-1} \\ &= \int_{V_2} (-\psi) R \wedge dd^c f_j \wedge \gamma^{n-p-1} + \int_{V_2 \backslash V_3} f_j R \wedge dd^c \psi \wedge \gamma^{n-p-1}. \end{aligned}$$

Now if we let j goes to $+\infty$, we get using the Fatou's lemma that

$$\int_{V_3} |f| R \wedge \gamma^{n-p} < \infty.$$

It follows that $f \in L^1_{loc}(R \wedge \gamma^{n-p})$ in Ω and this proves that (3) ⇒ (2). As the converse implication (2) ⇒ (3) is obvious, the proof of the theorem is completed. □

In the rest of this section, we take f_k a sequence of $m-$subharmonic functions and R_k a sequence of $m-$positive currents. We will treat the problem of convergence of $f_k.R_k \wedge \gamma^{n-m}$ to $f.R \wedge \gamma^{n-m}$. More precisely we look for a suitable type of convergence for f_k to f that guarantees the convergence of $f_k.R_k \wedge \gamma^{n-m}$ toward $f.R \wedge \gamma^{n-m}$. In the classic case $m = n$ Demailly [5], Fornaess and Sibony [7] obtained some convergence and approximation theorems. We will use the convergence in capacity to deal with this problem when the currents R_k are $m-$positive.

Definition 3.3. *Let $1 \leq r \leq m \leq n$ and $f_j, f \in SH^m(\Omega)$ for all $j \in \mathbb{N}$. We say that $(f_j)_j$ is convergent to f with respect to Cap_r on A if*

$$\forall \varepsilon > 0, \quad \lim_{j \to +\infty} Cap_r(A \cap \{|f - f_j| > \varepsilon\}) = 0.$$

Theorem 3.4. *(Main Result)*
Let R and R_k be $m-$positive currents defined on Ω with bidegree equal to (p, p) where $(p \leq m \leq n)$. Take f and f_k $m-$subharmonic functions satisfying $f_k \in \mathbb{L}^1_{loc}(R_k \wedge \gamma^{n-p})$ and $f \in \mathbb{L}^1_{loc}(R \wedge \gamma^{n-p})$. Assume that there is $1 \leq r \leq m$ such that

1. *f_k converges to f in Cap_r on each $A \Subset \Omega$,*
2. *$R_k \wedge \gamma^{n-m} \to R \wedge \gamma^{n-m}$ as currents in Ω,*
3. *$(R + R_k) \wedge \gamma^{n-p} \ll Cap_r$ on every $A \Subset \Omega$ and the convergence is uniforme for all k,*
4. *$|f| R \wedge \gamma^{n-p} \ll Cap_r$ on every $A \Subset \Omega$ and $|f_k| R_k \wedge \gamma^{n-p} \ll Cap_r$ on every $A \Subset \Omega$ uniformly for all k,*

then $f_kR_k \wedge \gamma^{n-m} \to fR \wedge \gamma^{n-m}$ *and hence* $dd^c f_k \wedge R_k \wedge \gamma^{n-m} \to dd^c f \wedge R \wedge \gamma^{n-m}$ *in the sense of currents.*

Proof. Without loss of generality, we can assume that the functions f_k and f are negative in Ω. Since for all $s > 0$, the function f_k can be written as $f_k = \max(f_k, -s) + f_k - \max(f_k, -s)$ hence the proof will be established in two steps. In the first step we will deal with the bounded part in the last decomposition (so one may suppose that the functions f_k and f are uniformly bounded in Ω) and in the second step we will focus on the unbounded part.

First step: We assume that the functions f_k and f are uniformly bounded in Ω.

