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Quasi-hemi slant submanifolds of para Hermitian manifolds

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Abstract. The aim of the present paper is to study quasi hemi slant submanifolds in a para Hermitian manifold. We study properties and condition of integrability of the distributions in the quasi hemi slant submanifold. In addition, we find the necessary and sufficient condition for a quasi-hemi slant submanifold of a para Kaehler manifold to be totally geodesic and study the geometry of foliations determined by distributions. Furthermore, we present some examples of quasi-hemi slant submanifolds of para Hermitian manifolds.

1. Introduction

The notion of slant submanifold was initiated by B. Y. Chen [6, 7] in 1990. He studied slant submanifolds in an almost Hermitian manifold. It is well known that this type of submanifolds are generalization of holomorphic (invariant) and totally real (anti-invariant) submanifolds. In 1996, A. Lotta [3] introduced the notion of slant immersion of Riemannian manifold into an almost contact metric manifold. Many geometers studied slant submanifolds in Riemannan manifolds equiped with different kind of structures [4, 5, 9]. P. Alegree and A. Carriazo [13, 14] extended the notion of slant submanifolds in para Hermitian manifold and studied bislant submanifolds.

The slant submanifolds were generalized as semi-slant submanifolds, hemi-slant (pseudo-slant) submanifolds, bi-slant submanifolds and quasi-hemi slant submanifolds. These genaralisations have been studied by several geometers [1, 2, 8, 10, 12, 20, 22]. Recently many geometers generalized these notions as quasi-bislant submanifolds. R. Prasad, A. Haseeb, S. Singh, S. K. Verma, M. A. Akyol, S. Y. Perktas, A. M. Blaga, S. Uddin and others studied quasi bislant submanifolds [11, 15–19, 21].

This paper is organised as follows. After introduction, section 2 contains some basic results and definitions related to para-Kaehler manifold. We mention some theorems regarding slant distributions and define quasi-hemi slant submanifold of a para Hermitian manifold in this section. Section 3 contains the study of quasi-hemi slant submanifolds. This section is devoted to the geometry of distributions. We study integrability conditions of distributions, geodesic foliations defined by distributions in this section. In section 4 we give examples of quasi-hemi slant submanifolds.

2. Preliminaries

A pair $(\overline{M}, J)$, where $\overline{M}$ is a smooth manifold and J is a $(1, 1)$-tensor field on $\overline{M}$ satisfying $J^2 = Id$ is called an almost product manifold and J is called an almost product structure on $\overline{M}$. It is called an almost para

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complex structure if two eigenbundles

$$T^+M := ker(Id - J), \quad T^-M := ker(Id + J)$$

have same dimension. In this case $\overline{M}$, which is even dimensional, is called an almost para complex manifold.

A para Hermitian manifold $\overline{M}$ is a para complex manifold $(\overline{M}, J)$ endowed with a pseudo-Riemannian metric g satisfying

$$g(JX, Y) + g(X, JY) = 0, \tag{1}$$

for any vector fields X, Y on $\overline{M}$. It is said to be a para Kaehler if, in addition

$$\overline{\nabla}J = 0, \tag{2}$$

i.e., J is parallel with respect to $\overline{\nabla}$, the Levi-Civita connection of g.

Let M be a pseudo-Riemannian submanifold isometrically immersed in a para Hermitian manifold $(\overline{M}, J, g)$. We have the following Gauss and Weingarten formulae:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{3}$$

$$\overline{\nabla}_X V = -\mathcal{A}_V X + \nabla^{\perp}_X V, \tag{4}$$

for any $X, Y \in TM$ and $V \in T^{\perp}M$, where ∇ and $\nabla^{\perp}$ are respectively induced connections on the tangent bundle TM and normal bundle $T^{\perp}M$. Here h is second fundamental form of M and $\mathcal{A}_V$ is the Weingarten endomorphism associated with V satisfying the following relation

$$g(h(X, Y), V) = g(\mathcal{A}_V X, Y). \tag{5}$$

For any $X \in TM$ and $V \in T^{\perp}M$, we put

$$JX = \phi X + \omega X, \qquad JV = \alpha V + \beta V, \tag{6}$$

where $\phi X, \alpha V \in TM$ ($\omega X, \beta V \in T^{\perp}M$) are respectively called tangential (normal) components of JX, JV. The covariant derivative of projection morphisms given in the equation (6) are defined by

$$(\overline{\nabla}_X \phi)Y = \nabla_X \phi Y - \phi \nabla_X Y, \tag{7}$$

$$(\overline{\nabla}_X \omega)Y = \nabla^{\perp}_X \omega Y - \omega \nabla_X Y, \tag{8}$$

$$(\overline{\nabla}_X \alpha)V = \nabla_X \alpha V - \alpha \nabla^{\perp}_X V, \tag{9}$$

$$(\overline{\nabla}_X \beta)V = \nabla^{\perp}_X \beta V - \beta \nabla^{\perp}_X V, \tag{10}$$

for any $X, Y \in TM$ and $V \in T^{\perp}M$.

