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On diagonal and off-diagonal splitting-based iteration method to solve absolute value equations

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Abstract. Lately, M. Dehghan et al. suggested a two-step iterative method for solving linear problems based on diagonal splitting and off-diagonal splitting (DOS) [Filomat 31:5 (2017) 1441–1452]. In this study, a two-step nonlinear DOS-like iteration method for solving absolute value equations is presented based on the DOS technique. Two linear subsystems need to be solved using the diagonal and lower triangular coefficient matrices in every iteration of the proposed approach. The convergence characteristics of the nonlinear DOS-like iteration technique are investigated under certain circumstances. Several examples are given to demonstrate the method efficacy.

1. Introduction

Studying the algorithms of finding the solutions of linear and nonlinear matrix equations, coupled matrix equations and linear system has wide applications in many engineering fields. The recursive or iterative search schemes are often used for finding the solutions of linear and nonlinear matrix equations. Many related works involve these methods, e.g., closed-form solution of non-symmetric algebraic Riccati matrix equation [1], iterative Tikhonov regularization of tensor equations based on the Arnoldi process and some of its generalizations [2], Hermitian and Skew Hermitian splitting-like iteration approach for solving complex continuous time algebraic Riccati matrix equation [3], analysis of an iterative algorithm to solve the generalized coupled Sylvester matrix equations [4], single step iterative method for linear system of equations with complex symmetric positive semi-definite coefficient matrices [5], inexact low-rank Newton-ADI method for large-scale algebraic Riccati equations [6].

One of the nonlinear matrix equations is the absolute value equations abbreviated as AVEs, whose form is

$$Ax - |x| = b, \tag{1}$$

in which $|x| = (|x_1|, \cdots, |x_n|)^T$, $\quad A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. A more general form of the AVE (GAVE)

$$Ax - B|x| = b, \tag{2}$$

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where $B \in \mathbb{R}^{n\times n}$, was presented by Rohn [7] and was analyzed in a more general context in [8–13]. Numerous fields of engineering applications and scientific computing deal with the absolute value equations (1). One can reduce convex quadratic programming, bimatrix games, and linear programming to a LCP (linear complementarity problem), and LCP is formulated as an AVE [14]. Bai in [15] proposed the modulus-based matrix splitting (MMS) iterative method for solving LCP fast and economically. Actually, in the MMS iteration methods, AVE (2) is considered as a solution of LCP [16, 17]. Various form of MMS iteration methods have been developed to solve LCPs [18–24].

Recently, some techniques have been presented to solve absolute value equations. Mangasarian in [25] gave a direct GN (generalized Newton) technique for absolute value equation (1), that in case the singular values of A surpass 1, it becomes globally convergent. Then, a GN technique was presented by Hu et al. [9] to solve the AVEs related to second order cones. Also, they demonstrated that under appropriate assumptions, the suggested technique is locally quadratically and globally linearly convergent. By separation of the non-differential and differential parts of the generalized AVEs, a class of modified Newton-type techniques (MN) were suggested in [26] and a globally and quadratically convergent method was also presented in [27]. Some other forms of GN method were also proposed to solve AVE [28, 29].

The expensive costs of the GN technique are ascribable to the varied coefficient matrix in each GN iteration. Two CSCS-based iteration methods for solving absolute value equations is proposed in [30]. An iteration technique has been developed by Rohn et al. [8] for absolute value equation. Practically, their technique is reducible to the well-known Picard technique [31]. In accordance with Hermitian and skew-Hermitian splitting (HSS) of the matrix A, Bai et al. in [32] investigated the efficient HSS iterative technique to solve $Ax = b$. Bai and Yang developed the Picard-HSS iteration technique with regard to the HSS approach for solving weakly nonlinear systems [33]. Salkuyeh deliberate on the Picard-HSS approach to solve absolute value equations [31]. In real computations, it is not easy to determine the number of inner HSS iterative steps, because it depends on the problem. Also, the iteration vector is not updatable in a timely manner. To conquer these shortcomings, the nonlinear HSS-like iteration technique for solving AVEs has been presented in [34] on the basis of the nonlinear HSS-like iteration technique [33]. As a generalization of the nonlinear HSS-like iteration technique, Zhang [16] developed a relaxed nonlinear PHSS-like iterative technique for solving AVEs. Li [35], based on MHSS method, suggested a nonlinear MHSS-like iteration technique to find solution of a category of AVEs. Also, he suggested the convergence features of the nonlinear MHSS-like iteration technique via a smoothing approximate function. Some classical matrix-splitting iterative methods have been developed to solve AVE, see for example [10, 36–38]. In case $B = 0$ in (2), it is reduced to a linear equations system with various applications in scientific calculations [39–42]. Dehghan et al. [42] presented the DOS iterative method to solve $Ax = b$, which each iterate of it alternates between a diagonal matrix and a lower triangular matrix. Besides, the DOS method for specific values of the parameters w_1 and w_2 reduces to Jacobi, Gauss-seidel and SOR methods.
Given the above mentioned benefits of the DOS technique, this paper suggests a two-step nonlinear DOS-like iterative technique used to solve absolute value equations (1). If A represents a nonsingular matrix with nonvanishing diagonal entries, and $A = D + \mathfrak{L} + \mathfrak{U}$ represents the splitting of A, (1) can be restated as two systems of fixed point equations as follows

$$\begin{aligned} Dx &= [w_1 D + (w_1 - 1)\mathfrak{L} + (w_1 - 1)\mathfrak{U}]x + (1 - w_1)(|x| + b), \\ (D + w_2\mathfrak{L})x &= [(1 - w_2)D - w_2\mathfrak{U}]x + w_2(|x| + b), \end{aligned} \tag{3}$$

in which w_1 and w_2 are given constants and $D = diag(A)$, $\mathfrak{U}$ represents a general matrix, and $\mathfrak{L}$ represents a strictly lower triangular matrix. In addition, we investigate the convergence of method when A is a diagonally dominant matrix or an H-matrix. Some examples are given to demonstrate the effectiveness of the nonlinear DOS-like method.

The outline of the paper is structured as follows. Section 2 presents some preliminaries and notations. Through a smooth approximation of a nonlinear function, the nonlinear DOS-like iteration technique and its convergence analysis are specified in section 3. Numerical results presented in Section 4, and finally, the paper is concluded by presenting a number of concluding remarks.

2. Preliminaries

This section presents the lemmas, some necessary definitions, and a concise analysis of the DOS technique.

For convenience, in whole of the paper we consider notations as follows. Assume matrix $A = (\mathsf{a}_{i,j}) \in \mathbb{R}^{n\times n}$, $D = diag(A)$, $x \in \mathbb{R}^n$ and $|x| = (|x_i|)$, $U = triu(A,1)$ and $L = tril(A,-1)$. $A \geq 0$ if $\mathsf{a}_{i,j} \geq 0$ holds for all $i, j = 1(1)n$. $\rho(A)$ represents the spectral radius of A. $\|.\|$ is the infinity norm, $\|.\|_2$ is the spectral norm, where $\|A\|_2 = [\rho(A^H A)]^{1/2}$.