In this case we can use the quasicontinuity property for bounded $m-$subharmonic function (see Proposition 2.8). Using the same argument as in [**?**] we deduce by the assumption (1) and quasicontinuity of $m-$sh function with respect to Cap_r for any $\varepsilon > 0$ we can write

$$f = f_1 + f_2 \textit{ and } f_k = f_{k,1} + f_{k,2} \in \Omega$$

such that

- (i) f_1 is a continuous on Ω
- (ii) $f_{k,2}(z) = f_2(z) = 0$ on $\Omega \setminus \mathcal{U}$ for some $\mathcal{U} \subset \Omega$ with $Cap_r(\mathcal{U}) < \varepsilon$.
- (iii) For each $A \Subset \Omega \setminus \mathcal{U}$, we have $|f_{k,1} - f_1| < \varepsilon$ on A if k is large enough.
- (iv) All the functions $f_{k,1}$, $f_{k,2}$, f_1 and f_2 are uniformly bounded by a constant that does not depend on ε.

A simple computation show that the following decompositions hold

$$\begin{aligned} f_kR_k \wedge \gamma^{n-m} - fR \wedge \gamma^{n-m} &= (f_{k,1} - f_1)R_k \wedge \gamma^{n-m} + f_1(R_k - R) \wedge \gamma^{n-m} \\ &+ (f_{k,2}R_k - f_2R) \wedge \gamma^{n-m} \\ &= a_k + b_k + c_k \end{aligned}$$

so the proof will be completed in this case if we show that all the sequences a_k, b_k and c_k tend weakly to 0. Since the following inequality

$$\int_A |f_{k,1} - f_1| R_k \wedge \gamma^{n-p} \le \varepsilon \int_{A\setminus\mathcal{U}} R_k \wedge \gamma^{n-p} + \sup_j |f_{j,1} - f_1| \int_{\mathcal{U}} R_k \wedge \gamma^{n-p},$$

holds for any $A \Subset \Omega$ and sufficently large k then using the assumption (3) we deduce that a_k goes weakly to zero uniformly for all k as $\varepsilon \to 0$.

For each fixed $\varepsilon > 0$ it follows from the assumption (2) that the sequence b_k tends weakly to zero as $k \to \infty$. Finally for c_k we have for a test form φ there exist two constants $\alpha, \beta > 0$ such that

$$\begin{aligned} | < c_k, \varphi > | &\le \alpha \left(\int_{\mathcal{U}} |f_k| R_k \wedge \gamma^{n-m} - |f| R \wedge \gamma^{n-m} \right) \\ &\le \beta\, cap_r(\mathcal{U}). \end{aligned}$$

By the assumption (4), we get that c_k goes weakly to zero and uniformly for all k when ε tends to 0. The result of the theorem in the stated first case follows.

Second step: The general case

It suffices to prove that for every $s > 0$

$$(f_k - \max(f_k, -s)) \wedge R_k \wedge \gamma^{n-p} \to (f - \max(f, -s)) \wedge R \wedge \gamma^{n-p}$$

weakly to finish the proof of the theorem. For each $A \Subset \Omega$ we have

$$\begin{aligned}
&\left|\int_A (f_kR_k - fR)\wedge\gamma^{n-p} - \int_A (\max(f_k,-s)R_k - \max(f,-s)R)\wedge\gamma^{n-p}\right| \\
&= \left|\int_{A\cap\{f_k<-s\}} (f_k+s)R_k\wedge\gamma^{n-p} - \int_{A\cap\{u<-s\}} (f+s)R\wedge\gamma^{n-p}\right| \\
&\le \left|\int_{A\cap\{f_k<-s\}} (f_k+s)R_k\wedge\gamma^{n-p}\right| + \left|\int_{A\cap\{f<-s\}} (f+s)R\wedge\gamma^{n-p}\right| \\
&\le \int_{A\cap\{f_k<-s\}} (-f_k-s)R_k\wedge\gamma^{n-p} + \int_{A\cap\{f<-s\}} (-f-s)R\wedge\gamma^{n-p} \\
&\le \int_{A\cap\{f_k<-s\}} |f_k|R_k\wedge\gamma^{n-p} + \int_{A\cap\{f<-s\}} |f|R\wedge\gamma^{n-p}.
\end{aligned}$$

