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Multivalued operators with finite essential ascent or essential descent and perturbations

Ezzeddine Chafai[a]

[a]*Department of Mathematics, Faculty of Sciences of Sfax, University of Sfax, Tunisia*

Abstract. The aim of this paper is to enlarge some known results from Fredholm and perturbation theory via measure of non-compactness. As applications, we focus on the study of the essential ascent and the essential descent spectra of an operator T defined on a given Banach space. Some perturbation results are also investigated.

1. Introduction

J. Von. Neumann introduced, in 1950, the notion of multi-valued linear operators in order to study adjoints of non-densely defined linear differential operators [25]. In fact, the adjoint of a non-densely defined operator and the inverse of a non invertible operator is not single-valued. So it is always required that the operators are densely defined or invertible, when one considers their adjoints or their inverses in the classical operator theory. Further, we found recently, that minimal and maximal operators generated by symmetric linear difference expressions are multi-valued or non-densely defined in general even though the corresponding definiteness condition is satisfied [18, 23]. So, the classical operator theory seems not available in some situations. For this reason, it is legitimate for us to extend some results to the multi-valued case in order to solve some connected problems. During the past last years, a number of papers have appeared on the spectral analysis of multi-valued linear operators.

Throughout this work, X will be an infinite dimensional Banach space on the field $\mathbb{C}$. A multi-valued linear operator T on X is a mapping from a subspace $D(T)$, called the domain of T, into 2^X (the set of all subsets of X) satisfying $T(\lambda x_1 + \mu x_2) \supset \lambda T x_1 + \mu T x_2$, for all scalars λ and μ, with equality if λ and μ are nonzero. We use the term linear relation, to refer to such a multi-valued linear operator denoted $T \in LR(X)$. The simplest naturally occurring examples of a multi-valued operators are the inverse, closure, completion and adjoint of single-valued operators. Similar to a single-valued linear transform, a multi-valued linear operator T is determined by its graph: $G(T) = \{(x, y) : x \in D(T), y \in Tx\}$. For this reason we identify T and $G(T)$ and we say that T is closed if its graph is closed in the space $X \times X$. The class of such linear relations will be denoted by $CR(X)$. The inverse of T is the linear relation T^{-1} defined by $T^{-1} := \{(y, x) \in X \times X : (x, y) \in T\}$. Clearly that T is closed if and only if T^{-1} is closed.

Let M and N be subspaces of X. Then the inverse image of N under T is defined to be the set $T^{-1}(N) := \{x \in D(T) : Tx \cap N \neq \emptyset\}$. When $M \cap D(T) \neq \emptyset$, the restriction $T_{/M}$ is given by $T_{/M} : \{(m, y) \in T : m \in M\}$, and

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Email address: ezzeddine.chafai@ipeis.rnu.tn (Ezzeddine Chafai)

we define the image of M under T by $T(M) := \{y \in Tx : x \in M \cap D(T)\}$. The set $T(0)$, called the multivalued part of T, is a subspace of X. One can easy see that T is single-valued if and only if $T(0) = \{0\}$. By Q_T we denote the natural quotient map from X into $X/\overline{T(0)}$ with kernel $\overline{T(0)}$. Clearly that $Q_T T$ is single-valued. For $x \in D(T)$, define $\|Tx\|$ by $\|Tx\| := \|Q_T Tx\|$, and let the quantity $\|T\|$ be defined as $\|T\| := \|Q_T T\|$. T is said to be continuous if $\|T\| < \infty$ and we say that T is open if $\gamma(T) := \sup\{\lambda \in \mathbb{R} : \lambda d(x, N(T)) \leq \|Tx\|, x \in D(T)\} > 0$. From [8, III.4.2], if T is closed, then T is open if and only if $R(T)$ is closed.

Let $S, T \in LR(X)$. If $D(T) \cap D(S) \neq \emptyset$, then the sum $S + T$ is defined as the linear relation

$$S + T := \{(x, y + z) : (x, y) \in S \text{ and } (x, z) \in T\}.$$

When $R(T) \cap D(S) \neq \emptyset$, the product ST is given by

$$ST := \{(x, z) : (x, y) \in T, (y, z) \in S \text{ for some } y \in X\}.$$

While, for $\lambda \in \mathbb{C}$, λT and $T + \lambda$ are the linear relations $(\lambda I)T$ and $T + \lambda I$ respectively, where I is the identity operator in X. Since the product of linear relations is clearly associative, if $n \in \mathbb{Z}$, T^n is defined as usual with $T^0 = I$, $T^1 = T$ and $T^{n+1} = T^n T$.

For $T \in LR(X)$, we will write $N(T) := T^{-1}(0)$ for its kernel and $R(T) := T(D(T))$ for its range and we denote by $\alpha(T)$ and $\beta(T)$ the dimension of $N(T)$ and the codimension of $R(T)$ respectively. If $T \in CR(X)$, then T is called upper semi-Fredholm, denoted $T \in \Phi_+(X)$, (respectively lower semi-Fredholm, denoted $T \in \Phi_-(X)$) if $R(T)$ is closed and $\alpha(T)$ (respectively $\beta(T)$) is finite. We also write $\overline{\beta}(T) := dim X/\overline{R(T)}$. If T is both upper and lower semi-Fredholm, we say that T is Fredholm denoted $T \in \Phi(X)$. The index of such linear relation is given by $ind(T) = \alpha(T) - \beta(T)$.

Recall that we have the following chains

$$N(T) \subset N(T^2) \subset ... \subset N(T^n) \subset ...., \quad \text{and} \quad R(T) \supset R(T^2) \supset ... \supset R(T^n) \supset ....$$

The chain singular of T, which play a fundamental role in the proofs of several main results of this paper, is given by

$$\mathcal{R}_c(T) := [\bigcup_{n=0}^{\infty} N(T^n)] \cap [\bigcup_{n=0}^{\infty} T^n(0)].$$

To notice that every single-valued operator T has a trivial singular chain, that is $\mathcal{R}_c(T) = \{0\}$. In the same way, we define the generalized kernel and the the generalized range of T by

$$N^\infty(T) := \bigcup_{n=0}^{\infty} N(T^n) \quad \text{and} \quad R^\infty(T) := \bigcap_{n=0}^{\infty} R(T^n).$$

For $i, n \in \mathbb{N}$, let as consider the quantities:

$$\alpha_n^i(T) := dim \frac{N(T^{n+i})}{N(T^n)} \text{ (the essential nullity)}, \quad \beta_n^i(T) := dim \frac{R(T^n)}{R(T^{n+i})} \text{ (the essential defect)},$$

$$\gamma_n^i(T) := dim \frac{T^{n+i}(0)}{T^n(0)} \text{ (the essential co-nullity)}, \quad \delta_n^i(T) := dim \frac{D(T^n)}{D(T^{n+i})} \text{ (the essential co-defect)},$$

$$s_n^i(T) := dim \frac{N(T^i) \cap R(T^n)}{N(T^i) \cap R(T^{n+i})}, \qquad k_n^i(T) := dim \frac{T^i(0) \cap D(T^n)}{T^i(0) \cap D(T^{n+1})}.$$

As useful we denote $\alpha_n(T) := \alpha_n^1(T)$ and $\beta_n(T) := \beta_n^1(T)$. Clearly that $\alpha(T) = \alpha_0(T)$ and $\beta(T) = \beta_0(T)$. Wherefore, we define the ascent, the descent, the essential ascent and the essential descent of T, respectively by

$$a(T) := \inf\{n \in \mathbb{N} : \alpha_n(T) = 0\}, \quad d(T) := \inf\{n \in \mathbb{N} : \beta_n(T) = 0\},$$

$$a_e(T) := \inf\{n \in \mathbb{N} : \alpha_n(T) < \infty\}, \quad d_e(T) := \inf\{n \in \mathbb{N} : \beta_n(T) < \infty\},$$

the infimum over the empty set is taken to be ∞. A classical example of operator with finite essential ascent (respectively essential descent) is the upper (respectively lower) semi-Fredholm operator, precisely, if $T \in \Phi_+(X)$ (respectively $T \in \Phi_-(X)$), then $a_e(T) = 0$ (respectively $d_e(T) = 0$). Define the following sets which are used in the sequel

$$\mathcal{C}(X) := \{T \in CR(X) : \mathcal{R}_c(T) = \{0\} \text{ and } D(T^r) + R(T^s) = X\ \forall r, s \in \mathbb{N}\},$$

$$\xi_+(X) := \{T \in \mathcal{C}(X) : n = a_e(T) < \infty \text{ and } R(T^{n+1}) \text{ is closed}\},$$
$$\xi_-(X) := \{T \in \mathcal{C}(X) : n = d_e(T) < \infty \text{ and } R(T^n) \text{ is closed}\},$$
$$\xi_\pm(X) := \xi_+(X) \cup \xi_-(X) \text{ and } \xi(X) := \xi_+(X) \cap \xi_-(X).$$