Now, first we recall the following definitions:

Definition 2.1 ([13]). *A submanifold M of a para Hermitian manifold $(\overline{M}, J, g)$ is called slant if for every space-like or time-like tangent vector field X, the quotient $\dfrac{g(\phi X, \phi X)}{g(JX, JX)}$ is constant.*

It is clear from the definition 2.1 that both complex and totally real submanifolds are particular cases of slant submanifolds. A neither complex nor totally real slant submanifold is known as proper slant submanifold.

Definition 2.2 ([14]). *Let M be a proper slant submanifold of a para Hermitian manifold $(\overline{M}, J, g)$. We say that it is of:*

Type 1 if for every space-like (time-like) vector field X, if ϕX is time-like (space-like), and $\frac{|\phi X|}{|JX|} > 1$, *Type 2 if for every space-like (time-like) vector field X, if ϕX is time-like (space-like), and* $\frac{|\phi X|}{|JX|} < 1$,

Type 3 if for every space-like (time-like) vector field X, if ϕX is space-like (time-like).

Definition 2.3 ([14]). *A differentiable distribution D on a para Hermitian manifold $(\overline{M}, J, g)$ is called a slant distribution if for every non-light-like $X \in D$, the quotient $\frac{g(P_D X, P_D X)}{g(JX, JX)}$ is constant. Where $P_D X$ is projection of JX over D.*

It is easy to see that for invariant and anti-invariant distributions are particular cases of slant distributions. A slant distribution is called proper if it is neither invariant nor anti-invariant distribution.

Definition 2.4 ([14]). *Let D be a proper slant distribution of a para Hermitian manifold (M, J, g) . We say that it is of:*

Type 1 if for every space-like (time-like) vector field X, if $P_D X$ is time-like (space-like), and $\frac{|P_D X|}{|JX|} > 1$,

Type 2 if for every space-like (time-like) vector field X, if $P_D X$ is time-like (space-like), and $\frac{|P_D X|}{|JX|} < 1$,

Type 3 if for every space-like (time-like) vector field X, if $P_D X$ is space-like (time-like).

Next, the following result gives a characterization of slant distributions on para Hermitian manifolds:

Theorem 2.5 ([14]). *Let D be a distribution of a para Hermitian manifold $(\overline{M}, J, g)$. Then,*

(1) D is a slant distribution of Type 1 if and only if for any space-like (time-like) vector field X, $P_D X$ is time-like(space-like), and there exists a constant $\lambda \in (1, +\infty)$ such that

$$P_D^2 = \lambda I.$$

Moreover, in such a case, $\lambda = cosh^2\theta$.

(2) D is a slant distribution of Type 2 if and only if for any space-like (time-like) vector field X, $P_D X$ is time-like(space-like), and there exists a constant $\lambda \in (0, 1)$ such that

$$P_D^2 = \lambda I.$$

Moreover, in such a case, $\lambda = cos^2\theta$.

(3) D is a slant distribution of Type 3 if and only if for any space-like (time-like) vector field X, $P_D X$ is space-like(time-like), and there exists a constant $\lambda \in (0, +\infty)$ such that

$$P_D^2 = -\lambda I.$$

Moreover, in such a case, $\lambda = sinh^2\theta$.

In each case, θ is called the slant angle of the distribution D.

Now we define quasi-hemi slant submanifold of a para Hermitian manifold.

Definition 2.6. *A submanifold M of a para Hermitian manifold $(\overline{M}, J, g)$ is called quasi hemi-slant submanifold if there exists distributions D, D_θ and $D^\perp$ such that*

(i) TM admits the orthogonal direct decomposition as $TM = D \oplus D_\theta \oplus D^\perp$,
(ii) the distribution D is invariant, i.e., $JD = D$,
(iii) the distribution D_θ is slant distribution,
(iv) the distribution $D^\perp$ is anti-invariant, i.e., $JD^\perp \subseteq T^\perp M$.

If θ denotes the slant angle of D_θ, we observe that

(a) If $\dim D \neq 0$, $\dim D_\theta = 0$ and $\dim D^\perp = 0$, then M is an invariant submanifold.
(b) If $\dim D \neq 0$, $\dim D_\theta \neq 0, 0 < \theta < \pi/2$ and $\dim D^\perp = 0$, then M is proper semi-slant submanifold.