Since $Cap_r(A\cap\{f_k<-s\}) \le C_A Cap_m(A\cap\{f_k<-s\}) \to 0$ as $s\to\infty$(see Corollary 1.6.10 in [3]), it turns out from the hypothesis (1) and (2) that the R.H.S in the last inequality tends uniformly to zero as $s\to\infty$. Hence we get the desired result. □

4. Application: The case $R_k = (dd^c f_k)^p$

This section is an application of the Theorem 3.4 since we will deal with a special family of $m-$positive currents: the family of currents that can be written as $R_k = (dd^c f_k)^p$. Following the proof of Theorem 3.2 in [1], one can prove the following lemma.

Lemma 4.1. *Let R_j be a sequence of $m-$positive currents of bidimension $(n-p, n-p)$, $(p\le m\le n)$ in Ω such that $R_j\wedge\gamma^{n-m}\to R\wedge\gamma^{n-m}$. The statements that follow are equivalent:*

1. *For all $m-$polar set $M\subset\Omega$ one has $R\wedge\gamma^{n-m}(M)=0$ and $f.R_j\wedge\gamma^{n-m}\to f.R\wedge\gamma^{n-m}$ for every locally bounded $m-$subharmonic function f on Ω;*
2. *the sequence $R_j\wedge\gamma^{n-m}$ has uniformly a small mass on sets of small $m-$capacity.*

Since the proof is completely similar to the proof of Theorem 3.2 in [1], we will omit it. We prove in the following example that when f_k decreases to f, the condition (4) of Theorem 3.4 will be satisfied.

Example 4.2. *Let $\xi_k\in SH^m(\Omega)$ a bounded sequence of functions that decrease to a function ξ in Ω. We consider $R=(dd^c\xi)^p\wedge\gamma^{n-m}$ and $R_k=(dd^c\xi_k)^p\wedge\gamma^{n-m}$. Take f and f_k $m-$subharmonic functions satisfying $f_k\in\mathbb{L}^1_{loc}(R_k\wedge\gamma^{n-p})$ and $f\in\mathbb{L}^1_{loc}(R\wedge\gamma^{n-p})$. It is easy to check that*

$$(-f_k)R_k\wedge\gamma^{n-p} \ll (dd^c\xi_k)^p\wedge\gamma^{n-p} \ll Cap_p$$

for each k. Using Lemma 4.1, we find that

$$(-f)R\wedge\gamma^{n-p} \ll Cap_p.$$

Hence the hypothesis (4) *of Theorem 3.4 holds.*

The previous example represents a motivation to study the converse implication for this special family of currents. In this direction we will show that the assumption (1) in Theorem 3.4 is necessary in some particular cases. This is the objective of the following theorem.

Theorem 4.3. *Suppose that $1\le p\le m$ and f, f_k are locally uniformly bounded $m-$sh functions in Ω satisfying all the following:*

1. *There exists $E\Subset\Omega$ such that $f_k=f$ in $\Omega\setminus E$ for all $k=1,2,....$*
2. *$f_k(dd^c f_k)^{m-p}\wedge\gamma^{n-m}$ tends weakly in the sense of currents to $f(dd^c f)^{m-p}\wedge\gamma^{n-m}$ in Ω.*
3. *$f_k\to f$ in $\mathbb{L}^1_{loc}(\Omega)$.*

Then $f_k \to f$ in Cap_{m-p} on Ω.