Lemma 2.1. *[14] The absolute value equation* (1) *can be uniquely solved for all $b \in \mathbb{R}^n$ if $\|A^{-1}\|_2 < 1$.*

Definition 2.2. *[43] Let us consider $\mathcal{F} : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n$, if an open neighborhood of $S \subset \mathbb{D}$ of $x^\star$ is considered so that for any $x^{(0)} \in S$, $\{x^{(\mathrm{k})}\}$ lie in $\mathbb{D}$ and converge to $x^\star$, then $x^\star$ is a point of attraction of the iteration*

$$x^{(k+1)} = \mathcal{F}x^{(k)},\ k = 0, 1, 2, \cdots, . \tag{4}$$

Lemma 2.3. *(Ostrowski Theorem 10.1.3 in [43]) Assume that $\mathcal{F} : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is F-differentiable at $x^\star$, in which $x^\star \in int(\mathbb{D})$ is a fixed point of $\mathcal{F}$. In case $\rho(\mathcal{F}'(x^\star)) < 1$, then $x^\star$ can be deemed as a point of attraction of the iteration* (4) *.*

The following lemma is a generalization of Lemma 3.2 in [35] and results in [27] to infinity norm.

Lemma 2.4. *Suppose $\vartheta : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is definable as*

$$\vartheta(x) = (\sqrt{x_1{}^2 + \epsilon^2}, \cdots, \sqrt{x_n{}^2 + \epsilon^2})^T, \quad \forall \epsilon > 0, \quad x = (x_1, ..., x_n) \in \mathbb{D} \tag{5}$$

then $\|\vartheta(x) - |x|\| \leq \epsilon$.

Proof. we start from the left hand of the equation as

$$\|\vartheta(x) - |x|\| = \underset{1\leq i\leq n}{max}| - \sqrt{x_i^2} + \sqrt{x_i^2 + \epsilon^2}| = \frac{\epsilon^2}{\sqrt{x_l^2} + \sqrt{x_l^2 + \epsilon^2}},\ 1 \leq l \leq n$$

and $\epsilon \leq \sqrt{x_l^2 + \epsilon^2} + \sqrt{x_l^2}$, so we have $\frac{\epsilon^2}{\sqrt{x_l^2+\epsilon^2}+\sqrt{x_l^2}} \leq \epsilon$, which completes the proof. □

In accordance with the preceding lemma, one can present the following properties of $\vartheta(x)$.

Lemma 2.5. *Let $\lambda = |\frac{x_s+\tilde{x}_s}{\sqrt{x_s^2+\epsilon^2}+\sqrt{\tilde{x}_s^2+\epsilon^2}}|, 1 \leq s \leq n$, then*

$$\|\vartheta(x) - \vartheta(\tilde{x})\| \leq \lambda\|x - \tilde{x}\|, \tag{6}$$

where $|\sqrt{x_s^2 + \epsilon^2} - \sqrt{\tilde{x}_s^2 + \epsilon^2}| = \underset{1\leq i\leq n}{max}|\sqrt{x_i^2 + \epsilon^2} - \sqrt{\tilde{x}_i^2 + \epsilon^2}|$.

Proof. we have

$$\begin{aligned}\|\vartheta(x) - \vartheta(\tilde{x})\| &= |\sqrt{x_s^2 + \epsilon^2} - \sqrt{\tilde{x}_s^2 + \epsilon^2}| = |\frac{(x_s + \tilde{x}_s)(x_s - \tilde{x}_s)}{\sqrt{x_s^2 + \epsilon^2} + \sqrt{\tilde{x}_s^2 + \epsilon^2}}| \\ &\leq \lambda|x_s - \tilde{x}_s| \leq \lambda \underset{1\leq i\leq n}{max}|x_i - \tilde{x}_i|,\end{aligned}$$

where $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$ and x_s is the sth component of x. □

Lemma 2.6. *[35] The Jacobian of $\vartheta(x)$ at $x \in \mathbb{R}^n$ is described as*

$$\vartheta'(x) = diag(\frac{x_i}{\sqrt{x_i{}^2 + \epsilon^2}}), i = 1(1)n,\ \forall \epsilon > 0. \tag{7}$$

Lemma 2.7. *[44] Suppose that $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^n$, then $\||x_1| - |x_2|\| \leq \|x_1 - x_2\|$.*

Lemma 2.8. *Suppose (1) is solvable. Then it has unique solution when $\|A^{-1}\| < 1$.*

Proof. Let us suppose $y^\star$ and $\mathcal{X}^\star$ as two different solutions of (1). If $\|A^{-1}\| < 1$, thus

$$\|y^\star - \mathcal{X}^\star\| \leq \|A^{-1}\|\||y^\star| - |\mathcal{X}^\star|\| < \|y^\star - \mathcal{X}^\star\|.$$

Therefore, $y^\star = \mathcal{X}^\star$. □

Definition 2.9. *The comparison matrix of $A = (\mathsf{a}_{ij}) \in \mathbb{R}^{n\times n}$ presented by $\langle A\rangle = (\iota_{ij}) \in \mathbb{R}^{n\times n}$ is described by* $\begin{cases} \iota_{ij} = |\mathsf{a}_{i,j}|, & j = i, \\ \iota_{ij} = -|\mathsf{a}_{i,j}|, & j \neq i. \end{cases}$

Definition 2.10. *The matrix $A \in \mathbb{R}^{n\times n}$ is named an $\mathcal{M}$-matrix if $\langle A\rangle = A$ and a vector $\mathbf{u} > 0$ exists with $A\mathbf{u} > 0$.*

Definition 2.11. *If the comparison matrix of the matrix $A \in \mathbb{R}^{n\times n}$ is an $\mathcal{M}$-matrix, then it is named an $\mathcal{H}$-matrix.*

Definition 2.12. *[45] One can call the matrix $A \in \mathbb{R}^{n\times n}$ strictly generalized diagonally dominant, if there is an entrywise positive vector $\mathbf{u} = (\mathsf{u}_k) \in \mathbb{R}^n$ in which*

$$|\mathsf{a}_{ii}|\mathsf{u}_i > \sum_{\substack{k=0 \\ k\neq i}}^{n} |\mathsf{a}_{ik}|\mathsf{u}_k, i = 1(1)n. \tag{8}$$

Lemma 2.13. *[45] Suppose that $A = (\mathsf{a}_{ij}) \in \mathbb{R}^{n\times n}$. Then, the statements that follow are equivalent.*
(i) $\langle A\rangle^{-1} \geq 0$.
(ii) $\langle A\rangle$ is a nonsingular $\mathcal{M}$-matrix.
(iii) A is a strictly generalized diagonally dominant matrix.
(iv) A is an $\mathcal{H}$-matrix.
(v) There is a vector $\mathbf{u} \in \mathbb{R}^n$ with $\mathbf{u} > 0$ so that $\langle A\rangle\mathbf{u} > 0$. Similarly, by supposing $\mathcal{D} = diag(\mathbf{u})$, $A\mathcal{D}$ will be strictly diagonally dominant.