(c) If dim $D = 0$, dim $D_\theta \neq 0, 0 < \theta < \pi/2$ and dim $D^\perp = 0$, then M is a proper slant submanifold with slant angle θ .

(d) If dim $D = 0$, dim $D_\theta = 0$ and dim $D^\perp \neq 0$, then M is anti-invariant submanifold .

(e) If dim $D \neq 0$, dim $D_\theta = 0$ and dim $D^\perp \neq 0$, then M is a semi-invariant submanifold.

(f) If dim $D = 0$, dim $D_\theta \neq 0, 0 < \theta < \pi/2$ and dim $D^\perp \neq 0$, then M is a proper hemi-slant submanifold.

(g) If dim $D \neq 0$, dim $D_\theta \neq 0, 0 < \theta < \pi/2$ and dim $D^\perp \neq 0$, then M is a proper quasi hemi-slant submanifold.

Definition 2.7. *Let M is a quasi hemi-slant submanifold of a para Hermitian manifold $(\overline{M}, J, g)$. We say M is of type 1, type 2 or type 3 according as the slant distribution D_θ is of type 1, type 2 or type 3.*

3. Quasi hemi-slant submanifolds of a para Hermitian manifold

Let M be a quasi hemi-slant submanifold of a para Hermitian manifold $(\overline{M}, J, g)$. Then for any $X \in TM$, we write

$$X = PX + QX + RX, \tag{11}$$

where P, Q and R are projections of TM onto the distributions D, D_θ and $D^\perp$ respectively. From equations (6) and (11), we have

$$JX = \phi PX + \phi QX + \omega QX + \omega RX,$$

which implies

$$J(TM) = D \oplus \phi D_\theta \oplus \omega D_\theta \oplus \omega D^\perp.$$

So we have

$$T^\perp M = \omega D_\theta \oplus \omega D^\perp \oplus \mu,$$

where μ is orthogonal complement of $\omega D_\theta \oplus \omega D^\perp$ in $T^\perp M$. It is invariant with respect to J. Now, it is easy to prove the following lemma.

Lemma 3.1. *Let M is a quasi hemi-slant submanifold of a para Hermitian manifold $(\overline{M}, J, g)$, then*

$$\phi D = D, \quad \phi D_\theta = D_\theta, \quad \phi D^\perp = \{0\}, \quad \alpha\omega D_\theta = D_\theta, \quad \text{and} \quad \alpha\omega D^\perp = D^\perp.$$

Now we prove the following lemma.

Lemma 3.2. *Let M is a quasi hemi-slant submanifold of a para Hermitian manifold $(\overline{M}, J, g)$, then for any $X \in TM$ and $V \in T^\perp M$, we have*

$$\phi^2 X + \alpha\omega X = X, \qquad \omega\phi X + \beta\omega X = 0$$
$$\phi\alpha V + \alpha\beta V = 0, \qquad \omega\alpha V + \beta^2 V = V.$$

Proof. The proof is straight forward from the equations (1) and (6). □

Lemma 3.3. *Suppose that M is a quasi hemi-slant submanifold of a para Hermitian manifold $(\overline{M}, J, g)$. If M is quasi hemi-slant submanifold of type 1, then we have*

$$\phi^2 X = (cosh^2\theta)X, \quad g(\phi X, \phi Y) = -(cosh^2\theta)g(X, Y)$$

$$\text{and} \qquad g(\omega X, \omega Y) = (sinh^2\theta)g(X, Y), \qquad \text{for all } X, Y \in \Gamma(D_\theta).$$

Proof. Since M is quasi hemi-slant submanifold of type 1, by definition 2.7 it follows that $TM = D \oplus D_\theta \oplus D^\perp$, where D_θ is slant distribution of type 1. Hence from theorem 2.5, there exists $\lambda \in (1, +\infty)$ such that

$$\phi^2 X = \lambda X,$$

or equivalently

$$\phi^2 X = (cosh^2\theta)X,$$

for all $X \in \Gamma(D_\theta)$ and θ is slant angle of D_θ. Now, for any $X, Y \in \Gamma(D_\theta)$ we have

$$g(\phi X, \phi Y) = g(JX, \phi Y) = -g(X, J\phi Y) = -g(X, \phi^2 Y),$$

and

$$-g(X, Y) = g(JX, JY) = g(\phi X, \phi Y) + g(\omega X, \omega Y).$$

Hence the lemma. □

In similar manner, we have the following lemmas:

Lemma 3.4. *Suppose that M is a quasi hemi-slant submanifold of a para Hermitian manifold $(\overline{M}, J, g)$. If M is quasi hemi-slant submanifold of type 2, then we have*

$$\phi^2 X = (cos^2\theta)X, \quad g(\phi X, \phi Y) = -(cos^2\theta)g(X, Y)$$

and $$g(\omega X, \omega Y) = (sin^2\theta)g(X, Y), \quad \text{for all } X, Y \in \Gamma(D_\theta).$$