Evidently, $\mathcal{C}(X)$ is a non-empty set because it contains all everywhere defined or surjective linear relations. Let $T \in \xi_\pm$ and $n \geq inf\{a_e(T), d_e(T)\}$. The i^{th} essential index of degree n of T is defined as

$$ind_n^i(T) := \alpha_n^i(T) - \beta_n^i(T) \in \mathbb{Z} \cup \{-\infty, \infty\}.$$

In particular, if T is semi-Fredholm, then $ind_0^1(T) = \alpha(T) - \beta(T) = ind(T)$ (the Fredholm index). For $T \in \xi_\pm$ we define the degree of stable iteration $p(T)$ and $q(T)$ by

$$p(T) := \inf\{p \in \mathbb{N} : \alpha_n(T) = \alpha_p(T), \forall n \geq p\},$$

respectively,

$$q(T) := \inf\{q \in \mathbb{N} : \beta_n(T) = \beta_p(T), \forall n \geq q\}.$$

The concept of ascent and descent of linear operators is introduced by F. Riesz in [19] in connection with his investigations of compact linear operators. Several authors are interested in studying this notion in the context of single-valued operators [2, 12, 13]. Recently, some works has been devoted to extend these concepts to the multi-valued case (see for instance [3–7, 21]). Further, the quantities $\alpha(T)$ and $\beta(T)$ appear in [10, 13, 14, 20] in connection with the perturbation theory of linear operators in Banach spaces. Our aim of this work is to study the essential index of a given linear relation T on X. We shall proved some results related to the stability of such index under perturbations. In Section 2, we give some preliminary properties of the essential nullity $\alpha_n^i(T)$ and the essential defect $\beta_n^i(T)$ of T. In particular we give relationships between $\alpha_n^i(ST)$ and $\beta_n^i(ST)$ and those of S and T, where S and T are linear relations on X. As consequence, a formula which relates the essential ascent and the essential descent of the relations S and T with those of the product relation ST is presented. Section 3 is devoted to focus on the essential index of a product of linear relations under some supplementary conditions. We finish by Section 4 where some perturbation results related to the essential ascent and essential descent of closed linear relation on X are given. In particular, we prove the stability of the essential index under perturbations.

2. Preliminaries

Always X is an infinite dimensional Banach space on the field $\mathbb{C}$. The following algebraic results, together with their proofs, can be found in [13, 24].

Lemma 2.1. *Let M and N be subspaces of a linear space E. Then*

(*i*) *$M/M \cap N \simeq (M + N)/N$.*
(*ii*) *If moreover $M \subset N$, then $dim(E/M) = dim(E/N) + dim(N/M)$.*

Lemma 2.2. *Let M_1, M_2 and N be subspaces of a linear space E. If $M_1 \subset M_2$, then*

$$dim \frac{M_1}{M_1 \cap N} \leq dim \frac{M_2}{M_2 \cap N}.$$

The next lemma is an improvement of [21, Lemma 4.4].

Lemma 2.3. *Let $A \in LR(X)$ and let $i, n \in \mathbb{N}$. Then*

$$\frac{N(A^{n+i})}{N(A^n)} \simeq \frac{N(A^i) \cap R(A^n)}{N(A^i) \cap A^n(0)}.$$

In particular, if $\mathcal{R}_c(A) = \{0\}$, then $\frac{N(A^{n+i})}{N(A^n)} \simeq N(A^i) \cap R(A^n)$.

As consequences of Lemma 2.3 and [21, Lemmas 4.1, 4.2, 4.4], we have the following properties.

$$\alpha_n^i(A) = dim[\frac{N(A^i)\cap R(A^n)}{N(A^i)\cap A^n(0)}]. \quad (1)$$

$$\beta_n^i(A) = dim[\frac{D(A^n)}{(N(A^n)+R(A^i))\cap D(A^n)}]. \quad (2)$$

$$\gamma_n^i(A) = dim[\frac{A^i(0)\cap D(A^n)}{A^i(0)\cap N(A^n)}]. \quad (3)$$

$$\delta_n^i(A) = dim[\frac{R(A^n)}{(A^n(0)+D(A^i))\cap R(A^n)}]. \quad (4)$$

$$s_n^i(A) = dim[\frac{N(A^{n+i})}{(N(A^n)+R(A^i))\cap N(A^{n+i})}]. \quad (5)$$

$$k_n^i(A) = dim[\frac{A^{n+i}(0)}{(A^n(0)+D(A^i))\cap A^{n+i}(0)}]. \quad (6)$$

Lemma 2.4. *Let $A \in LR(X)$. Then*

(*i*) $\alpha_n^i(A) \geq \alpha_{n+i}^i(A) + s_n^i(A)$, *with equality if $\mathcal{R}_c(A) = \{0\}$.*
(*ii*) $\beta_n^i(A) \geq \beta_{n+i}^i(A) + s_n^i(A)$, *with equality if $X = D(A^n) + R(A^i)$.*
(*iii*) $\gamma_n^i(A) \geq \gamma_{n+i}^i(A) + k_n^i(A)$, *with equality if $\mathcal{R}_c(A) = \{0\}$.*
(*iv*) $\delta_n^i(A) \geq \delta_{n+i}^i(A) + k_n^i(A)$, *with equality if $X = D(A^i) + R(A^n)$.*

Proof. (*i*)

$$\begin{aligned}
\alpha_n^i(A) = &\ dim\frac{N(A^{n+i})}{N(A^n)} \\
= &\ dim[\frac{N(A^i)\cap R(A^n)}{N(A^i)\cap A^n(0)}] && (from\ (1)) \\
= &\ dim\frac{N(A^i)\cap R(A^n)}{N(A^i)\cap R(A^{n+i})} + dim\frac{N(A^i)\cap R(A^{n+i})}{N(A^i)\cap A^n(0)} && (Lemma\ 2.1) \\
\geq &\ dim\frac{N(A^i)\cap R(A^n)}{N(A^i)\cap R(A^{n+i})} + dim\frac{N(A^i)\cap R(A^{n+i})}{N(A^i)\cap A^{n+i}(0)} \\
= &\ s_n^i(A) + \alpha_{n+i}^i(A).
\end{aligned}$$

If $\mathcal{R}_c(A) = \{0\}$, then

$$\begin{aligned}
\alpha_n^i(A) = &\ dim[N(A^i)\cap R(A^n)] \\
= &\ dim\frac{N(A^i)\cap R(A^n)}{N(A^i)\cap R(A^{n+i})} + dim N(A^i)\cap R(A^{n+i}) \\
= &\ s_n^i(A) + \alpha_{n+i}^i(A).
\end{aligned}$$

(*ii*)

$$\begin{aligned}
\beta_n^i(A) = &\ dim\frac{D(A^n)}{[N(A^n)+R(A^i)]\cap D(A^n)} && (from\ (2)) \\
= &\ dim\frac{D(A^n)+R(A^i)}{N(A^n)+R(A^i)} && (Lemma 2.1(i)) \\
= &\ dim\frac{D(A^n)+R(A^i)}{N(A^{n+i})+R(A^i)} + dim\frac{N(A^{n+i})+R(A^i)}{N(A^n)+R(A^i)} && (Lemma\ 2.1(ii)) \\
\geq &\ dim\frac{D(A^{n+i})+R(A^i)}{N(A^{n+i})+R(A^i)} + dim\frac{N(A^{n+i})+R(A^i)}{N(A^n)+R(A^i)} \\
= &\ dim\frac{D(A^{n+i})+R(A^i)}{N(A^{n+i})+R(A^i)} + dim\frac{N(A^{n+i})}{(N(A^n)+R(A^i))\cap N(A^{n+i})} && (Lemma\ 2.1(i)) \\
= &\ dim\frac{D(A^{n+i})}{[N(A^{n+i})+R(A^i)]\cap D(A^{n+i})} + dim\frac{N(A^{n+i})}{(N(A^n)+R(A^i))\cap N(A^{n+i})} \\
= &\ \beta_{n+i}^i(A) + s_n^i(A).
\end{aligned}$$