Lemma 2.14. *Consider a generalized strictly diagonally dominant matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, then, a positive vector $\hat{\mathbf{u}} = (\hat{\mathsf{u}}_i) > 0$ exists such that $0 < \hat{\mathsf{u}}_i < 1, i = 1(1)n$ and $\langle A\rangle\hat{\mathbf{u}} > 0$.*

Proof. Since matrix A is a generalized strictly diagonally dominant matrix, thus a positive vector $\mathbf{u}$ exists in which (8) holds, so

$$|\mathsf{a}_{ii}|\frac{\mathsf{u}_i}{\mathsf{u}_l + 1} > \sum_{\substack{k=0 \\ k\neq i}}^{n} |a_{ik}|\frac{\mathsf{u}_k}{\mathsf{u}_l+1}, i = 1(1)n,$$

where $\mathsf{u}_l = \max_{1\leq i\leq n}|\mathsf{u}_i|$. We define $\hat{\mathsf{u}}_i = \frac{\mathsf{u}_i}{\mathsf{u}_l+1}$, so $0 < \hat{\mathsf{u}}_i < 1$, $i = 1(1)n$, therefore,

$$|\mathsf{a}_{ii}|\hat{\mathsf{u}}_i > \sum_{\substack{k=0 \\ k\neq i}}^{n} |a_{ik}|\hat{\mathsf{u}}_k, i = 1(1)n.$$

□

The DOS iterative method. Assume that matrix A is a nonsingular with nonvanishing diagonal entries, $A = D + \mathfrak{L} + \mathfrak{U}$ and $x^{(0)} \in \mathbb{R}^n$ is an arbitrary primary guess. For $k = 0, 1, 2, \cdots$ until $\{x^{(k)}\}$ converges, calculate

$$\begin{cases} Dx^{(k+\frac{1}{2})} = [w_1 D + (w_1 - 1)\mathfrak{L} + (w_1 - 1)\mathfrak{U}]x^{(k)} + (1 - w_1)b, \\ (D + w_2\mathfrak{L})x^{(k+1)} = [D - w_2 D - w_2\mathfrak{U}]x^{(k+\frac{1}{2})} + w_2 b, \end{cases}$$

the symbols are the same as those in (3). The DOS method solves a system in the first half iteration with a diagonal matrix and in the second system with a triangular coefficient matrix.

It should be noted that when $\mathfrak{U} = triu(A, 1)$, the DOS method reduces to the well-known methods for specific values of the parameters w_1 and w_2. For instance, with $w_1 = w_2 = 0$, the DOS method will be Jacobi method, with $w_1 = w_2 = 1$ will be Gauss-Seidel method, with $w_1 = 1 - w_1$ and $w_2 = 0$ will be the Simultaneous Overrelaxation method and for $w_1 = 1$ and free w_2 will be SOR method [42].

Lemma 2.15. *[42] Suppose that $A = (\mathsf{a}_{ij}) \in \mathbb{R}^{n\times n}$ is a diagonally dominant matrix and $\sum_{j=2}^{n} |\mathsf{a}_{ij}| < |\mathsf{a}_{11}|$, if $\mathfrak{L} = (\mathsf{l}_{ij})$, $\mathfrak{U} = (u_{ij})$ and $\mathsf{l}_{ij} u_{ij} \geq 0$, $0 \leq w_1 \leq 1$ and $0 < w_2 \leq 1$, then the sequence of DOS iteration method is convergent to $x^\star \in \mathbb{R}^n$ for the whole primary guesses, in which $x^\star \in \mathbb{R}^n$ is the unique solution of $Ax = b$.*

Corollary 2.16. *[42] Consider a strictly diagonally dominant matrix $A \in \mathbb{R}^{n\times n}$, thus the DOS iteration converges for $0 \leq w_1 \leq 1$, $0 < w_2 \leq 1$ and appropriate choices of matrices $\mathfrak{L}$ and $\mathfrak{U}$ that satisfy the Lemma 2.15 requirements.*

In accordance with the statements discussed above, to construct a general matrix $\mathfrak{U}$ and a strictly lower triangular matrix $\mathfrak{L}$ satisfying the requirements of Lemma 2.15, we set

$$\begin{cases} \mathsf{l}_{ij} = 0, \; u_{ij} = \mathsf{a}_{ij}, \; \; i \leq j, \\ \mathsf{l}_{ij} + u_{ij} = \mathsf{a}_{ij}, \mathsf{l}_{ij} u_{ij} \geq 0, \; \; i > j, \end{cases}$$

in which one of the simplest choices is $\mathsf{l}_{ij} = u_{ij} = \dfrac{\mathsf{a}_{ij}}{2}$.

3. The nonlinear DOS-like iterative method

This section, via the system of nonlinear fixed-point equations (3), gives the nonlinear DOS-like iteration method to solve AVE. Also, we take advantage of the techniques in [33] to analyze the convergence of the introduced method.

Supposing $x^{(0)} \in \mathbb{R}^n$ as an arbitrary primary guess, we calculate $x^{(k+1)}$ for $k = 0, 1, \cdots$, according to the procedure

$$\begin{cases} D(x^{(k+\frac{1}{2})}) = [w_1 D + (w_1 - 1)\mathfrak{L} + (w_1 - 1)\mathfrak{U}]x^{(k)} + (1 - w_1)|x^{(k)}| + (1 - w_1)b, \\ (D + w_2\mathfrak{L})x^{(k+1)} = [D - w_2 D - w_2\mathfrak{U}]x^{(k+\frac{1}{2})} + w_2(|x^{(k+\frac{1}{2})}| + b). \end{cases} \tag{9}$$

At each iteration, we solve two subsystems with coefficient matrices D and $(D + w_2\mathfrak{L})$ exactly. The first linear subsystem with the diagonal coefficient matrix D is solved simply. For the second subsystem with the lower triangular coefficients matrix $(D + w_2\mathfrak{L})$, we can employ the forward substitution method [42, 46]. Note that the scheme (9) can be reformulated as

$$x^{(\mathrm{j}+1)} = \Gamma(x^{(\mathrm{j})}), \; \mathrm{j} = 0, 1, 2, \cdots. \tag{10}$$

Where

$$\Gamma(x) = \mathcal{V} o \mathcal{U}(x) = \mathcal{V}(\mathcal{U}(x)),$$

and

$$\begin{cases} \mathcal{U}(x) = D^{-1}([w_1 D + (w_1 - 1)\mathfrak{L} + (w_1 - 1)\mathfrak{U}]x + (1 - w_1)(|x| + b)), \\ \mathcal{V}(x) = (D + w_2\mathfrak{L})^{-1}([(1 - w_2)D - w_2\mathfrak{U}]x + w_2(|x| + b)). \end{cases} \tag{11}$$

In order to evaluate the convergence analysis of iteration (10), we indirectly use Lemma 2.3. Since the term $|x| + b$ in (10) is non-differentiable, we substitute $|x|$ with the smoothing approximate function $\vartheta(x)$, defined in

[27] and according to Lemma 2.4. Then, we will have the following smoothing nonlinear DOS-like iteration scheme

$$\tilde{x}^{(k+1)} = \tilde{\Gamma}(\tilde{x}^{(k)}),\ k = 0, 1, 2, ..., \tag{12}$$

in which

$$\tilde{\Gamma}(x) = \tilde{\mathcal{V}} o \tilde{\mathcal{U}} = \tilde{\mathcal{V}}(\tilde{\mathcal{U}}(x)),$$

and

$$\begin{cases} \tilde{\mathcal{U}}(x) = D^{-1}([w_1 D + (w_1 - 1)\mathfrak{L} + (w_1 - 1)\mathfrak{U}]x + (1 - w_1)(\vartheta(x) + b)), \\ \tilde{\mathcal{V}}(x) = (D + w_2\mathfrak{L})^{-1}([(1 - w_2)D - w_2\mathfrak{U}]x + w_2(\vartheta(x) + b)). \end{cases} \tag{13}$$

Now, via Lemma 2.3 and the next theorems, we give the convergence of the nonlinear DOS-like scheme. It should be noted that the following theorems and their proofs are generalization of the Theorem 4.1, Corollary 4.2 in [33] and Theorems 3.2, 3.3 and 3.4 in [35] to nonlinear DOS-like method.