Lemma 3.5. *Suppose that M is a quasi hemi-slant submanifold of a para Hermitian manifold $(\overline{M}, J, g)$. If M is quasi hemi-slant submanifold of type 3, then we have*

$$\phi^2 X = (-sinh^2\theta)X, \quad g(\phi X, \phi Y) = (sinh^2\theta)g(X, Y)$$

and $$g(\omega X, \omega Y) = (-cosh^2\theta)g(X, Y), \quad \text{for all } X, Y \in \Gamma(D_\theta).$$

Next, we obtain covariant derivative of projection morphisms.

Lemma 3.6. *Let M is a quasi hemi-slant submanifold of a para Kaehlerian manifold $(\overline{M}, J, g)$, then, we have*

$$(\overline{\nabla}_X \phi)Y = \mathcal{A}_{\omega Y}X + \alpha h(X, Y),$$
$$(\overline{\nabla}_X \omega)Y = \beta h(X, Y) - h(X, \phi Y),$$
$$(\overline{\nabla}_X \alpha)V = \mathcal{A}_{\beta V}X - \phi(\mathcal{A}_V X),$$
$$(\overline{\nabla}_X \beta)V = -\omega(\mathcal{A}_V X) - h(X, \alpha V),$$

for any $X, Y \in TM$ and $V \in T^\perp M$.

Proof. The proof follows from the equations (2), (3), (4), (7), (8), (9) and (10). □

Now, we study integrability of distributions on M. Suppose $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D_\theta \oplus D^\perp)$, then from equations (1), (2) and (3) we have

$$\begin{aligned} g([X, Y], Z) &= -g(\overline{\nabla}_X JY - \overline{\nabla}_Y JX, JZ) \\ &= -g(\nabla_X \phi Y - \nabla_Y \phi X, \phi QZ) - g(h(X, \phi Y) - h(\phi X, Y), \omega Z). \end{aligned}$$

Therefore, we have the following theorem.

Theorem 3.7. *Let M is a quasi hemi-slant submanifold of a para Kaehlerian manifold $(\overline{M}, J, g)$, then the invariant distribution D is integrable if and only if*

$$g(\nabla_X \phi Y - \nabla_Y \phi X, \phi QZ) = g(h(\phi X, Y) - h(X, \phi Y), \omega Z),$$

for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D_\theta \oplus D^\perp)$.

Now, consider $X, Y \in \Gamma(D^\perp)$ and $Z \in \Gamma(D \oplus D_\theta)$, then from equations (1), (2) and (4) we have

$$\begin{aligned} g([X, Y], Z) &= -g(\overline{\nabla}_X JY - \overline{\nabla}_Y JX, JZ) \\ &= -g(\overline{\nabla}_X \omega Y - \overline{\nabla}_Y \omega X, \phi Z + \omega Z) \\ &= g(\mathcal{A}_{\omega Y} X - \mathcal{A}_{\omega X} Y, \phi Z) - g(\nabla^\perp_X \omega Y - \nabla^\perp_Y \omega X, \omega Z). \end{aligned}$$

Therefore, we state the following result.

Theorem 3.8. *Let M is a quasi hemi-slant submanifold of a para Kaehlerian manifold $(\overline{M}, J, g)$, then the anti-invariant distribution $D^\perp$ is integrable if and only if*

$$g(\mathcal{A}_{\omega Y} X - \mathcal{A}_{\omega X} Y, \phi Z) = g(\nabla^\perp_X \omega Y - \nabla^\perp_Y \omega X, \omega Z),$$

for any $X, Y \in \Gamma(D^\perp)$ and $Z \in \Gamma(D \oplus D_\theta)$.

Next, we have the following criteria for integrability of slant distribution D_θ.

Theorem 3.9. *Let M is a proper quasi hemi-slant submanifold of a para Kaehlerian manifold $(\overline{M}, J, g)$, then the slant distribution D_θ is integrable if and only if*

$$g(\mathcal{A}_{\omega\phi X} Y - \mathcal{A}_{\omega\phi Y} X, Z) + g(\mathcal{A}_{\omega Y} X - \mathcal{A}_{\omega X} Y, \alpha Z) = g(\nabla^\perp_Y \omega X - \nabla^\perp_X \omega Y, \beta Z),$$

for $X, Y \in \Gamma(D_\theta)$ and $Z \in \Gamma(D \oplus D^\perp)$.