Suppose, moreover, that $X = D(A^n) + R(A^i)$. It follows that

$$\beta_n^i(A) = dim\frac{X}{N(A^{n+i}) + R(A^i)} + dim\frac{N(A^{n+i}) + R(A^i)}{N(A^n) + R(A^i)} = \beta_{n+i}^i(A) + s_n^i(A).$$

To show (*iii*) and (*iv*) it suffices to replace A^{-1} instead to A in (*i*) and (*ii*) respectively.
□

Lemma 2.5. *Let $A \in C(X)$. Then*

(*i*) *For $i \in \mathbb{N}$, $(\alpha_n^i(A))_n$, $(\beta_n^i(A))_n$, $(\gamma_n^i(A))_n$ and $(\delta_n^i(A))_n$ are decreasing sequences.*
(*ii*) *For $n \in \mathbb{N}$, $(\alpha_n^i(A))_i$, $(\beta_n^i(A))_i$, $(\gamma_n^i(A))_i$ and $(\delta_n^i(A))_i$ are increasing sequences.*

Proof. (*i*) Since $R(A^{n+1}) \subset R(A^n)$ and $A \in C(X)$, then $\alpha_n^i(A) = dim[N(A^i) \cap R(A^n)] \geq dim[N(A^i) \cap R(A^{n+1})] = \alpha_{n+1}^i(A)$. Combining Formula (2) together with Lemma 2.1(i) and the fact that $A \in C(X)$, one has $\beta_n^i(A) = dim\frac{D(A^n)+R(A^i)}{N(A^n)+R(A^i)} = dim\frac{X}{N(A^n)+R(A^i)} \geq dim\frac{X}{N(A^{n+1})+R(A^i)} = \beta_{n+1}^i(A)$. Furthermore, clearly that $A^{-1} \in C(X)$ whenever $A \in C(X)$, and since $\gamma_n^i(A) = \alpha_n^i(A^{-1})$ and $\delta_n^i(A) = \beta_n^i(A^{-1})$, then it follows that $(\gamma_n^i(A))_n$ and $(\delta_n^i(A))_n$ are also decreasing sequences.
(*ii*) Since $N(A^i) \subset N(A^{i+1})$ and $A \in C(X)$, then $\alpha_n^i(A) = dim[N(A^i) \cap R(A^n)] \leq dim[N(A^{i+1} \cap R(A^n)] = \alpha_n^{i+1}(A)$. In the same way we prove that $\beta_n^i(A) \leq \beta_n^{i+1}(A)$. To prove that $(\gamma_n^i(A))_i$ and $(\delta_n^i(A))_i$ are increasing sequences it suffices to take A^{-1} instead to A in $\alpha_n^i(A)$ and $\beta_n^i(A)$ respectively.
□

The next proposition is an improvement of [21, Lemma 5.4]. Our techniques used in this proof are different from those used in [21, Lemma 5.4].

Proposition 2.6. *Let $A \in C(X)$ and $n \in \mathbb{N}, m \in \mathbb{N}\backslash\{0\}$. Then*

(*i*) $\alpha_{nm}^i(A) \leq \alpha_n^i(A^m) \leq mi\alpha_{nm}^1(A)$.
(*ii*) $\beta_{nm}^i(A) \leq \beta_n^i(A^m) \leq mi\beta_{nm}^1(A)$.

Proof. The use of Lemma 2.1(ii) together with the fact that $(\alpha_n^i(A))_n$ is a decreasing sequence, one can deduce that

$$\begin{aligned}
\alpha_n^i(A^m) &= dim\frac{N(A^{nm+mi})}{N(A^{nm})} \\
&= \sum_{j=0}^{mi-1} dim\frac{N(A^{mn+mi-j})}{N(A^{mn+mi-j-1})} \\
&= \sum_{j=0}^{mi-1} \alpha_{nm+mi-j-1}^1 \\
&\leq mi\alpha_{nm}^1(A).
\end{aligned}$$

On the other hand, since $N(A^{nm+i}) \subset N(A^{nm+mi})$, then

$$\alpha_{nm}^i(A) = dim\frac{N(A^{nm+i})}{N(A^{nm})} \leq dim\frac{N(A^{nm+mi})}{N(A^{nm})} = \alpha_n^i(A^m).$$

This proves (*i*). In the same way we prove (*ii*). □

Corollary 2.7. *Let $A \in C(X)$ and let $m \in \mathbb{N}$. Then*

(*i*) $a(A^m) \leq a(A) \leq ma(A^m)$.
(*ii*) $a_e(A^m) \leq a_e(A) \leq ma_e(A^m)$.
(*iii*) $d(A^m) \leq d(A) \leq md(A^m)$.
(*iv*) $d_e(A^m) \leq d_e(A) \leq md_e(A^m)$.

Proof. (*i*) Since $(\alpha_n(A))_n$ is a decreasing sequence, one can deduce from Proposition 2.6(*i*), that $\alpha_{nm}(A) \leq \alpha_n(A^m) \leq m\alpha_{nm}(A) \leq m\alpha_n(A)$. Therefore the result is trivial if $a(A) = \infty$. Assume now that $r = a(A) < \infty$. Then $\alpha_r(A) = 0$ and hence $\alpha_r(A^m) \leq m\alpha_r(A) = 0$. This implies that $s = a(A^m) \leq r$. The fact that $\alpha_{sm}(A) \leq \alpha_s(A^m) = 0$ gives that $r \leq ms$. Consequently, $a(A^m) \leq a(A) \leq ma(A^m)$.

(*ii*) First observe that, for $T \in LR(X)$, $a_e(T) \leq k$ if and only if $\alpha_n(T) < \infty$ for all $n \geq k$. Let us suppose that $p = a_e(A) < \infty$. Then $\alpha_p(A^m) \leq \alpha_{pm}(A) < \infty$ (Proposition 2.6). Hence $q = a_e(A^m) \leq p$. On the other hand, $\alpha_{mq}(A) \leq \alpha_q(A^m) < \infty$, which implies that $p \leq mq$. The result is evident when $a_e(A) = \infty$. I omit the proofs of (*iii*) and (*iv*) because they are similar to those of (*i*) and (*ii*).

□

The next lemma gives a relationship between the essential nullity and the essential defect of a linear relation which extend the classical very well known result $\alpha(T) \leq \beta(T)$ (see [21, Lemma 5.3]).

Lemma 2.8. *Let $A \in LR(X)$ and $i, n \in \mathbb{N}$. If $N(A) \cap R(A^{nr}) = \{0\}$ for some $r \in \mathbb{N}$, then $\alpha_n^i(A) \leq \beta_n^i(A)$.*

Proof. For $k \in \mathbb{N}$, let A_k be the restriction of A viewed as a map from $R(A^k)$ into $R(A^k)$. Since $N(A) \cap R(A^{nr}) = \{0\}$, then $N(A_{n+j}) \cap R((A_{n+j})^r) = N(A) \cap R(A^{(n+j)r}) = \{0\}$, for all $j \in \mathbb{N}$. Now, from [21, Lemma 5.3], it follows that

$$\alpha_n^i(A) = \sum_{j=0}^{i-1} \alpha_{n+j}^1(A) = \sum_{j=0}^{i-1} \alpha(A_{n+j}) \leq \sum_{j=0}^{i-1} \beta(A_{n+j}) = \sum_{j=0}^{i-1} \beta_{n+j}^1(A) = \beta_n^i(A).$$

□

Lemma 2.9. *Let $A \in LR(X)$. Then*

(*i*) *$a_e(A) \leq a(A)$,*
(*ii*) *$d_e(A) \leq d(A)$,*
(*iii*) *if $a(A) < \infty$ and $d(A) < \infty$, then $a(A) \leq d(A)$, with equality if $D(A^m) \subset R(A) + D(A^n)$, for some $m, n \in \mathbb{N}$,*
(*iv*) *if $a_e(A) < \infty$ and $d_e(A) < \infty$, then $a_e(A) \leq d_e(A)$, with equality if $R(A) + D(A^n) = X$, for some $n \in \mathbb{N}$.*