Theorem 3.1. *Suppose that matrix A satisfy in the conditions of Lemma 2.15 and $\vartheta(x)$ is F-differentiable in $x^\star \in \mathbb{D}$ with $Ax^\star = \vartheta(x^\star) + b$, thus $x^\star$ is a point of attraction of iteration* (12), *if*

$$\varrho_{w_2} < \sqrt{2 - \Theta_{w_1,w_2}} - 1, \tag{14}$$

where $0 \le w_1 \le 1$, $0 < w_2 \le 1$, $\varrho_{w_2} = max\{\|D^{-1}\|, \|(D + w_2\mathfrak{L})^{-1}\|\}$, $\Theta_{w_1,w_2} = \|M_{w_1,w_2}\|$, and $M_{w_1,w_2} = (D + w_2\mathfrak{L})^{-1}[(1 - w_2)D - w_2\mathfrak{U}]D^{-1}[w_1 D + (w_1 - 1)\mathfrak{L} + (w_1 - 1)\mathfrak{U}]$.

Proof. Since

$$\begin{aligned} \tilde{\mathcal{V}}'(x^\star) =& (D + w_2\mathfrak{L})^{-1}([(1 - w_2)D - w_2\mathfrak{U}] + w_2\vartheta'(x^\star)), \\ \tilde{\mathcal{U}}'(x^\star) =& D^{-1}([w_1 D + (w_1 - 1)\mathfrak{L} + (w_1 - 1)\mathfrak{U}] + (1 - w_1)\vartheta'(x^\star)), \textit{and} \\ \tilde{\Gamma}'(x^\star) =& \tilde{\mathcal{V}}'(\tilde{\mathcal{U}}(x^\star))(\tilde{\mathcal{U}}'(x^\star)) = \tilde{\mathcal{V}}'(x^\star)\tilde{\mathcal{U}}'(x^\star). \end{aligned}$$

So

$$\begin{aligned} \|\tilde{\Gamma}'(x^\star)\| \le& \|M_{w_1,w_2}\| + \|(D + w_2\mathfrak{L})^{-1}w_2\vartheta'(x^\star)(1 - w_1)D^{-1}\vartheta'(x^\star)\| \\ &+ \|(D + w_2\mathfrak{L})^{-1}[(1 - w_2)D - w_2\mathfrak{U}](1 - w_1)D^{-1}\vartheta'(x^\star)\| \\ &+ \|(D + w_2\mathfrak{L})^{-1}w_2\vartheta'(x^\star)D^{-1}[w_1\mathsf{D} + (w_1 - 1)\mathfrak{L} + (w_1 - 1)\mathfrak{U}]\| \\ \le& \|M_{w_1,w_2}\| + (w_2 - w_1 w_2)\|D^{-1}\|\|(D + w_2\mathfrak{L})^{-1}\|\|\vartheta'(x^\star)\|^2 \\ &+ (1 - w_1)\|D^{-1}\|\|[D - w_2 D - w_2\mathfrak{U}](D + w_2\mathfrak{L})^{-1}\|\|\vartheta'(x^\star)\| \\ &+ w_2\|\vartheta'(x^\star)\|\|(D + w_2\mathfrak{L})^{-1}\|\|D^{-1}[w_1 D + (w_1 - 1)\mathfrak{L} + (w_1 - 1)\mathfrak{U}]\|. \end{aligned}$$

By considering Lemma 2.6, we get $\|\vartheta'(x)\| < 1$. Now assume that

$$L_{w_1} = D^{-1}[w_1 D + (w_1 - 1)\mathfrak{L} + (w_1 - 1)\mathfrak{U}],\ L_{w_2} = (D + w_2\mathfrak{L})^{-1}[(1 - w_2)\mathsf{D} - w_2\mathfrak{U}], \tag{15}$$

we acquire $\|L_{w_1}\| \le 1$, $\|L_{w_2}\| < 1$ and $\Theta_{w_1,w_2} = \|M_{w_1,w_2}\| < 1$ from proof of Theorem 2.1 in [42], so

$$\|\tilde{\Gamma}'(x^\star)\| \le \Theta_{w_1,w_2} + w_2(1 - w_1)\varrho_{w_2}^2 + (w_2 + 1 - w_1)\varrho_{w_2} \le \Theta_{\mathsf{w}_1,\mathsf{w}_2} + \varrho_{w_2}^2 + 2\varrho_{w_2} \tag{16}$$

Therefore $\rho(\tilde{\Gamma}'(x^\star)) < 1$ and based on Ostrowski theorem, the proof is completed. □

Theorem 3.2. *Suppose that the Lemma 2.15 conditions are satisfied. The iterative sequence $\{x^{(k)}\}_{k=0}^{\infty}$ of the nonlinear DOS-like technique* (10), *can be approximated via* (12) *for the whole primary guesses $x^{(0)} \in \mathbb{R}^n$ and any arbitrary $\xi > 0$, provided that*

$$\epsilon < \frac{\xi(1 - (\Theta_{w_1,w_2} + 2\varrho_{w_2} + \varrho_{w_2}^2))}{1 - \Theta_{w_1,w_2}}, \tag{17}$$

where $0 \le w_1 \le 1$, $0 < w_2 \le 1$ and $\varrho_{w_2} < \sqrt{2 - \Theta_{w_1,w_2}} - 1$.

Proof. By using the definitions of $\Gamma(x^{(j)})$ and $\tilde{\Gamma}(\tilde{x}^{(j)})$ in (10) and (12), respectively, gets

$$\|x^{(k+1)} - \tilde{x}^{(k+1)}\| \leq \|\Gamma(x^{(k)}) - \tilde{\Gamma}(x^{(k)})\| + \|\tilde{\Gamma}(x^{(k)}) - \tilde{\Gamma}(\tilde{x}^{(k)})\|, \tag{18}$$