Proof. Consider $X, Y \in \Gamma(D_\theta)$ and $Z \in \Gamma(D \oplus D^\perp)$. Then from equations (1) and (2), we have

$$\begin{aligned} g([X, Y], Z) &= -g(\overline{\nabla}_X JY - \overline{\nabla}_Y JX, JY) \\ &= g(\overline{\nabla}_X J\phi Y - \overline{\nabla}_Y J\phi X, Z) - g(\overline{\nabla}_X \omega Y - \overline{\nabla}_Y \omega X, JZ) \\ &= g(\overline{\nabla}_X \phi^2 Y - \overline{\nabla}_Y \phi^2 X, Z) + g(\overline{\nabla}_X \omega\phi Y - \overline{\nabla}_Y \omega\phi X, Z) \\ &- g(\overline{\nabla}_X \omega Y - \overline{\nabla}_Y \omega X, JZ). \end{aligned}$$

Last equation implies

$$\begin{aligned} g([X, Y], Z) - g(\overline{\nabla}_X \phi^2 Y - \overline{\nabla}_Y \phi^2 X, Z) &= g(\mathcal{A}_{\omega\phi X} Y - \mathcal{A}_{\omega\phi Y} X, Z) \\ &+ g(\mathcal{A}_{\omega Y} X - \mathcal{A}_{\omega X} Y, \alpha Z) \\ &- g(\nabla^\perp_Y \omega X - \nabla^\perp_X \omega Y, \beta Z). \end{aligned} \tag{12}$$

Now we have the following cases. If D_θ is slant distribution of type 1, then using theorem 2.5 in the equation (12), we obtain

$$\begin{aligned} (-sinh^2\theta) g([X, Y], Z) &= g(\mathcal{A}_{\omega\phi X} Y - \mathcal{A}_{\omega\phi Y} X, Z) + g(\mathcal{A}_{\omega Y} X - \mathcal{A}_{\omega X} Y, \alpha Z) \\ &- g(\nabla^\perp_Y \omega X - \nabla^\perp_X \omega Y, \beta Z). \end{aligned}$$

Hence the statement of the theorem. Similarly, If D_θ is slant distribution of type 2, then using theorem 2.5 in the equation (12), we calculate

$$\begin{aligned} (sin^2\theta) g([X, Y], Z) &= g(\mathcal{A}_{\omega\phi X} Y - \mathcal{A}_{\omega\phi Y} X, Z) + g(\mathcal{A}_{\omega Y} X - \mathcal{A}_{\omega X} Y, \alpha Z) \\ &- g(\nabla^\perp_Y \omega X - \nabla^\perp_X \omega Y, \beta Z), \end{aligned}$$

and if D_θ is slant distribution of type 3, once again equation (12) yields

$$(cosh^2\theta)g([X,\,Y],\,Z) = g(\mathcal{A}_{\omega\phi X}Y - \mathcal{A}_{\omega\phi Y}X,\,Z) + g(\mathcal{A}_{\omega Y}X - \mathcal{A}_{\omega X}Y,\,\alpha Z)$$
$$- g(\nabla^{\perp}_{Y}\omega X - \nabla^{\perp}_{X}\omega Y,\,\beta Z).$$

This completes the proof. □

Theorem 3.10. *Let M be a proper quasi-hemi slant submanifold of a para Kaehler manifold $(\overline{M},\,J,\,g)$, then M is totally geodesic if and only if the following conditions hold for any vector fields $X,\,Y \in TM$, $V \in T^{\perp}M$:*

$$g(h(X,\,PY),\,V) + \lambda g(h(X,\,QY),\,V) + g(\nabla^{\perp}_{X}\omega\phi QY,\,V)$$
$$+ g(\mathcal{A}_{\omega QY}X + \mathcal{A}_{\omega RY}X,\,\alpha V) - g(\nabla^{\perp}_{X}\omega Y,\,\beta V) = 0,$$

where λ is equal to $cosh^2\theta$, $cos^2\theta$ or $-sinh^2\theta$ according as M is of type 1, type 2 or type 3 and θ denotes slant angle of the slant distribution.

Proof. Consider $X,\,Y \in TM$ and $V \in T^{\perp}M$. Then equations (1), (2) and (11) imply

$$g(\overline{\nabla}_X Y,\,V) = -g(\overline{\nabla}_X JPY,\,JV) - g(\overline{\nabla}_X JQY,\,JV) - g(\overline{\nabla}_X JRY,\,JV)$$
$$= g(\overline{\nabla}_X PY,\,V) + g(\overline{\nabla}_X \phi^2 QY,\,V) + g(\overline{\nabla}_X \omega\phi QY,\,V)$$
$$- g(\overline{\nabla}_X \omega QY,\,JV) - g(\overline{\nabla}_X \omega RY,\,JV).$$