Proof. The parts (*i*) and (*ii*) are trivial and the part (*iii*) is proved in [21, Theorem 5.7]. In [5, Theorem 2.1] the authors proved that $a_e(A) \leq d_e(A)$ whenever these quantities are finite and we have equality when A is everywhere defined. We will prove that the equality in Part (*iv*) holds when $R(A) + D(A^n) = X$, for some $n \in \mathbb{N}$. For this, let $m = a_e(A)$ and $n = d_e(A)$. Since $m \leq n$, then $n = m + j$ for some $j \in \mathbb{N}$. It follows, from Lemma 2.4, that

$$\begin{aligned}
\beta_m(A) &= \beta_{m+1}(A) + s_m(A) \\
&= \beta_{m+2}(A) + s_{m+1}(A) + s_m(A) \\
&= \ldots\ldots \\
&= \beta_{m+j}(A) + s_{m+j-1}(A) + \ldots + s_m(A) \\
&= \beta_n(A) + s_{m+j-1}(A) + \ldots + s_m(A).
\end{aligned}$$

Now, since $\alpha_m(A) < \infty$, then $\alpha_k(A) < \infty$ for all $k \geq m$, and hence one can deduce from Lemma 2.4 (*i*), that $s_k(A) < \infty$, for all $k \geq m$. It follows, since $\beta_n(A) < \infty$, that $\beta_m(A) = \beta_n(A) + s_{m+j-1}(A) + \ldots + s_m(A) < \infty$, so that $d_e(A) \leq m$. Consequently, $n = m$. □

It is very well known, for $A, B \in LR(X)$, that $\alpha(AB) \leq \alpha(A) + \alpha(B)$ and $\beta(AB) \leq \beta_(A) + \beta(B)$. In the next proposition we show that we have, under supplementary conditions, a similar relation in terms of essential nullity and essential defect.

Proposition 2.10. *Let $A, B \in LR(X)$ be such that $AB \in \mathcal{C}(X)$. Suppose that $AB = BA$, $N(A^kB^k) = N(A^k) + N(B^k)$, $R(A^kB^k) = R(A^k) \cap R(B^k)$, $N(B^i) \subset R(A^k)$ and $N(A^i) \subset R(B^k)$, $\forall i, k \in \mathbb{N}$. Then we have the following:*

(*i*) $\max\{\alpha_n^i(A), \alpha_n^i(B)\} \leq \alpha_n^i(AB) \leq \alpha_n^i(A) + \alpha_n^i(B)$.
(*ii*) $\max\{\beta_n^i(A), \beta_n^i(B)\} \leq \beta_n^i(AB) \leq \beta_n^i(A) + \beta_n^i(B)$.

Proof. Since $\mathcal{R}_c(AB) = \{0\}$, then one can deduce that $\mathcal{R}_c(A) = \mathcal{R}_c(B) = \{0\}$.

(*i*)

$$\begin{aligned}
\alpha_n^i(AB) &= dim\frac{N((AB)^i)}{N((AB)^n)} \\
&= dim[N(A^iB^i) \cap R(A^nB^n)] \\
&= dim[N(A^i) + N(B^i)] \cap R(A^n) \cap R(B^n) \\
&= dim[N(A^i) \cap R(A^n) + N(B^i)] \cap R(B^n) \quad \text{(equality since } N(B^i) \subset R(A^n)) \\
&= dim[N(A^i) \cap R(A^n) + N(B^i) \cap R(B^n)] \quad \text{(equality since } N(A^i) \cap R(A^n) \subset R(B^n)) \\
&\leq dim[N(A^i) \cap R(A^n)] + dim[N(B^i) \cap R(B^n)] \\
&= \alpha_n^i(A) + \alpha_n^i(B).
\end{aligned}$$

Now we show that $\alpha_n^i(A) \leq \alpha_n^i(AB)$. We have

$$\begin{aligned}
\alpha_n^i(A) = & \; dim N(A^i) \cap R(A^n) \\
&= dim N(A^i) \cap N(A^iB^i) \cap R(A^n) \\
&\leq dim N(A^iB^i) \cap R(B^n) \cap R(A^n) \quad \text{(since } N(A^i) \subset R(B^n)) \\
&= dim N((AB)^i) \cap R((AB)^n) \\
&= \alpha_n^i(AB).
\end{aligned}$$

In the same way (by interchanging the roles of A and B) we show that $\alpha_n^i(B) \leq \alpha_n^i(AB)$ and this completes the proof of the part (i).

(ii) By using (2) and Lemma 2.1 we have that

$$\begin{aligned}
\beta_n^i(AB) &= dim \frac{D((AB)^n) + R((AB)^i)}{N((AB)^n) + R((AB)^i)} \\
&= dim \frac{X}{N(A^n) + N(B^n) + R(A^i) \cap R(B^i)} \\
&= dim \frac{X}{N(A^n) + N(B^n) + R(A^i) \cap R(B^i)} \\
&= dim \frac{X}{[N(A^n) + R(A^i)] \cap R(B^i) + N(B^n)} \qquad \text{(since } N(A^n) \subset R(B^i)) \\
&= dim \frac{X}{[N(A^n) + R(A^i)] \cap [R(B^i) + N(B^n)]} \qquad \text{(since } N(B^n) \subset R(A^i)) \\
&= dim \frac{X}{[N(A^n) + R(A^i)]} + dim \frac{N(A^n) + R(A^i)}{[N(A^n) + R(A^i)] \cap [N(B^n) + R(B^i)]} \\
&= dim \frac{X}{[N(A^n) + R(A^i)]} + dim \frac{N(A^n) + R(A^i) + N(B^n) + R(B^i)}{N(B^n) + R(B^i)} \qquad \text{(Lemma2.1)} \\
&\leq dim \frac{X}{[N(A^n) + R(A^i)]} + dim \frac{X}{N(B^n) + R(B^i)} \\
&= \beta_n^i(A) + \beta_n^i(B).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\beta_n^i(A) &= dim \frac{X}{N(A^n) + R(A^i)} \\
&\leq dim \frac{X}{[N(A^n) + R(A^i)] \cap [N(B^n) + R(B^i)]} \\
&= dim \frac{X}{[N(A^n) + R(A^i)] \cap N(B^n) + [N(A^n) + R(A^i)] \cap R(B^i)} \quad \text{(since } N(A^n) \subset N(B^n) + R(B^i)) \\
&= dim \frac{X}{[N(B^n) + N(A^n) + R(A^i) \cap R(B^i)]} \quad \text{(since } N(B^n) \subset N(A^n) + R(B^i) \text{ and } N(A^n) \subset R(B^i)) \\
&= dim \frac{X}{[N(A^n) + N(B^n) + R(A^i) + R(B^i)]} \\
&= dim \frac{X}{N(A^nB^n) + R(A^iB^i)} \\
&= dim \frac{X}{N((AB)^n) + R((AB)^i)} \\
&= \beta_n^i(AB).
\end{aligned}$$

In the same way we show that $\beta_n^i(B) \leq \beta_n^i(AB)$. This complete the proof.

□

By substituting A and B in the Proposition 2.10 by A^{-1} and B^{-1} respectively and by observing that $AB \in \mathcal{C}(X)$ if, and only if $A^{-1}B^{-1} \in \mathcal{C}(X)$, we have the immediate next corollary.

Corollary 2.11. *Let $A, B \in LR(X)$ such that $AB \in \mathcal{C}(X)$. Suppose that $AB = BA$, $A^kB^k(0) = A^k(0) + B^k(0)$, $D(A^kB^k) = D(A^k) \cap D(B^k)$, $B^i(0) \subset D(A^k)$ and $A^i(0) \subset D(B^k)$, $\forall i, k \in \mathbb{N}$. Then the following assertions hold:*

(i) $\max\{\gamma_n^i(A), \gamma_n^i(B)\} \leq \gamma_n^i(AB) \leq \gamma_n^i(A) + \gamma_n^i(B)$.
(ii) $\max\{\delta_n^i(A), \delta_n^i(B)\} \leq \delta_n^i(AB) \leq \delta_n^i(A) + \delta_n^i(B)$.