hence

$$\begin{aligned}\|\Gamma(x^{(k)}) - \tilde{\Gamma}(x^{(k)})\| \leq & \|(D + w_2\mathfrak{L})^{-1}([(1 - w_2)D - w_2\mathfrak{U}]D^{-1}(1 - w_1)(\vartheta(x^{(k)}) - |x^{(k)}|))\| \\ & + \|(D + w_2\mathfrak{L})^{-1}w_2(\vartheta(\tilde{\mathcal{U}}(x^{(k)})) - |\mathcal{U}(x^{(k)})| + |\tilde{\mathcal{U}}(x^{(k)})| - |\tilde{\mathcal{U}}(x^{(k)})|)\| \\ & \leq \|(D + w_2\mathfrak{L})^{-1}[(1 - w_2)D - w_2\mathfrak{U}]D^{-1}\|(1 - w_1)\|(\vartheta(x^{(k)}) - |x^{(k)}|)\| \\ & + w_2\|(D + w_2\mathfrak{L})^{-1}\|\,\|\,|\tilde{\mathcal{U}}(x^{(k)})| - \vartheta(\tilde{\mathcal{U}}(x^{(k)}))\| \\ & + w_2\|(D + w_2\mathfrak{L})^{-1}\|\,\|\,|\mathcal{U}(x^{(k)})| - |\tilde{\mathcal{U}}(x^{(k)})|\|\| \\ & \leq \|(D + w_2\mathfrak{L})^{-1}[(1 - w_2)D - w_2\mathfrak{U}]\|(1 - w_1)\|D^{-1}\|\|\|(\vartheta(x^{(k)}) - |x^{(k)}|)\| \\ & + w_2\|(D + w_2\mathfrak{L})^{-1}\|\,\|\|\,\tilde{\mathcal{U}}(x^{(k)})| - \vartheta(\tilde{\mathcal{U}}(x^{(k)}))\| \\ & + w_2\|(D + w_2\mathfrak{L})^{-1}\|\,\|D^{-1}\|(1 - w_1)\|\vartheta(x^{(k)}) - |x^{(k)}|\|\|.\end{aligned}$$

According to Lemma 2.4, considering $\|L_{w_2}\| < 1$ and the definition of ϱ_{w_2}, we get

$$\begin{aligned}\|\Gamma(x^{(k)}) - \tilde{\Gamma}(x^{(k)})\| & \leq \epsilon(\|D^{-1}\| - w_1\|D^{-1}\| + w_2\|(D + w_2\mathfrak{L})^{-1}\|(\|D^{-1}\| - w_1\|D^{-1}\| + 1)) \\ & \leq \epsilon(\varrho_{w_2}(w_2 + 1 - w_1) - \varrho_{w_2}^2(w_1 - 1)w_2),\end{aligned}$$

so by (16) and (14), we conclude

$$\|\Gamma(x^{(k)}) - \tilde{\Gamma}(x^{(k)})\| < \epsilon(1 - \Theta_{w_1,w_2}). \tag{19}$$

Moreover, according to Lemma 2.5 and considering $\lambda < 1$, we obtain

$$\begin{aligned}\|\tilde{\Gamma}(x^{(k)}) - \tilde{\Gamma}(\tilde{x}^{(k)})\| \leq & \\ & \|(D + w_2\mathfrak{L})^{-1}[(1 - w_2)D - w_2\mathfrak{U}]D^{-1}[w_1 D + (w_1 - 1)(\mathfrak{L} + \mathfrak{U})]\|\|\|x^{(k)} - \tilde{x}^{(k)}\| \\ & + \|(D + w_2\mathfrak{L})^{-1}([(1 - w_2)D - w_2\mathfrak{U}]D^{-1}(1 - w_1)\|\|\|\vartheta(x^{(k)}) - \vartheta(\tilde{x}^{(k)})\| \\ & + \|(D + w_2\mathfrak{L})^{-1}w_2\|\|\|\vartheta(\tilde{\mathcal{U}}(x^{(k)})) - \vartheta(\tilde{\mathcal{U}}(\tilde{x}^{(k)})\| \\ & \leq \Theta_{w_1,w_2}\|x^{(k)} - \tilde{x}^{(k)}\| + \|L_{w_2}\|\varrho_{w_2}(\lambda - \lambda w_1)\|x^{(k)} - \tilde{x}^{(k)}\| \\ & + \varrho_{w_2}\lambda w_2\|x^{(k)} - \tilde{x}^{(k)}\|\|\|L_{w_1}\| + \varrho_{w_2}{}^2\lambda^2 w_2 - \varrho_{w_2}{}^2\lambda^2 w_2 w_1\|x^{(k)} - \tilde{x}^{(k)}\| \\ \leq \ & (\Theta_{w_1,w_2} + 2\varrho_{w_2} + \varrho_{w_2}{}^2)\|x^{(k)} - \tilde{x}^{(k)}\|.\end{aligned} \tag{20}$$

Therefore, from the fact that $x^{(0)} = \tilde{x}^{(0)}$ and from (17), (18), (20), and using the mathematical induction, it follows that

$$\begin{aligned}\|x^{(k+1)} - \tilde{x}^{(k+1)}\| & \leq \epsilon(1 - \Theta_{w_1,w_2}) + (\Theta_{w_1,w_2} + 2\varrho_{w_2} + \varrho_{w_2}{}^2)\|x^{(k)} - \tilde{x}^{(k)}\| \\ & \leq \epsilon(1 - \Theta_{w_1,w_2})(1 + (\Theta_{w_1,\mathrm{w}_2} + 2\varrho_{w_2} + \varrho_{w_2}{}^2) + \cdots + (\Theta_{\mathrm{w}_1,\mathrm{w}_2} + 2\varrho_{w_2} + \varrho_{w_2}{}^2)^k) \\ & = \epsilon(1 - \Theta_{w_1,w_2})\frac{1 - (\Theta_{w_1,w_2} + 2\varrho_{w_2} + \varrho_{w_2}{}^2)^{k+1}}{1 - (\Theta_{w_1,w_2} + 2\varrho_{w_2} + \varrho_{w_2}^2)} \\ & \leq \frac{\epsilon(1 - \Theta_{w_1,w_2})}{1 - (\Theta_{w_1,w_2} + 2\varrho_{w_2} + \varrho_{w_2}{}^2)} < \xi.\end{aligned}$$

□

Theorem 3.3. *Let AVE* (1) *be solvable,* $\|A^{-1}\| < 1$ *and conditions of Theorem 3.1 and 3.2 are satisfied. Then* (1) *possess a unique solution* $x^\star$ *and the iterative sequence* $\{x^{(k)}\}_{k=0}^{\infty}$ *created via the nonlinear DOS-like iteration technique* (9)*, for any desirable initial point* $x^{(0)}$*, converges to a solution of* (1).

Proof. Since $\|A^{-1}\| < 1$, according to lemma 2.8, the equation (1) has a unique solution. Besides

$$\|x^{(k+1)} - x^{\star}\| \leq |\tilde{x}^{(k+1)} - x^{\star}\| + \|x^{(k+1)} - \tilde{x}^{(k+1)}\|.$$

According to Theorem 3.1 and Definition 2.2 for any $\xi_1 > 0$ we get $\|\tilde{x}^{(k+1)}) - x^{\star}\| < \xi_1$. In addition, it is obvious that according to the previous theorem, for all $\xi_2 > 0$ we have $\|x^{(k+1)} - x^{\star}\| < \xi_2$ where $\xi + \xi_1 < \xi_2$. □

In the following part, the nonlinear DOS-like iterative method for AVE with $\mathcal{H}$-matrix coefficient is proposed.