Using equations (3) and (4), last equation implies

$$g(\overline{\nabla}_X Y,\,V) = g(h(X,\,PY),\,V) + g(h(X,\,\phi^2 QY),\,V) + g(\nabla^{\perp}_{X}\omega\phi QY,\,V)$$
$$+ g(\mathcal{A}_{\omega QY}X + \mathcal{A}_{\omega RY}X,\,\alpha V) - g(\nabla^{\perp}_{X}\omega Y,\,\beta V). \tag{13}$$

Now, first we consider that M is a proper quasi-hemi slant submanifold of type 1, then D_θ is slant distribution of type 1. From lemma 3.3 there exists a constant $\lambda \in (0,\,+\infty)$ such that $\phi^2 QX = \lambda QX$ or $\phi^2 QX = (cosh^2\theta)QX$, for any $X \in TM$, where θ is slant angle of D_θ. Hence equation (13) yields

$$g(\overline{\nabla}_X Y,\,V) = g(h(X,\,PY),\,V) + (cosh^2\theta)g(h(X,\,QY),\,V) + g(\nabla^{\perp}_{X}\omega\phi QY,\,V)$$
$$+ g(\mathcal{A}_{\omega QY}X + \mathcal{A}_{\omega RY}X,\,\alpha V) - g(\nabla^{\perp}_{X}\omega Y,\,\beta V).$$

Hence the statement.

If M is a proper quasi-hemi slant submanifold of type 2, then D_θ is slant distribution of type 2. Using lemma 3.4 in the equation (13), we obtain

$$g(\overline{\nabla}_X Y,\,V) = g(h(X,\,PY),\,V) + (cos^2\theta)g(h(X,\,QY),\,V) + g(\nabla^{\perp}_{X}\omega\phi QY,\,V)$$
$$+ g(\mathcal{A}_{\omega QY}X + \mathcal{A}_{\omega RY}X,\,\alpha V) - g(\nabla^{\perp}_{X}\omega Y,\,\beta V),$$

where θ is slant angle of the distribution D_θ. Hence the statement.

Finally if M is a proper quasi-hemi slant submanifold of type 3, then lemma 3.5 and equation (13) imply

$$g(\overline{\nabla}_X Y,\,V) = g(h(X,\,PY),\,V) - (sinh^2\theta)g(h(X,\,QY),\,V) + g(\nabla^{\perp}_{X}\omega\phi QY,\,V)$$
$$+ g(\mathcal{A}_{\omega QY}X + \mathcal{A}_{\omega RY}X,\,\alpha V) - g(\nabla^{\perp}_{X}\omega Y,\,\beta V),$$

where θ is slant angle of the distribution D_θ. This completes the proof. □

Next, we discuss geodesic foliations defined by the distributions on M. First we have the following result.

Theorem 3.11. *Let M be a proper quasi-hemi slant submanifold of a para Kaehler manifold $(\overline{M}, J, g)$, then the invariant distribution D defines a totally geodesic foliation on M if and only if the following conditions holds for any vector fields $X, Y \in \Gamma(D)$, $Z \in \Gamma(D_\theta \oplus D^\perp)$ and $V \in T^\perp M$:*

$$g(\nabla_X \phi Y, \phi Z) = g(h(X, \phi Y), \omega Z),$$

and

$$g(\nabla_X \phi Y, \alpha V) + g(h(X, \phi Y), \beta V) = 0.$$

Proof. The proof is straight forward from equations (2), (3) and (4) and using the fact $\omega(D) = \{0\}$. □

Theorem 3.12. *Let M be a proper quasi-hemi slant submanifold of a para Kaehler manifold $(\overline{M}, J, g)$, then the invariant distribution $D^\perp$ defines a totally geodesic foliation on M if and only if the following conditions holds for any vector fields $X, Y \in \Gamma(D^\perp)$, $Z \in \Gamma(D \oplus D_\theta)$ and $V \in T^\perp M$:*

$$g(\nabla^\perp_X \omega Y, \omega Z) = g(\mathcal{A}_{\omega Y} X, \phi Z),$$

and

$$g(\nabla^\perp_X \omega Y, \beta V) = g(\mathcal{A}_{\omega Y} X, \alpha V).$$

Proof. The proof is similar to the proof of previous theorem. □

Theorem 3.13. *Let M be a proper quasi-hemi slant submanifold of a para Kaehler manifold $(\overline{M}, J, g)$, then the invariant distribution D_θ defines a totally geodesic foliation on M if and only if the following conditions holds for any vector fields $X, Y \in \Gamma(D_\theta)$, $Z \in \Gamma(D \oplus D^\perp)$ and $V \in T^\perp M$:*

$$g(\mathcal{A}_{\omega Y} X, \phi Z) = g(\mathcal{A}_{\omega\phi Y} X, Z) + g(\nabla^\perp_X \omega Y, \omega Z),$$

and

$$g(\nabla^\perp_X \omega Y, \beta V) = g(\nabla^\perp_X \omega\phi Y, V) + g(\mathcal{A}_{\omega Y} X, \alpha V),$$

where θ denotes the slant angle of the distribution D_θ.