Example 2.12. *Let $T \in LR(X)$ be everywhere defined with trivial singular chain and let $\lambda \neq \mu \in \mathbb{C}$. Consider the linear relations $A = T - \lambda$ and $B = T - \mu$. From [22, Corollary 21,Theorems 3.2,3.3 and 3.4], we have $AB = BA$, $N(A^kB^k) = N(A^k) + N(B^k)$, $R(A^kB^k) = R(A^k) \cap R(B^k)$ and $N(A^i) \subset R(B^k)$, $\forall i, k \in \mathbb{N}$. Moreover $AB \in \mathcal{C}(X)$. Now, let $P(X) = \prod_{j=1}^{k}(X - \lambda_j)^{m_j}$ be a complex polynomial and define the polynomial in T as the linear relation*

$$P(T) := \prod_{j=1}^{k}(T - \lambda_j)^{m_j}.$$

Then one has

$$max_j\{\alpha_{nm_j}^i(T - \lambda_j)\} \leq \alpha_n^i(P(T)) \leq \sum_{j=1}^{k} m_j\alpha_{nm_j}^i(T - \lambda_j).$$

$$max_j\{\beta_{nm_j}^i(T - \lambda_j)\} \leq \beta_n^i(P(T)) \leq \sum_{j=1}^{k} m_j\beta_{nm_j}^i(T - \lambda_j).$$

Example 2.13. *Let A, B, C and D be bounded mutually commuting operators on X such that $AC + DB = I$. Then A and B satisfy the conditions of Proposition 2.10. Indeed, A and B are bounded, hence $AB \in \mathcal{C}(X)$. Moreover, the fact that $AC + DB = I$ and that A, B, C and D are mutually commuting operators allows us to easily show (using the binomial expansion for commuting operators), that all conditions in Proposition 2.10 are satisfied.*

Corollary 2.14. *Let $A, B \in LR(X)$ defined as in Proposition 2.10. Then*

(i) $a_e(AB) = max\{a_e(A), a_e(B)\}$.
(ii) $a(AB) = max\{a(A), a(B)\}$.
(iii) $d_e(AB) = max\{d_e(A), d_e(B)\}$.
(iv) $d(AB) = max\{d(A), d(B)\}$.

Proof. (i) Suppose that $a_e(AB) = \infty$, then $\alpha_n(AB) = \infty$, for all $n \in \mathbb{N}$. Hence, from Proposition 2.10 (i), $max\{\alpha_n(A), \alpha_n(B)\} = \infty$, for all $n \in \mathbb{N}$. This means that $max\{a_e(A), a_e(B)\} = \infty$. Now, suppose that $n = a_e(AB) < \infty$. Then $max\{\alpha_n(A), \alpha_n(B)\} \leq \alpha_n(AB) < \infty$. It follows that $p = max\{a_e(A), a_e(B)\} \leq n$. On the other hand, $\alpha_p(AB) \leq \alpha_p(A) + \alpha_p(B) < \infty$, hence $a_e(AB) \leq p$. Consequently, $a_e(AB) = max\{a_e(A), a_e(B)\}$.

(ii) If AB has an infinite ascent, then $\alpha_n(AB) > 0$, for all $n \in \mathbb{N}$, which implies that $max\{\alpha_n(A), \alpha_n(B)\} \geq \frac{1}{2}[\alpha_n(A) + \alpha_n(B)] \geq \frac{1}{2}\alpha_n(AB) > 0$, for all $n \in \mathbb{N}$. Therefore $max\{a(A), a(B)\} = \infty$. Assume now that $n = a(AB) < \infty$. Then, $max\{\alpha_n(A), \alpha_n(B)\} \leq \alpha_n(AB) = 0$, so that $p = max\{a(A), a(B)\} \leq n$. On the other hand, $\alpha_p(AB) \leq \alpha_p(A) + \alpha_p(B) = 0$. Hence $n = a(AB) \leq p$. This proves (ii). In the same way we prove the Parts (iii) and (iv).
□

3. The essential index

Let $A, B \in LR(X)$. Our aim in this section is to give relationship between the essential index of AB and those of A and B. Also the index of the power A^m, $m \in \mathbb{N}$ is studied. From Lemma 2.4, it is easy to see that $s_n(A) < \infty$ if and only if $\alpha_n(A)$ or $\beta_n(A)$ is finite.

Theorem 3.1. *Let $A, B \in LR(X)$ such that $AB \in \mathcal{C}(X)$. Suppose that $AB = BA$, $N(A^kB^k) = N(A^k) + N(B^k)$, $R(A^kB^k) = R(A^k) \cap R(B^k)$, $N(B^i) \subset R(A^k)$ and $N(A^i) \subset R(B^k)$, $\forall i, k \in \mathbb{N}$. If A and B have finite indices $ind_n^i(A)$ and $ind_n^i(B)$, for some $n, i \in \mathbb{N}$, then $ind_n^i(AB)$ is finite and*

$$-\inf\{\alpha_n^i(A), \alpha_n^i(B)\} \leq ind_n^i(AB) - ind_n^i(A) - ind_n^i(B) \leq \inf\{\beta_n^i(A), \beta_n^i(B)\}.$$

Proof. Clearly, since $ind_n^i(A)$ and $ind_n^i(B)$ are finite, then $\alpha_n^i(A), \beta_n^i(A), \alpha_n^i(B)$ and $\beta_n^i(B)$ are finite. Hence, from Proposition 2.10, one can deduce that AB has a finite i^{th}-essential index of degree n. The result follows easily from Proposition 2.10. □

Corollary 3.2. *Let $A, B \in LR(X)$ as in Theorem 3.1. Suppose that A has finite ascent and descent. If $n = d(A)$, then*

$$ind_n^i(AB) = ind_n^i(A) + ind_n^i(B).$$

Proof. First observe, since $a(A)$ and $d(A)$ are finite, then $a(A) \leq d(A)$ (Lemma 2.9). Hence $\alpha_n^i(A) = \beta_n^i(A) = 0$. The result follows from Theorem 3.1. □

Theorem 3.3. *Let $A \in LR(X)$ and let $m \in \mathbb{N}\backslash\{0\}$. If A finite index $ind_n^i(A)$, for some $i, n \in \mathbb{N}$. Then A^m has a finite index $ind_n^i(A^m)$, moreover*

$$-(mi-1)\alpha_{nm}^i(A) \leq ind_n^i(A^m) - mi\, ind_{nm}^i(A) \leq (mi-1)\beta_{nm}^i(A).$$

Proof. From Proposition 2.6, since $\alpha_n^i(A^m)$ and $\beta_n^i(A^m)$ are finite, then $\alpha_{nm}^i(A)$ and $\beta_{nm}^i(A)$ are both finite. Hence, again by using Proposition 2.6, we deduce that

$$\alpha_{nm}^i(A) - mi\beta_{nm}^i(A) \leq \alpha_n^i(A^m) - \beta_n^i(A^m) \leq mi\alpha_{nm}^i(A) - \beta_{nm}^i(A) \tag{7}$$

The result follows immediately from (7). □

Suppose that A has a finite ascent and descent. Then $a_e(A) \leq a(A) \leq d(A) = q$ (Lemma 2.9) and hence $\alpha_n^i(A) = \beta_n^i(A) = 0$ for all $n \geq q$. Which implies, from the above theorem, that $ind_n^i(A^m) - mi\, ind_{nm}^i(A) = 0$. The next corollary improved a result in [20, Proposition 6.2].

Corollary 3.4. *Let $A \in LR(X)$ with finite ascent and descent. If $n = d(A)$, then, for $i, m \in \mathbb{N}\backslash\{0\}$,*

$$ind_n^i(A^m) = mi\, ind_{nm}^i(A).$$

4. Stability of the essential index

We start by some lemmas which will be needed to obtain the main results of this section. The next lemma extend the result proved in [26, Proposition 1.6] to the multi-valued case.

Lemma 4.1. *Let $A \in \mathcal{C}(X)$.*

(*i*) *If $a(A) < \infty$, then $N^\infty(A) \cap R^\infty(A) = \{0\}$.*
(*ii*) *If $a_e(A) < \infty$ and $N^\infty(A) \cap R^\infty(A) = \{0\}$, then $a(A) < \infty$.*
(*iii*) *If $q = d(A) < \infty$, then $D(A^q) \subset N^\infty(A) + R^\infty(A)$.*
(*iv*) *If $q = d_e(A) < \infty$ and $D(A^q) \subset N^\infty(A) + R^\infty(A)$, then $d(A) \leq q < \infty$.*

Proof. (*i*) Suppose that $p = a(A) < \infty$ and let $x \in N^\infty(A) \cap R^\infty(A)$. Then $x \in N(A^p) \cap R(A^p)$. Hence $x \in A^p y$, for some $y \in D(A)$, so that $y \in N(A^{2p}) = N(A^p)$. It follows that $x = x + 0 \in A^p y - A^p y = A^p(0)$. Thus $x \in A^p(0) \cap N(A^p) \subset \mathcal{R}_c(A) = \{0\}$.