As a consequence of Lemma 2.13, A is an $\mathcal{H}$-matrix if and only if there is a matrix $\mathcal{D} = diag(\mathsf{u}_1, \mathsf{u}_2, ..., \mathsf{u}_n)$ so that $A\mathcal{D}$ is strictly diagonally dominant. This enables us to apply matrix $\mathcal{D}$ as a right preconditioner in (1) and solve both $A\mathcal{D}x - |\mathcal{D}x| = \mathsf{b}$ and $\mathcal{D}x = \mathsf{y}$ instead of only (1). This approach needs some assumptions, which are as follows. Let $\mathfrak{A} = D + \mathfrak{L} + \mathfrak{U}$ where $\mathfrak{A} = A\mathcal{D}$ and

$$\begin{cases} \mathcal{U}(x) = D^{-1}([w_1 D + w_1\mathfrak{L} - \mathfrak{L} + (w_1 - 1)\mathfrak{U}]x + (1 - w_1)(|\mathcal{D}x| + b)), \\ \mathcal{V}(x) = (D + w_2\mathfrak{L})^{-1}([(1 - w_2)D - w_2\mathfrak{U}]x + w_2(|\mathcal{D}x| + b)), \end{cases}$$

besides

$$\Gamma(x) = \mathcal{V}(\mathcal{U}(x)).$$

We define $\varphi : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ with

$$\varphi(x) = (\sqrt{\mathsf{u}_1{}^2 x_1{}^2 + \epsilon^2}, \cdots, \sqrt{\mathsf{u}_n{}^2 x_n{}^2 + \epsilon^2})^T,$$

where $\epsilon > 0$ and $x \in \mathbb{D}$. Now we have $\varphi'(x) = diag(\frac{\mathsf{u}_i{}^2 x_i}{\sqrt{\mathsf{u}_i{}^2 x_i{}^2 + \epsilon^2}}), i = 1, 2, ..., n$. Furthermore by using Lemma 2.14, we can conclude that $\|\varphi'(x)\| < 1$. Finally, by similar definition for smoothing nonlinear DOS-like method (12), i.e.
$\tilde{x}^{(j+1)} = \tilde{\Gamma}(\tilde{x}^{(j)}),\ j = 0, 1, 2, ...,$ where $\tilde{\Gamma}(x) = \tilde{\mathcal{V}}(\tilde{\mathcal{U}}(x))$,

$$\begin{cases} \tilde{\mathcal{U}}(x) = D^{-1}([w_1 D + (w_1\mathfrak{L} - \mathfrak{L}) + (w_1 - 1)\mathfrak{U}]x + (1 - w_1)(\varphi(x) + b)), \\ \tilde{\mathcal{V}}(x) = (D + w_2\mathfrak{L})^{-1}([(1 - w_2)D - w_2\mathfrak{U}]x + w_2(\varphi(x) + b)), \end{cases} \tag{21}$$

we conclude the next theorem.

Theorem 3.4. *Assume that equation* (1) *be uniquely solvable, matrix A be an $\mathcal{H}$-matrix, and $\varphi(x)$ be F-differentiable in $x^{\star} \in \mathbb{D}$ with $\mathfrak{A}x^{\star} = \varphi(x^{\star}) + b$. The iteration sequance $\{x^{(k)}\}_{k=0}^{\infty}$ defined by the process* (9), *with any initial guess $x^{(0)} \in \mathbb{R}^n$, converges to $x^{\star}$ when*

$$\varrho_{w_2} < \sqrt{2 - \Theta_{w_1,w_2}} - 1,$$

and

$$\epsilon < \frac{\xi(1 - (\Theta_{w_1,w_2} + 2\varrho_{w_2} + \varrho_{w_2}{}^2))}{1 - \Theta_{w_1,w_2}},$$

where $\varrho_{w_2} = max\{\|D^{-1}\|, \|(D + w_2\mathfrak{L})^{-1}\|\}, \Theta_{w_1,w_2} = \|M_{w_1,w_2}\|$, $0 \leq w_1 \leq 1$ and $0 < w_2 \leq 1$.

Proof. A is an $\mathcal{H}$-matrix, therefore there exists a matrix $\mathcal{D}$ so that $\mathfrak{A} = A\mathcal{D}$ where $\mathfrak{A}$ is strictly diagonally dominant matrix. Therefore

$$\begin{aligned} \|\tilde{\Gamma}'(x^{\star})\| \leq\ & \|M_{w_1,w_2}\| + \|(D + w_2\mathfrak{L})^{-1}\|\|D^{-1}\|\|\varphi'(x^{\star})\|^2 \\ & + \|(D + w_2\mathfrak{L})^{-1}[(1 - w_2)D - w_2\mathfrak{U}]\|\|D^{-1}\|\|\varphi'(x^{\star})\| \\ & + \|(D + w_2\mathfrak{L})^{-1}\|\|\varphi'(x^{\star})\|\|D^{-1}[w_1 D + (w_1 - 1)\mathfrak{L} + (w_1 - 1)\mathfrak{U}]\|, \end{aligned}$$

where the symbols are as defined in Theorem 3.1. According to Lemma 2.16, we have $\|L_{w_1}\| \leq 1$, $\|L_{w_2}\| < 1$ and $\Theta_{w_1,w_2} < 1$. Since the rest of the proof is the same as Theorem 3.3, rewriting it would be redundant. □

It should be noted that according to Theorems 2.1 and 2.2 in [47], for every nonsingular matrix A there exists nonsingular matrices P and Q such that PAQ is strictly diagonally dominant matrix. Hence, for general nonsingular matrices, one can use preconditioned techniques [48].

4. Numerical experiments

This section provides several examples to express the efficiency of DOS-like iterative technique to solve system (1). We examine the numerical properties of nonlinear DOS-like, nonlinear HSS-like [34], relaxed nonlinear PHSS-like (RPHSS) [16], generalization Gauss-Seidel (GGS) [38], AOR [10], modified Newton-type (MN)[26] and SSOR iterative methods using some test problems in terms of processing time and iteration number for different problem sizes n.

In numerical examples, we use SSOR techniques to solve AVEs, which is based on the SSOR method proposed in [22, 49, 50]. Let matrix $A = D + L + U \in \mathbb{R}^{n\times n}$ and $w \in (0, 2)$, we have

$$\begin{aligned} (D + wL)x^{(k+\frac{1}{2})} &= [-w(D + U) + D]x^{(k)} + w|x^{(k)}| + wb, \\ (D + wU)x^{(k+1)} &= [-w(D + L) + D]x^{(k+\frac{1}{2})} + w|x^{(k+\frac{1}{2})}| + wb. \end{aligned} \tag{22}$$

The SSOR method requires solving two half subsystems in each iteration with triangular coefficient matrices.

All the numerical experiments have been carried out using MATLAB 2017 (64-bit) on a system with Intel® Core^{TM} i5-10210U processor @ 1.60GHz 2.11 GHz, 8GB RAM, running Windows 10. The process is run by using the zero vector as an initial guess, while we consider the maximum number of iterations to 500 or

$$\frac{\|Ax^{(k)} - |x^{(k)}| - b\|}{\|b\|} \le 10^{-7},$$

as a stopping criterion. Results are presented for three examples in four tables. For all examples, we choose $\Omega = wI$ in the MN iteration technique and $\mathfrak{U} = U$. In the tables, we denote the iterations number and CPU time by Iter and CPU, respectively and RES is the norm of absolute residual. Given the fact that finding the optimal parameters is often dependent on the problem and it is not easily determined, the optimal parameters are decided experimentally and denoted by w_{1exp}, w_{2exp}, w_{exp}, r_{exp} and α_{exp} in tables. CPU time in the methods that fails to converge in 500 iterations is listed by "fail". The Cholesky factorization and LU factorization are used to solve subsystems of the nonlinear HSS-like and RPHSS methods. Subsystems with the diagonal matrix are solved exactly, and subsystems with the triangular coefficient matrices $(D + wL)$ and $(D + wU)$ are solved using the substitution methods.