Proof.

$$\begin{aligned} g(\overline{\nabla}_X Y, Z) &= -g(\overline{\nabla}_X \phi Y, JZ) - g(\overline{\nabla}_X \omega Y, JZ) \\ &= g(\overline{\nabla}_X \phi^2 Y, Z) + g(\overline{\nabla}_X \omega\phi Y, Z) - g(\overline{\nabla}_X \omega Y, JZ) \end{aligned}$$

implies

$$g(\overline{\nabla}_X Y, Z) - g(\overline{\nabla}_X \phi^2 Y, Z) = -g(\mathcal{A}_{\omega\phi Y} X, Z) + g(\mathcal{A}_{\omega Y} X, \phi Z) - g(\nabla^\perp_X \omega Y, \omega Z)$$

Now, if D_θ is slant distribution of type 1, using lemma 3.3, last equation implies

$$(-sinh^2\theta) g(\overline{\nabla}_X Y, Z) = -g(\mathcal{A}_{\omega\phi Y} X, Z) + g(\mathcal{A}_{\omega Y} X, \phi Z) - g(\nabla^\perp_X \omega Y, \omega Z), \tag{14}$$

where θ is the slant angle of D_θ. Similarly we find

$$(-sinh^2\theta) g(\overline{\nabla}_X Y, V) = g(\nabla^\perp_X \omega\phi Y, V) + g(\mathcal{A}_{\omega Y} X, \alpha V) - g(\nabla^\perp_X \omega Y, \beta V), \tag{15}$$

where θ denotes the slant angle of D_θ. The statement of the theorem follows from equations (14) and (15).

In similar manner one can write the proof when D_θ is slant distribution of type 2 or type 3. This completes the proof. □

Now, we study parallelism of projection morphism. We begin with the following theorem.

Theorem 3.14. *Let M be a proper quasi-hemi slant submanifold of a para Kaehler manifold $(\overline{M}, J, g)$, then ϕ is parallel if and only if the shape operator $\mathcal{A}$ satisfies*

$$\mathcal{A}_{\omega Y}Z = \mathcal{A}_{\omega Z}Y,$$

for all $Y, Z \in TM$.

Proof. Using equations (3) and (5) in the equation (7), we obtain

$$\begin{aligned} g((\overline{\nabla}_X\phi)Y, Z) &= g(\mathcal{A}_{\omega Y}X + \alpha h(X, Y), Z) \\ &= g(h(X, Z), \omega Y) - g(h(X, Y), \omega Z) \\ &= g(\mathcal{A}_{\omega Y}Z - \mathcal{A}_{\omega Z}Y, X), \end{aligned}$$

for all $X, Y, Z \in TM$. Hence the theorem. □

Theorem 3.15. *Let M be a proper quasi-hemi slant submanifold of a para Kaehler manifold $(\overline{M}, J, g)$, then ω is parallel if and only if the shape operator $\mathcal{A}$ satisfies*

$$\mathcal{A}_{\beta V}Y + \mathcal{A}_V\phi Y = 0,$$

for all $Y \in TM$ and $V \in T^\perp M$.

Proof. Let $X, Y \in TM$ and $V \in T^\perp M$. From equations (5) and (8), we have

$$\begin{aligned} g((\overline{\nabla}_X\omega Y), V) &= g(\beta h(X, Y) - h(X, \phi Y), V) \\ &= -g(h(X, Y), \beta V) - g(\mathcal{A}_V\phi Y, X) \\ &= -g(\mathcal{A}_{\beta V}Y + \mathcal{A}_V\phi Y, X). \end{aligned}$$

Hence the statement follows from the last equation. □

Theorem 3.16. *Let M be a proper quasi-hemi slant submanifold of a para Kaehler manifold $(\overline{M}, J, g)$, then ω is parallel if and only if α is parallel.*

Proof. From equations (5) and (9), we obtain

$$\begin{aligned} g((\overline{\nabla}_X\omega Y), V) &= g(\beta h(X, Y) - h(X, \phi Y), V) \\ &= -g(h(X, Y), \beta V) - g(\mathcal{A}_V X, JY) \\ &= -g(\mathcal{A}_{\beta V}X - \phi(\mathcal{A}_V X), Y) \\ &= -g((\overline{\nabla}_X\alpha)V, Y), \end{aligned}$$

for all $X, Y \in TM$ and $V \in T^\perp M$. Hence the assertion. □

Theorem 3.17. *Let M be a proper quasi-hemi slant submanifold of a para Kaehler manifold $(\overline{M}, J, g)$, then β is parallel if and only if the shape operator $\mathcal{A}$ satisfies*

$$\mathcal{A}_V\alpha U = \mathcal{A}_U\alpha V,$$

for all $U, V \in T^\perp M$.