(*ii*) Since $a_e(A) < \infty$, then the decreasing sequence $\alpha_n(A)$ terminates. Hence, there exists $p \in \mathbb{N}$ such that $\alpha_p(A) = dim N(A) \cap R(A^p) = dim N(A) \cap R^\infty(A) \leq dim N^\infty(A) \cap R^\infty(A) = 0$. This implies that $a(A) \leq p < \infty$.

(*iii*) Let $q = d(A) < \infty$. Then $N(A^q) + R(A^q) = N(A^q) + R^\infty(A) \subset N^\infty(A) + R^\infty(A)$. Let $x \in D(A^q)$ and let $y \in A^q x \subset R(A^q) = R(A^{2q})$. So that, $y \in A^{2q}z$, for some $z \in D(A^q)$. Hence $z \in A^q t$, for some $t \in A^q z$. It follows that $0 \in A^q x - A^q t = A^q(x-t)$ and consequently $x - t \in N(A^q)$.

(*iv*) Suppose now that $d_e(A) = q < \infty$ and $D(A^q) \subset N^\infty(A) + R^\infty(A)$. Then $X = D(A^q) + R(A) \subset N^\infty(A) + R^\infty(A) + R(A) = N^\infty(A) + R(A)$. It follows that

$$\begin{aligned}\beta_n(A) &= dim\frac{D(A^n)}{[N(A^n)+R(A)]\cap D(A^n)}\\ &= dim\frac{D(A^n)+R(A)}{N(A^n)+R(A)} \qquad \text{(Lemma 2.1)}\\ &= dim\frac{X}{N(A^n)+R(A)}.\end{aligned}$$

Since $q = d_e(A) < \infty$, then the decreasing sequence $(\beta_n(A))_n$ terminates and hence $N(A^q) + R(A) = N^\infty(A) + R(A)$. It follows that

$$\beta_q(A) = dim\frac{X}{N(A^q)+R(A)} = dim\frac{X}{N^\infty(A)+R(A)} = 0.$$

Consequently, $d(A) \leq q$.

□

Lemma 4.2. *Let $A \in CR(X)$ be continuous.*

(*i*) *If $R_c(A) = \{0\}$, $a_e(A) < \infty$ and $R^{a_e(T)+1}$ is closed, then $R(A^n)$ is closed for all $n \geq a_e(A)$.*
(*ii*) *If $d_e(A) < \infty$ and $R^{d_e(T)}$ is closed, then $R(A^n)$ is closed for all $n \geq d_e(A)$.*

Proof. (*i*) Let $n = a_e(A) + 1$. Arguing as in the proof of [7, Lemma 4.2], we prove that $R(A^{n+1})$ and $R(A^{n-1})$ is closed. This prove (*i*).

(*ii*) First suppose that $d_e(A) = 0$. This means that $\beta(A) = 0$, so that $X = M \oplus R(T)$ for some subspace M of X. Consider the linear relation $\widehat{A} : \frac{X}{N(A)} \oplus M \to X$ defined by $\widehat{A}(\overline{x} + m) = Ax + m$. Since A is continuous, then $\frac{X}{N(A)} \oplus M$ is closed and hence $\widehat{A}$ is closed (as A is closed). Furthermore, A is surjective and hence, by the open mapping theorem for linear relations, $\widehat{A}$ is bounded below. It follows that $R(A) = \widehat{A}(\frac{X}{N(A)} \oplus \{0\})$ is closed. Now, suppose that $n = d_e(A) > 0$ and that $R(A^n)$ is closed. It suffices to prove that $R(A^{n+1})$ is closed. Let A_n the linear relation induced by A to the Banach space $R(A^n)$. Then $\beta(A_n) = \beta_n(A) = 0$, so that $d_e(A_n) = 0$ and, since A and $R(A^n)$ is closed, then A_n is closed. Moreover, A_n is continuous (as A is continuous). It follows, from preceding, that $R(A^{n+1}) = R(A_n)$ is closed.
□

Lemma 4.3. *[8, Theorem III.7.4,Corollaries III.7.5 and III.7.6]*
Let $A, B \in LR(X)$. If $\|B\| < \gamma(A)$, then

(*i*) *$\alpha(A + B) \leq \alpha(A)$ and $\overline{\beta}(A + B) \leq \overline{\beta}(A)$.*
(*ii*) *If A is injective, then $\gamma(A + B) \geq \gamma(A) - \|B\|$.*
(*iii*) *If A is surjective, then so is $A + B$.*

Lemma 4.4. *[1, Lemma 14]. Let $A, B \in CR(X)$. If B is continuous with $B(0) \subset A(0)$ and $D(A) \subset D(B)$. Then $A + B$ is closed.*

The next lemma extend the result of [7, Theorem 3.2].

Lemma 4.5. *Let $A \in CR(X)$ and let $B \in CR(X)$ be nonzero and bounded satisfying $B(0) \subset A(0)$ and $AB^{-1} = B^{-1}A$.*

(*i*) *If $A \in \Phi_+(X)$ and $\|B\| < \gamma(A)$, then $A + B \in \Phi_+(X)$ with $\alpha(A + B) = \alpha(A)$ and $ind(A + B) = ind(A)$.*
(*ii*) *If $A \in \Phi_-(X)$ and $\|B\| < \gamma(A)$, then $A + B \in \Phi_-(X)$ with $\beta(A + B) = \beta(A)$ and $ind(A + B) = ind(A)$.*

Proof. Since A is closed, then $A + B$ is closed (by Lemma 4.4), $A + B \in \Phi_+(X)$ (by [8, V.3.2])and $ind(A + B) = ind(A)$ ([8, V.15.7]). We shall prove that

$$N(A + B) \subset R^{\infty}(A). \tag{8}$$

Let $x \in N(A+B)$. Then $Ax+Bx = A(0)$, which implies that $Bx \subset A(-x)+A(0) \subset R(A)$. Hence $x+N(B) \subset B^{-1}(R(A)) \subset R(A)$ (as $AB^{-1} = B^{-1}A$). This means that $x \in R(A)$ and consequently, $N(A + B) \subset R(A)$. Suppose that $N(A + B) \subset R(A^m)$ for some $m \in \mathbb{N}$ and let $x \in N(A + B)$. Then $x \in A^m y$ for some $y \in D(A^m)$. It follows that $Bx \subset A^{m+1}y + A(0) \subset A^{m+1}y + A^{m+1}(0) \subset R(A^{m+1})$. Therefore, $x + N(B) \subset B^{-1}R(A^{m+1}) \subset R(A^{m+1})$. This prove (8). Now, define $A_\infty := A_{/R^{\infty}(A)}$ and $B_\infty := B_{R^{\infty}(A)}$. Since A is open, then so is A_∞. Moreover, $\|B_\infty\| < \|B\| < \gamma(A) \leq \gamma(A_\infty)$. According to [8, III.7.4 and III.7.5], $A_\infty + B_\infty$ is surjective (as A_∞ is surjective) and hence $\beta(A_\infty + B_\infty) = ind(A_\infty + B_\infty)$. It follows that

$$\begin{aligned}
\alpha(A + B) = dimN(A + B) &= dimN(A_\infty + B_\infty) \qquad (as\ N(A + B) \subset R^{\infty}(A)) \\
&= ind(A_\infty + B_\infty) \\
&= ind(A_\infty) \qquad ([8, V.15.7]) \\
&= dimN(A_\infty) \\
&= dimN(A) \\
&= \alpha(A).
\end{aligned}$$

The proof of (*ii*) may be achieved by the same reasoning as for (*i*) by using Lemma 2.9. □

The next lemma shows the stability of the generalized kernel and range of a regular linear relation with finite essential ascent or descent under small perturbation.

Lemma 4.6. *Let $A \in \xi_{\pm}(X)$ be regular and let $B \in LR(X)$ be bounded with $B(0) \subset A(0)$. Assume that $AB = BA$ and $\|B\| < \frac{1}{2}\gamma(A)$, then*

(*i*) $R^\infty(A+B) = R^\infty(A)$.
(*ii*) $\overline{N^\infty(A+B)} = \overline{N^\infty(A)}$.