Example 4.1. *[18] Consider the AVE* (1)*, in which* $A = tridiag(-1.5I, S, -0.5I) + 4I \in \mathbb{R}^{n\times n}$, $b = -Az^* \in \mathbb{R}^n$, $S = tridiag(-1.5, 4, -0.5) \in \mathbb{R}^{m\times m}$, $n = m^2$ *and* $z^* = (1, 2, 1, 2, ..., 1, 2, ...)^T \in \mathbb{R}^n$.

Table 1 presents the numerical results of Example 4.1 for the different values of n. It is easy to see that all of the mentioned methods in this table are convergent. It should be noted that in this example $\Theta_{\mathsf{w}_1,\mathsf{w}_2} + \varrho^2_{w_2} + 2\varrho_{w_2} = 0.5722$ *and* $\|A^{-1}\| = 0.25$.

Example 4.2. *[16, 31, 34] Consider*

$$\begin{cases} -(\nu_{xx} + \nu_{yy}) + (\nu_x + \nu_y) + 2\nu = f(x, y), & (x, y) \in \omega, \\ \nu(x, y) = 0, & (x, y) \in \partial\omega, \end{cases}$$

where $\mathsf{p} \in \mathbb{R}$, q *is a positive constant,* $\omega = (0, 1) \times (0, 1)$ *and* $\partial\omega$ *is boundary of* ω*. The central difference scheme to the convective terms and the five-point finite difference scheme to the diffusive terms are used to acquire the linear equations system* $Ax = \mathsf{d}$*, where*

$$A = \mathcal{T}_x \otimes I_m + I_m \otimes \mathcal{T}_{\mathsf{y}} + 2I_n \in \mathbb{R}^{n\times n}, \tag{23}$$

$n = m^2$, $\otimes$ *represent the Kronecker product,* $\mathcal{T}_x = tridiag(-\mathcal{R}e - 1, 4, \mathcal{R}e - 1)$, $\mathcal{T}_y = tridiag(-\mathcal{R}e - 1, 0, \mathcal{R}e - 1)$, $h = \frac{1}{m+1}$ *and* $\mathcal{R}e = \frac{h}{2}$*. Now, in system* (1)*, suppose that matrix A is as* (23)*. The vector* b *is also adopted as if* $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$ *with* $x_l = (-1)^l i, l = 1(1)n$ *would be the solution, where i denotes the imaginary unit.*

Numerical experiments of Example 4.2 are listed in Table 2, and $\Theta_{\mathsf{w}_1,\mathsf{w}_2} + \varrho^2_{w_2} + 2\varrho_{w_2} = 0.9720$ *and* $\|A^{-1}\| = 0.5$.

Table 1: Numerical experiments of Example 4.1.

Method	n	10000	40000	90000	250000
DOS-like	Iter.	5	5	5	5
	CPU	0.0046	0.0155	0.0555	0.0715
	RES	$1.40267e-07$	$1.40267e-07$	$1.40267e-07$	$1.40267e-07$
	w_{1exp}, w_{2exp}	0.1875, 0.9711	0.1875, 0.9711	0.1875, 0.9711	0.1875, 0.9711
SSOR	Iter.	5	5	5	5
	CPU	0.0109	0.02596	0.1755	0.2793
	RES	$3.62052e-06$	$5.77528e-06$	$7.782301e-06$	$1.16566e-05$
	w_{exp}	0.96	0.96	0.96	0.96
AOR	Iter.	8	8	8	8
	CPU	0.0282	0.0404	0.2193	0.3177
	RES	$3.54011e-05$	$4.6113e-05$	$5.52029e-05$	$7.09377e-05$
	w_{exp}, r_{exp}	0.8, 1.2	0.8, 1.2	0.8, 1.2	0.8, 1.2
RPHSS	Iter.	9	8	8	8
	CPU	0.0701	0.8925	1.9126	32.2329
	RES	$1.5866e-05$	$1.42251e-04$	$1.96039e-4$	$26.1516e-04$
	α_{exp}, w_{exp}	8.1, 1.2	8.1, 1.2	8.1, 1.2	8.1, 1.2
HSS-like	Iter.	10	10	10	9
	CPU	0.0843	0.9747	2.1446	40.0113
	RES	$3.32596e-05$	$4.75901e-05$	$5.86912e-05$	$2.98187e-04$
	α_{exp}	6.9	6.9	6.9	6.9
MN	Iter.	4	4	4	4
	CPU	0.1380	1.0158	1.8774	7.0569
	RES	$7.490458e-07$	$1.50981e-06$	$2.27059e-06$	$3.792147e-06$
	w_{exp}	1.03	1.03	1.03	1.03
GGS	Iter.	9	9	9	9
	CPU	0.0393	0.14403	0.3464	1.1146
	RES	$5.67150e-05$	$1.16252e-04$	$1.7579e-04$	$2.94865e-04$

Table 2: Numerical experiments of Example 4.2.

Method	n	2500	10000	40000	250000
DOS-like	Iter.	9	9	9	9
	CPU	0.0053	0.0134	0.03378	0.2534
	RES	$4.26023e-07$	$4.56024e-07$	$4.71190e-07$	$4.81701e-07$
	w_{1exp}, w_{2exp}	0.12, 1.24	0.12, 1.24	0.12, 1.24	0.12, 1.24
SSOR	Iter.	8	8	8	8
	CPU	0.0089	0.0155	0.04667	0.5110
	RES	$4.87836e-06$	$8.92217e-06$	$1.67461e-05$	$4.04956e-05$
	w_{exp}	1.17	1.17	1.17	1.17
AOR	Iter.	14	14	13	13
	CPU	0.0078	0.0417	0.0680	0.6093
	RES	$1.93119e-05$	$2.45639e-05$	$1.14185e-04$	$2.79883e-04$
	w_{exp}, r_{exp}	1.05, 1.45	1.05, 1.45	1.05, 1.45	1.05, 1.45
RPHSS	Iter.	9	9	9	9
	CPU	0.0168	0.1067	1.0421	27.3804
	RES	$2.12981e-05$	$1.6281e-05$	$3.3010e-05$	$4.1052e-05$
	α_{exp}, w_{exp}	4.3, 1.45	4.3, 1.45	4.3, 1.45	4.3, 1.45
HSS-like	Iter.	14	14	14	14
	CPU	0.0458	0.1573	1.5165	35.2495
	RES	$2.42739e-05$	$4.741546e-05$	$9.36804e-05$	$2.32441e-04$
	α_{exp}	4.3	4.3	4.3	4.3
MN	Iter.	5	5	5	5
	CPU	0.0451	0.1826	0.7541	7.4054
	RES	$2.40539e-06$	$5.06521e-06$	$1.03852e-05$	$2.63452e-05$
	w_{exp}	0.01	0.01	0.01	0.01
GGS	Iter.	$fail$	$fail$	$fail$	$fail$
	CPU	–	–	–	–
	RES	–	–	–	–