Proof. The proof is similar to the proof of theorem 3.13. □

4. Examples

Now, we construct examples of quasi-bi-slant submersions using the examples given in [13, 14]. We consider the following para Kaehler structure on a pseudo-Euclidean space $\mathbb{R}^{2n}_n$ with coordinates $(x_1, x_2, \dots x_{2n})$.

$$J_1\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_{i+1}}, \quad J_1\left(\frac{\partial}{\partial x_{i+1}}\right) = \frac{\partial}{\partial x_i}, \qquad i = 1, 3, 5, \dots, (2n-1)$$

and g_1 be the pseudo-Riemannian metric on $\mathbb{R}^{2n}_n$ defined by

$$g_1\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = (-1)^{i+1}\delta_{ij}, \quad \text{where } \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases},$$

i.e., $\mathbb{R}^{2n}_n$ is pseudo-Eucllidean space with signature $(+,-,+,-,\cdots)$ with respect to the cannonical basis $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_{2n}})$.

Example 4.1. *Let M be submanifold of $\mathbb{R}^{10}$, equipped with para Kaehlerian structure (J_1, g_1), defined by*

$$(r, s, t, u, v, w,) \mapsto (ar, s, br, r, 0, u, 2t, t, v, w), \tag{16}$$

where $a, b \in \mathbb{R}$ satisfying $(a^2 + b^2 \neq 1)$.
If we define

$$X_1 = \frac{ae_1 + be_3 + e_4}{\sqrt{(a^2 + b^2 - 1)}}, X_2 = e_2, X_3 = e_6, X_4 = \frac{2e_7 + e_8}{\sqrt{3}}, X_5 = e_9, X_6 = e_{10},$$

where e_i denotes $\frac{\partial}{\partial x_i}$ for each i and if we choose

$$D = Span\{X_5, X_6\}, D^\perp = Span\{X_3, X_4\} \text{ and } D_\theta = Span\{X_1, X_2\},$$

then it is easy to verify that $TM = D \oplus D_\theta \oplus D^\perp$ and D is invariant distribution, $D^\perp$ is anti-invariant distribution and D_θ is slant distribution. For any $X \in \Gamma(D_\theta)$ we easily find

$$\phi^2 X = \frac{a^2}{a^2 + b^2 - 1} X.$$

Moreover we have the following cases:

1. *M is quasi-hemi slant submanifold of type 1 if $a^2 + b^2 > 1$ and $b^2 < 1$.*
2. *M is quasi-hemi slant submanifold of type 2 if $a^2 + b^2 > 1$ and $b^2 > 1$.*
3. *M is quasi-hemi slant submanifold of type 3 if $a^2 + b^2 < 1$.*

Example 4.2. *Let M be submanifold of $\mathbb{R}^{10}$ equipped with para Kaehlerian structure (J_1, g_1) defined by*

$$(r, s, t, u, v, w,) \mapsto (r, bs, as, s, 0, u, t, 2t, v, w), \tag{17}$$

where $a, b \in \mathbb{R}$ satisfying $(a^2 - b^2 \neq 1)$.
If we define

$$X_1 = \frac{be_2 + ae_3 + e_4}{\sqrt{(a^2 - b^2 - 1)}}, X_2 = e_1, X_3 = e_6, X_4 = \frac{e_7 + 2e_8}{\sqrt{3}}, X_5 = e_9, X_6 = e_{10},$$

if we choose

$$D = Span\{X_5, X_6\}, D^\perp = Span\{X_3, X_4\} \text{ and } D_\theta = Span\{X_1, X_2\},$$

then it is easy to verify that $TM = D \oplus D_\theta \oplus D^\perp$ and D is invariant distribution, $D^\perp$ is anti-invariant distribution and D_θ is slant distribution. For any $X \in \Gamma(D_\theta)$ we easily find

$$\phi^2 X = \frac{b^2}{-a^2 + b^2 + 1} X.$$

Moreover we have the following cases:

1. *M is quasi-hemi slant submanifold of type 1 if* $a^2 - b^2 < 1$ *and* $a^2 > 1$.
2. *M is quasi-hemi slant submanifold of type 2 if* $a^2 - b^2 < 1$ *and* $a^2 < 1$.
3. *M is quasi-hemi slant submanifold of type 3 if* $a^2 - b^2 > 1$.

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