If, moreover, $AB^{-1} = B^{-1}A$, *then*

$$\alpha_n(A+B) = \alpha_n(A), \beta_n(A+B) = \beta_n(A) \text{ and } ind_n(A+B) = ind_n(A).$$

Proof. Suppose that $A \in \xi_+(X)$ and let $p = a_e(A) < \infty$. Then $R(T^n)$ is closed for all $n \geq p$, so that $R^\infty(A)$ is also closed. Let A_∞ and B_∞ the maps induced by A and B to the Banach space $R^\infty(A)$. The fact that $A(D(A) \cap R^\infty(A)) = R^\infty(A)$ and A and B commute ensure that A_∞ and B_∞ are well defined. Further, A_∞ is surjective and closed (as A is closed and $R^\infty(A)$ is closed), and $\|B_\infty\| \leq \|B\|$. Now, since $A_{/N(A)+D(A)\cap R^\infty(A)}$ and A have the same kernel and A is regular, then

$$0 < \gamma(A) \leq \gamma(A_{/N(A)+D(A)\cap R^\infty(A)}) = \gamma(A_{/N(A)R^\infty(A)}) = \gamma(A_{/R^\infty(A)}) = \gamma(A_\infty).$$

On the other hand, the use of Lemma 4.4 leads to $A_\infty + B_\infty$ is closed. Furthermore, since A_∞ is surjective, so is $A_\infty + B_\infty$ (Lemma 4.3). This implies that $R^\infty(A) = R^n(A_\infty + B_\infty) = (A+B)^n(R(A^\infty)) \subset R(A+B)^n$, $\forall n \in \mathbb{N}$. Thus

$$R^\infty(A) \subset R^\infty(A+B). \tag{9}$$

Now, consider the maps $\widehat{A}$ and $\widehat{B}$ induced by A and B on the Banach space $X/R^\infty(A)$. It is easy to see that $\widehat{A}$ and $\widehat{B}$ are correctly defined and single valued (as $B(0) \subset A(0) \subset R^\infty(A)$. As above, we show that

$$\|\widehat{B}\| \leq \|B\| \text{ and } \gamma(A) \leq \gamma(\widehat{A}). \tag{10}$$

Since $\widehat{A}$ is injective, it follows, from Lemma 4.3(*ii*), that

$$\gamma(\widehat{A} + \widehat{B}) \geq \gamma(\widehat{A}) - \|\widehat{B}\|. \tag{11}$$

The formula (10) together with (11) leads to

$$\|\widehat{B}\| \leq \|B\| < \frac{1}{2}\gamma(A) \leq \frac{1}{2}\gamma(\widehat{A}) \leq \frac{1}{2}[\gamma(\widehat{A} + \widehat{B}) + \|\widehat{B}\|].$$

It follows, from this, that $\|\widehat{B}\| \leq \gamma(\widehat{A} + \widehat{B})$. Hence we can apply formula (9), with $A + B$ replaced by $\widehat{A}$ and A replaced by $\widehat{A} + \widehat{B}$, to obtain $R^\infty(\widehat{A} + \widehat{B}) \subset R^\infty(\widehat{A}) = \{0\}$. Hence

$$R^\infty(A+B) \subset R^\infty(A). \tag{12}$$

The formulas (9) and (12) leads to $R^\infty(A+B) = R^\infty(A)$. This prove the part (*i*). We omit the proof of the part (*ii*) because they follow from an argument very similar to the above, with the maps induced on $X/R^\infty(A)$ replaced by the maps induced on $\overline{N^\infty(A)}$. Suppose now that $AB^{-1} = B^{-1}A$. Since $A \in \xi_+(X)$, then $A_\infty \in \Phi_+(R^\infty(A))$. By using Lemma 4.5, one can deduce that $A_\infty + B_\infty \in \Phi_+(R^\infty(A))$ and $\alpha(A_\infty + B_\infty) = \alpha(A_\infty)$. It follows that

$$\alpha_n(A+B) = \alpha(A_\infty + B_\infty) = \alpha(A_\infty) = \alpha_n(A) < \infty,$$

and

$$ind_n(A+B) = ind(A_\infty + B_\infty) = ind(A_\infty) = ind_n(A).$$

Consequently,

$$\beta_n(A+B) = ind_n(A+B) - \alpha_n(A+B) = ind_n(A) - \alpha_n(A) = \beta_n(A).$$

□

Now we are ready to state the first main result of this section.

Theorem 4.7. *Let* $A \in C(X)$ *and let* $B \in CR(X)$ *be nonzero and bounded such that* $B(0) \subset A(0)$ *and* $AB^{-1} = B^{-1}A$.

(*i*) *If* $A \in \xi_+(X)$, *then there exists* $\varepsilon > 0$ *such that* $A + B \in \Phi_+(X)$ *whenever* $\|B\| < \varepsilon$. *Moreover* $\alpha_n^i(A) = i\alpha(A+B)$, *for all* $i \in \mathbb{N}$ *and* $n \geq p(A)$.
(*ii*) *If* $A \in \xi_-(X)$, *then there exists* $\varepsilon > 0$ *such that* $A + B \in \Phi_-(X)$ *whenever* $\|B\| < \varepsilon$. *Moreover* $\beta_n^i(A) = i\beta(A+B)$, *for all* $i \in \mathbb{N}$ *and* $n \geq q(A)$.

(*iii*) *If $A \in \xi(X)$, then there exists $\varepsilon > 0$ such that $A + B \in \Phi(X)$ whenever $\|B\| < \varepsilon$. Moreover $ind_n^i(A) = i[ind(A + B)]$, for all $i \in \mathbb{N}$ and $n \geq q(A)$.*

Proof. (*i*) Let $n \geq a_e(A)$ and define the linear relations $A_n := A_{/R(A^n)}$ and $B_n := B_{R(A^n)}$. Then, according to Lemma 4.2, $R(A^n)$ is closed and $A_n \in \Phi_+(R(A^n))$. Furthermore, there exists, from Lemma 4.5, $\varepsilon > 0$, for which $A_n + B_n$ is upper semi-Fredholm whenever $\|B_n\| < \varepsilon$, moreover, $\alpha(A_n + B_n) = \alpha(A_n)$. It follows that

$$\begin{aligned}\alpha(A + B) = dimN(A + B) &= dimN(A + B) \cap R(A^n) \qquad (by\ 8)\\ &= \alpha(A_n + B_n)\\ &= \alpha(A_n)\\ &= dimN(A) \cap R(A^n)\\ &= \alpha_n(A).\end{aligned}$$

Hence

$$\alpha_n^i(A) = \Sigma_{j=1}^i \alpha_{n+j-1}(A) = \sum_{j=1}^{i} \alpha_n(A) = i\alpha_n(A) = i\,\alpha(A + B).$$

(*ii*) Suppose that $A \in \xi_-(X)$ and let $n \geq d_e(A)$. Then $R(A^n)$ is closed (Lemma 4.2), and the relation $A_n = A_{/R(A^n)}$ viewed as a linear relation on the Banach space $R(A^n)$ is closed. Since $d_e(A) < \infty$, then A_n is lower semi-Fredholm. Indeed, $dim(R(A_n)) = dim\frac{R(A^n)}{R(A^{n+1})} = \beta_n(A) < \infty$. Now, by using Lemma 4.5, it follows that

$$\begin{aligned}\beta(A + B) = dim\frac{D(A + B)}{R(A + B)} &= dim\frac{D(A)}{R(A + B)} \qquad (as\ B\ is\ bounded)\\ &= dim\frac{T^n(D(A))}{T^n(R(A + B))}\\ &= dim\frac{R(A^n)}{R(A_n + B_n)}\\ &= \beta(A_n + B_n)\\ &= \beta(A_n)\\ &= \beta_n(A).\end{aligned}$$

Consequently,

$$\beta_n^i(A) = \sum_{j=1}^{i} \beta_{n+j-1}(A) = \sum_{j=1}^{i} \beta_n(A) = i\beta_n(A) = i\beta(A + B).$$

(*iii*) Immediately consequence of Parts (*i*) and (*ii*).
□

The next corollary is a direct consequence of Theorem 4.7.