Proof. Let $t > 0$ and fix $g \in SH^m(\Omega)$ with value in $[0, 1]$. By combining the well-known Schwarz inequality and integration by parts, we get that

$$\begin{aligned}
&\int_{\{|f_k-f|>t\}} (dd^c g)^{m-p} \wedge \gamma^{n+p-m} \\
&\le \frac{1}{t^2}\int_\Omega (f_k - f)^2 (dd^c g)^{m-p} \wedge \gamma^{n+p-m} \\
&= \frac{-1}{t^2}\int_\Omega d(f_k - f)^2 \wedge d^c g \wedge (dd^c g)^{m-p-1} \wedge \gamma^{n+p-m} \\
&\le C_1 \left(\int_\Omega d(f_k - f)^2 \wedge d^c (f_k - f)^2 \wedge (dd^c g)^{m-p-1} \wedge \gamma^{n+p-m}\right)^{\frac{1}{2}} \\
&\le 2C_1C_2 \left(\int_\Omega d(f_k - f) \wedge d^c (f_k - f) \wedge (dd^c g)^{m-p-1} \wedge \gamma^{n+p-m}\right)^{\frac{1}{2}} \\
&\le 2C_1C_2 \left(\int_\Omega d(f_k - f) \wedge d^c g \wedge dd^c (f_k - f) \wedge (dd^c g)^{m-p-2} \wedge \gamma^{n+p-m}\right)^{\frac{1}{2}}
\end{aligned}$$

where the constant $C_1 := \frac{1}{t^2}\left(\int_\Omega dg \wedge d^c g \wedge (dd^c g)^{m-p-1} \wedge \gamma^{n+p-m}\right)^{\frac{1}{2}}$ and C_2 is a constant greater than $\|f_k - f\|_\infty < \infty$ for all k and $z \in \Omega$. By the Chern-Levine-Nirenberg estimate in [3], we get that C_1 is uniformly bounded for all functions $g \in SH^m(\Omega)$ with $0 < g < 1$. As

$$dd^c(f_k - f) \wedge T \wedge \gamma^{n-m} \le dd^c(f_k + f) \wedge T \wedge \gamma^{n-m}$$

and if we repeat the same operation $(m - p - 2)$−times, we obtain that there exists $S > 0$ such that

$$\int_{\{|f_k-f|>t\}} (dd^c g)^{m-p} \wedge \gamma^{n+p-m} \le S\left(\int_\Omega (f_k - f) dd^c (f_k - f) \wedge (dd^c(f_k + f))^{m-p-1} \wedge \gamma^{n+p-m}\right)^{\frac{1}{2^{m-p}}}$$

for all k, t and such functions g.

By combining the Hartogs' Lemma and the Proposition 2.8 we obtain that for every $\varepsilon > 0$ there exists $l > 0$ and $A \subset E$ such that $Cap_{m-p}(A) < \varepsilon$ and $f_k(z) \le f(z) + \varepsilon$ in $\Omega \setminus A$ for all $k \ge l$. As the sequence f_k is locally uniformly bounded, then for $k \ge l$ one has

$$\begin{aligned}
&\int_\Omega (f_k - f) dd^c (f_k - f) \wedge (dd^c(f_k + f))^{m-p-1} \wedge \gamma^{n+p-m} \\
&\le \int_{\Omega\setminus A} (\varepsilon + f - f_k) dd^c (f - f_k) \wedge (dd^c(f_k + f))^{m-p-1} \wedge \gamma^{n+p-m} + O(Cap_{m-p}(A) + \varepsilon) \\
&\le (m-p)! \int_{E\setminus A} (\varepsilon + f - f_k) \sum_{i=0}^{m-p} (dd^c f_k)^i \wedge (dd^c f)^{m-p-i} \wedge \gamma^{n+p-m} + O(\varepsilon) \\
&\le (m-p)! \int_\Omega |z|^2 ((dd^c f)^{m-p+1} - (dd^c f_k)^{m-p+1}) \wedge \gamma^{n+p-m-1} + O(\varepsilon).
\end{aligned}$$

Since $(dd^c f_k)^{m-p+1} \wedge \gamma^{n-m} - (dd^c f)^{m-p+1} \wedge \gamma^{n-m}$ is supported by a compact subset of Ω and

$$(dd^c f_k)^{m-p+1} \wedge \gamma^{n-m} - (dd^c f)^{m-p+1} \wedge \gamma^{n-m} \to 0$$

weakly, we get that $f_k \to f$ in Cap_{m-p} on Ω and the proof is completed.