Example 4.3. *[10, 18, 26] Consider the LCP(q, M) as*

$$\zeta \geq 0, \mathcal{W} = M\zeta + q \geq 0, \mathcal{W}^T \zeta = 0 \tag{24}$$

to finding two unknown vectors $\mathcal{W}$ and $\zeta \in \mathbb{R}^n$, in which $\mathcal{W}$ is real vector, $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. We rewrite (24) *as*

$$(I + M)x + (I - M)|x| = q, \tag{25}$$

which $x = \frac{1}{2}((M - I)\zeta + q)$ and (25) *is a generalized AVE [14]. We consider $A = I + M$ and $B = -I + M$, in which $M = \bar{M} + \mu I$ and $q = -M\zeta^\star$, where $\bar{M} = tridiag(-I, S, -I) \in \mathbb{R}^{m^2\times m^2}$, $S = tridiag(-1, 4, -1) \in \mathbb{R}^{m\times m}$ and $\zeta^\star = 1.2\tau$, where $\tau = (1, 1, \cdots, 1)^T \in \mathbb{R}^n$, is the unique solution of the LCP* (24)*. $x^\star = -0.6\tau \in \mathbb{R}^n$ is the exact solution of the GAVE* (25)*. Since $det(M) \neq 1$,* (25) *can be converted to* (1)*, but here we extend the nonlinear DOS-like method for GAVE, and we have*

$$\begin{cases} D(x^{(k+\frac{1}{2})}) = [w_1 D + (w_1 - 1)L + (w_1 - 1)U]x^{(k)} + (1 - w_1)(B|x^{(k)}| + b), \\ (D + w_2 L)x^{(k+1)} = [(1 - w_2)D - w_2 U]x^{(k+\frac{1}{2})} + w_2(B|x^{(k+\frac{1}{2})}| + b). \end{cases} \tag{26}$$

Since the GGS method, relaxed nonlinear PHSS-like and the nonlinear HSS-like methods have not developed to GAVE, we didn't list their numerical results in Table 3 and 4. In Table 3 we have $\Theta_{w_1,w_2} + \varrho^2_{w_2} + 2\varrho_{w_2} \leq 0.6319$ and $||A^{-1}|| = 0.2$. These values for Table 4 are 0.8532 *and* 0.33*, respectively.*

Table 3: Numerical experiments of Example 4.3 for $\mu = 4$.

Method	n	10000	40000	90000	250000
DOS-like	Iter.	7	7	7	7
	CPU	0.0086	0.0310	0.0802	0.2390
	RES	$1.6628e-07$	$1.4582e-07$	$1.45545e-07$	$1.45545e-07$
	w_{1exp}, w_{2exp}	.5214, 0.7890	0.5294, 0.7925	0.5304, 0.7927	0.5304, 0.7927
SSOR	Iter.	7	7	7	7
	CPU	0.0124	0.0457	0.1547	0.4434
	RES	$2.6969e-05$	$4.7677e-05$	$6.9993e-05$	$1.1572e-04$
	w_{exp}	0.67	0.67	0.67	0.67
AOR	Iter.	16	17	17	17
	CPU	0.0184	0.0825	0.2692	0.7932
	RES	$1.8944e-05$	$3.9319e-05$	$5.9694e-05$	$1.00444e-04$
	w_{exp}, r_{exp}	0.6, 0.7	0.6, 07	0.6, 07	0.6, 07
MN	Iter.	11	11	11	11
	CPU	0.1363	0.8352	1.8620	6.7701
	RES	$1.4032e-05$	$2.9098e-05$	$4.4167e-05$	$7.43064e-05$
	w_{exp}	5.1	5.1	5.1	5.1

The numerical results of examples presented in 4 tables for different values of n. These tables show that all listed schemes can rightly generate an approximate solution for all examples, except for the GGS method that fails to converge in 500 iterations in Table 2. Also, we found that the iterations number for the entire techniques is independent of the problem size as increases in Table 1, and 2, while the processing time in all tables increases with the increase in the problem size. It is also observed that in terms of CPU time, the nonlinear DOS-like, SSOR and AOR methods are the best ones among the listed methods, and the CPU time required by the nonlinear DOS-like technique is less than or equivalent to that required by the SSOR method. Additionally, the CPU time required by the SSOR technique is less than or equivalent to that required by the AOR method. Regarding CPU time in Table 1 and 2, we have the following results,

$$CPU_{DOS-like} \leq CPU_{SSOR} \leq CPU_{AOR} < CPU_{RPHSS} < CPU_{HSS-like}.$$

Regarding CPU time and the number of iteration in Table 3 and 4, we have the following results

$$CPU_{DOS-like} \leq CPU_{SSOR} \leq CPU_{AOR} < CPU_{MN}.$$

Table 4: Numerical experiments of Example 4.3 for $\mu = 2$.

Method	n	10000	40000	90000	250000
DOS-like	Iter.	10	10	10	10
	CPU	0.0104	0.0439	0.1157	0.3332
	RES	$8.8018e-08$	$8.806734e-08$	$8.8067e-08$	$8.80673e-08$
	w_{1exp}, w_{2exp}	0.5436, 0.9604	0.5437, 0.9600	0.5436, 0.9600	.5436, 0.9600
SSOR	Iter.	12	12	12	12
	CPU	0.0216	0.0717	0.2400	0.7647
	RES	$1.3100e-05$	$2.7337e-05$	$4.1664e-05$	$7.02660e-05$
	w_{exp}	0.7	0.7	0.7	0.7
AOR	Iter.	22	22	23	23
	CPU	0.0232	0.0997	0.3272	1.0217
	RES	$2.3452e-05$	$4.7957e-05$	$3.5080e-05$	$5.90154e-05$
	w_{exp}, r_{exp}	0.7, 0.8	0.7, 08	0.7, 08	0.7, 08
MN	Iter.	15	15	15	15
	CPU	0.1776	0.7712	2.5073	8.44133
	RES	$1.5239e-05$	$3.1965e-05$	$4.8690e-05$	$8.21427e-05$
	w_{exp}	3	3	3	3

$$Iter_{DOS-like} \leq Iter_{SSOR} < Iter_{MN} < Iter_{AOR}.$$

Where CPU_x signifies the CPU time of the x method and $Iter_x$ signifies the number of iterations for the x method.

In our numerical experiments, the optimal parameters of the DOS-like method have been obtained experimentally. The upper bound $\Theta_{w_1,w_2} + \varrho_{w_2}^2 + 2\varrho_{w_2}$ at Theorems 1 and 2 has role as the upper bound for the spectral radius of the iteration matrix of propose method. The theoretical optimal parameters w_1 and w_2 can be determined by minimizing this bound in the future, although it is a very difficult task.

5. Conclusion

This paper has investigated the nonlinear DOS-like iteration technique on the basis of diagonal and off diagonal splitting of the coefficient matrix A to solve the AVEs. In addition, we investigated the convergence features of the suggested technique. According to numerical experiments, the nonlinear DOS-like technique is a efficient and practical solver for absolute value equations.

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