Corollary 4.8. *Let $A \in C(X)$ and let B be a nonzero bounded operator such that $AB^{-1} = B^{-1}A$.*

(*i*) *If $a(A) < \infty$, then there exists $\varepsilon > 0$ such that $A + B$ is bounded below, whenever $\|B\| < \varepsilon$. Moreover, $\beta(A + B) = \beta_n(A)$, for all $n \geq a(A)$.*
(*ii*) *If $d(A) < \infty$, then there exists $\varepsilon > 0$ such that $A + B$ is surjective, whenever $\|B\| < \varepsilon$. Moreover, $\alpha(A + B) = \alpha_n(A)$, for all $n \geq d(A)$.*
(*iii*) *If A has finite ascent and descent, then there exists $\varepsilon > 0$ such that $A + B$ is bijective, whenever $\|B\| < \varepsilon$.*

Corollary 4.9. *Let $A \in C(X)$ and let B a nonzero bounded operator such that $AB^{-1} = B^{-1}A$.*

(*i*) *If $A \in \xi_+(X)$ and $B \in \mathcal{P}(\Phi_+(X))$, then $A + B \in \Phi_+(X)$ and $\alpha_n^i(A) = i\alpha(A + B)$, for all $i \in \mathbb{N}$ and $n \geq p(A)$.*
(*ii*) *If $A \in \xi_-(X)$ and $B \in \mathcal{P}\Phi_-(X)$, then $A + B \in \Phi_-(X)$ and $\beta_n^i(A) = i\beta(A + B)$, for all $i \in \mathbb{N}$ and $n \geq q(A)$.*
(*iii*) *If $A \in \xi(X)$ and $B \in \mathcal{P}(\Phi(X))$, then $A + B \in \Phi(X)$ and $ind_n^i(A) = i[ind(A + B)]$, for all $i \in \mathbb{N}$ and $n \geq q(A)$.*

Proof. Suppose that $A \in \xi_+(X)$, then there exists, from Theorem 4.7, $\varepsilon_1 > 0$ such that $A + \lambda B \in \Phi_+(X)$, with $\alpha_n^i(A) = i\alpha(A + \lambda B)$, for all $0 < \lambda < \varepsilon_1$. It follows that $A + \mu B \in \Phi_+(X)$, for all $0 < \mu \leq 1$ (as $B \in \mathcal{P}(\Phi_+(X))$ and hence so is μB). Moreover, the function $f(\lambda) = \alpha(A + \lambda B)$ is locally constant in the compact connected set $[\varepsilon, 1]$, for all $0 < \varepsilon < 1$, and so it is constant. Thus $\alpha(A + \varepsilon B) = \alpha(A + B)$, for all $0 < \varepsilon < 1$. Let $0 < \lambda_1 < \varepsilon_1$, then $\alpha_n^i(A) = i\alpha(A + \lambda_1 B) = i\alpha(A + B)$. This prove (*i*). The proof of (*ii*) is in the same way as (*i*) and the part (*iii*) is a trivial consequence of Parts (*i*) and (*ii*) and the fact that $a_e(A) \leq d_e(A)$ when $a_e(A)$ and $d_e(A)$ are finite. □

Lemma 4.10. *Let $A, B \in LR(X)$ be everywhere defined such that $AB = BA$. Assume that B is a bounded operator and bijective. Then*

(*i*) $AB^{-1} = B^{-1}A$.
(*ii*) $(A + \lambda B)B = B(A - \lambda B)$, *for all* $\lambda \in \mathbb{C}$.
(*iii*) *If* $A \in \mathcal{C}(X)$, *then* $A + \lambda B \in \mathcal{C}(X)$, *for all* $\lambda \in \mathbb{C}$.

Proof. (*i*) First observe that AB^{-1} and $B^{-1}A$ are everywhere defined. Indeed, $D(AB^{-1}) = B(D(A)) = B(X) = R(B) = X$ (as B is surjective), and $D(B^{-1}A) = A^{-1}(D(B^{-1})) = A^{-1}(R(B)) = A^{-1}(X) = D(X) = X$. Now, let $x \in X$. Then, there exits $y \in X$ such that $y = Bx$ which equivalent to $x = B^{-1}y$. It follows that $AB^{-1}y = Ax$ and $B^{-1}Ay = B^{-1}ABx = B^{-1}BAx = Ax \cap D(B) + B^{-1}(0) = Ax = AB^{-1}y$. Consequently, $AB^{-1} = B^{-1}A$.
(*ii*) Follows immediately from [8, I.4.2 (d) and (e)].
(*iii*) Suppose that $A \in \mathcal{C}(X)$. Then, from [7, Lemma 2.2], $\mathcal{R}_c(A + \lambda B) = \{0\}$, and according to Lemma 4.4, $A + \lambda B$ is closed. The identity $D(A + \lambda B)^r + R(A + \lambda B)^s = X$ is trivial. Consequently $A + \lambda B \in \mathcal{C}(X)$.
□

Theorem 4.11. *Let $A \in \mathcal{C}(X)$ be regular and everywhere defined and let B be a bijective and bounded operator on X such that $AB = BA$.*

(*i*) *If* $a(A) < \infty$ *and* $B \in \mathcal{P}(\xi_+(X))$, *then* $a(A + B) < \infty$ *and* $\alpha_n^i(A + B) = \beta_n^i(A)$ *for all* $i \in \mathbb{N}$ *and* $n \geq max\{p(A), p(A + B)\}$.
(*ii*) *If* $d(A) < \infty$ *and* $B \in \mathcal{P}(\xi_-(X)$, *then* $d(A + B) < \infty$ *and* $\beta_n^i(A + B) = \beta_n^i(A)$ *for all* $i \in \mathbb{N}$ *and* $n \geq max\{q(A), q(A + B)\}$.
(*iii*) *If A has finite ascent and descent and* $B \in \mathcal{P}(\xi(X)$, *then $A + B$ has finite ascent and descent and* $ind_n^i(A + B) = ind_n^i(A)$ *for all* $i \in \mathbb{N}$ *and for some* $n \in \mathbb{N}$.

Proof. We only prove (*i*). The proof of (*ii*) may be achieved by the same reasoning as for (*i*) and the part (*iii*) is a direct consequence of Parts (*i*) and (*ii*). First, observe by Lemma 4.10, that $A + \lambda B \in \mathcal{C}(X)$, for all $\lambda \in \mathbb{C}$. Suppose that A has a finite ascent, then $a_e(A) < \infty$ (Lemma 2.9(*i*)), and hence $A + \lambda B \in \xi_+(X)$ for all $\lambda \in \mathbb{C}$. Moreover, from Lemma 4.10, $A + \lambda B \in \mathcal{C}$, for every $\lambda \in \mathbb{C}$. From ([7, Theorem 3.1]) together with Lemma 4.10, for $\lambda, \mu \in \mathbb{C}$ such that $|\lambda - \mu|$ is sufficiently small, we have that

$$R^\infty(A + \lambda B) = R^\infty(A - \mu B),\ \overline{N^\infty(A + \lambda B)} = \overline{N^\infty(A - \mu B)},\ \alpha_n(A + \lambda B) = \alpha_n(A + \mu B). \tag{13}$$

On the other hand, since $[0, 1]$ is compact, then $[0, 1] \subset \bigcup_{j=1}^{m} I_j$, for some $m \in \mathbb{N}$, where $I_j =]\lambda_j - \varepsilon_j, \lambda_j + \varepsilon_j[$ satisfying $I_j \cap I_{j+1} \neq \emptyset$ and ε_j is sufficiently small. If follows, from (13), that

$$R^\infty(A + \lambda_j B) = R^\infty(A),\ \overline{N^\infty(A + \lambda_j B)} = \overline{N^\infty(A)},\ \alpha_n(A + \lambda_j B) = \alpha_n(A),\ \text{for } 1 \leq j \leq m.$$

This means that

$$R^\infty(A + B) = R^\infty(A),\ \overline{N^\infty(A + B)} = \overline{N(A)},\ \alpha_n(A + B) = \alpha_n(A). \tag{14}$$

The formula (14) together with Lemma 4.1 lead to $R^\infty(A + B) \cap \overline{N^\infty(A + B)} = \{0\}$. Thus $a(A + B) < \infty$. Let $n \geq max\{p(A), p(A + B)\}$. Then $\alpha_k(A) = \alpha_n(A)$ and $\alpha_k(A + B) = \alpha_n(A + B)$ for all $k \geq n$. It follows that

$$\alpha_n^i(A + B) = \sum_{j=1}^{i} \alpha_{n+j-1}(A + B) = i\alpha_n(A + B) = i\alpha_n(A) = \sum_{j=1}^{i} \alpha_{n+j-1}(A) = \alpha_n^i(A).$$

□

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