□

We give, in the following theorem, another version of Theorem 3.4 where the fourth condition will be replaced by another condition.

Theorem 4.4. *Let $R, R_k \in \mathcal{D}'_p(\Omega)$ for $p \leq m \leq n$. Assume that $f_k, f \in SH^m(\Omega)$ such that $f \in \mathbb{L}^1_{loc} R \wedge \gamma^{n-p}$. If there is $r \in \mathbb{N}$ $(1 \leq r \leq m)$ such that*

1. *$f_k \to f$ in Cap_r on every $A \Subset \Omega$,*
2. *$R_k \wedge \gamma^{n-m} \to R \wedge \gamma^{n-m}$ as currents in Ω,*
3. *$(R + R_k) \wedge \gamma^{n-p} \ll Cap_r$ on every $A \Subset \Omega$ uniformly for all k,*
4. *For all k, $\sum_{j=0}^{\infty} |R_k \wedge \gamma^{n-m}|_E(f_k < -j) < \infty$ is uniformly convergent on every $A \Subset \Omega$.*

then $f_k R_k \wedge \gamma^{n-m} \to f R \wedge \gamma^{n-m}$ as currents in Ω.

Proof. Using Theorem 3.2, the assumption (4) implies that for all k, the current $f_k R_k \wedge \gamma^{n-m}$ has locally bounded mass in Ω. As $|f| R \wedge \gamma^{n-p} \ll R \wedge \gamma^{n-p} \ll Cap_r$ on every $A \Subset \Omega$, Theorem 3.4 says that " it suffices to show that $|f_k| R_k \wedge \gamma^{n-p} \ll Cap_r$ uniformly for every k and $A \Subset \Omega$" which is a direct consequence of hypothesis (3) in the case when all f_k are uniformly bounded on A.
We can suppose, by Hartog's Lemma and proposition 2.8, that f_k are uniformly bounded from above on E. If J is a sufficiently large integer then we have

$$\int_{\{f_k < -J\} \cap E} |f_k| R_k \wedge \gamma^{n-p} \leq 2 \sum_{j=J}^{\infty} j |R_k \wedge \gamma^{n-m}|_E(-j-1 \leq f_k < -j)$$

$$= 2J \sum_{j=J}^{\infty} |R_k \wedge \gamma^{n-m}|_E(-j-1 \leq f_k < -j) + 2 \sum_{i=J+1}^{\infty} \sum_{j=i}^{\infty} |R_k \wedge \gamma^{n-m}|_E(-j-1 \leq f_k < -j)$$

$$= 2J |R_k \wedge \gamma^{n-m}|_E(f_k < -J) + 2 \sum_{i=J+1}^{\infty} |R_k \wedge \gamma^{n-m}|_E(f_k < -i) \leq 4 \sum_{i=[\frac{J}{2}]-1}^{\infty} |R_k \wedge \gamma^{n-m}|_E(f_k < -i)$$

which tends uniformly to zero for all k as $J \to \infty$ using hypothesis (4). The desired result follows. □

Conclusion : In conclusion, this work represents an extension on the study of the problem of the convergence in capacity for complex operators. In the classical case, several works have studied this problem to find sufficient conditions so that the convergence in capacity will ensure the convergence of the associated operator (See [1, 2, 6, 11]). This work deals with a more general and more complicated case as the operators are associated with a closed positive current R and the functions are $m-$subharmonics it suffices to take the trivial current $R = 1$ to get the results in [3] or $m = n$ and $R = 1$ to recover the results found in [1, 11]. The application of the main result for the particular currents $R_k = (dd^c f_k)^p$ is more general and shows the importance of the found result. Such application may be also useful to solve the Dirichlet problem whenever this problem is associated to a given closed current.

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