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Constructing self-dual complexes and self-dual triangulations of manifolds

Marinko Timotijević[a], Rade Živaljević[b]

[a] *Faculty of Science, University of Kragujevac*
[b] *Mathematical Institute SASA, Belgrade*

Abstract. Simplicial complexes K, which are equal to their Alexander dual K^Λ are known as self-dual simplicial complexes. We prove that topological and combinatorial properties of any self-dual simplicial complex, are fully determined by topological and combinatorial properties of the link of any of it's vertices which happens to be sub-dual in smaller combinatorial ambient. Using this observation, we describe a general method for constructing self-dual triangulations of given topological spaces and focus on self-dual triangulations of compact manifolds. We show that there exist only 4 types of self-dual combinatorial manifolds and provide a general method for their construction.

1. Introduction

A simplicial complex $K \subset 2^{[n]}$ is (Alexander) self-dual if $K = K^\Lambda := \{[n] \setminus A \mid A \notin K\}$. Self-dual simplical complexes appear in many branches of mathematics as fundamental geometrical objects. In combinatorial (algebraic) topology, self-dual simplicial complexes provide fundamental examples of triangulated geometrical objects which are not embeddable in Euclidean spaces of prescribed dimension. More explicitly, see [1, Section 5], self-dual complexes on n vertices cannot be embedded in the Euclidean $(n-3)-$dimensional space. Moreover, as demonstrated by S. A. Melikhov in [2], self-dual complexes are subset-minimal examples of simplicial complexes which are not embeddable in $\mathbb{R}^{n-3}$ in the sense that every proper subcomplex of a self-dual complex in the ambient $[n]$ can be embedded in $\mathbb{R}^{n-3}$. Similar property holds for joins of self-dual complexes.

U. Brehm and W. Kühnel had proven in [9] that if $d-$dimensionad manifold different from sphere has a triangulation with n vertices than

$$n \geq 3\lceil d/2 \rceil + 3 \tag{1}$$

where equality holds for $d = 2, 4, 8, 16$ when manifold has homology structure like a real, complex, quaternionic and octanionic projective plane. For $d = 2$, we have a 6-vertex triangulation of the real projective plane (hemi-icosahedron, exhibited in Figure 1).

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Email addresses: marinko.timotijevic@pmf.kg.ac.rs (Marinko Timotijević), rade@turing.mi.sanu.ac.rs (Rade Živaljević)

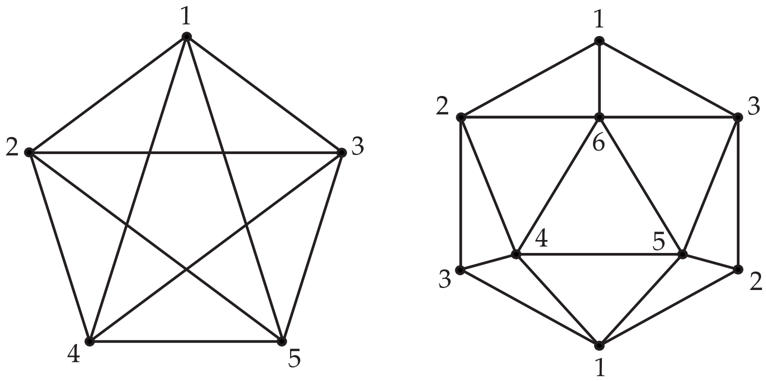

Figure 1: Complete graph K_5 and the hemi-icosahedron.

For $d = 4$ there is a unique 9-vertex triangulation of the complex projective plane, see [3–5, 17]. For $d = 8$, several fitting triangulations where described in [13] and in [14] was shown that one of them is actually a triangulation of the quaternionic projective plane. For $d = 16$, many suitable triangulations where discovered in [6] and it is conjectured that every one of them is a triangulation of the octanionic projective plane. As it turns out, every mentioned triangulation is a self-dual simplicial complex. Here we prove that there are no other self-dual complexes that are combinatorial manifolds and provide a general method for obtaining self-dual and consequently minimal triangulations of the projective planes.

1.1. *Overview of the Paper*

Chapter 2 gives a brief outline of the results published in [21]. There, we define sub-dual and self-dual simplicial complexes, provide several tools for their analysis and prove the main structural theorem (Theorem 2.8) which states that for any self-dual simplicial complex $K \subsetneq 2^{[n]}$ and any vertex $v \in [n]$, simplicial complex $\mathrm{Lk}(v)$ (i.e the link of a vertex $\{v\} \in K$) is sub-dual in the ambient $[n] \setminus \{v\}$ and if $\mathrm{Lk}(v)^\Lambda$ is the Alexander dual of $\mathrm{Lk}(v)$ in the ambient $[n] \setminus \{v\}$ then:

$$K = \mathrm{Lk}(v)^\Lambda \cup \mathrm{C}\,\mathrm{Lk}(v). \tag{2}$$

Thus, topological and combinatorial properties of self-dual simplicial complex are fully determined by topological and combinatorial properties of the link of any of its vertices. This in turn allows us to construct an operator referred to as "dual upgrade" which transforms sub dual simplicial complex $K \subsetneq 2^{[n]}$ into self-dual simplicial complex $\Lambda K \subsetneq 2^{[n]}$ given by

$$\Lambda K = K^\Lambda \cup \mathrm{C}K \tag{3}$$

where $\mathrm{C}K = K * \Big\{\emptyset, \{n\}\Big\}$ is the cone of a simplicial complex K.

In Chapter 3 we analyze the relations between f−vectors and consequently Euler characteristic of a simlicial complex and its Alexander dual (Proposition 3.2), describe f−vectors and Euler characteristic of self-dual simplicial complexes (Corollary 3.3) and analyze f−vector and Euler characteristic of self-dual upgrade ΛK in relation to f−vector and Euler characteristic of sub-dual complex K (Proposition 3.4). As it turns out, $f(K)$ and $\chi(K)$ completely determine $f(\Lambda K)$ and $\chi(\Lambda K)$. Also, if $K \subsetneq 2^{[n]}$ is a simlicial complex in an ambient of odd cardinality, then $\chi(K)$ determines $\chi(\mathrm{Lk}(v))$ for all $v \in [n]$ (Corollary 3.5).

In Chapter 4, using The Combinatorial Alexander duality (Theorem 4.1), we investigate the relations between homology and cohomology of a given sub-dual simplicial complex and homology and cohomology of its dual upgrade. Main result is the exact sequence (23) linking homology of a sub-dual simplicial complex K and its Alexander dual K^Λ with homology of its dual-upgrade ΛK. It is shown that homology (and consequently cohomology) of a sub-dual complex K and its placement within K^Λ determine the homology of ΛK and that, under specific circumstances, it is possible to construct a self-dual simplicial complex with prescribed homology groups (Theorem 4.3).

In Section 5, we analyze the existence of self-dual simplicial complexes which are combinatorial manifolds. Main result is Theorem 5.2 where we prove that there are only four types of such manifolds: one nonorientable in dimension 2 on 6 vertices which is isomorphic to the hemi icosahedron i.e. minimal triangulation of the real projective plane and three types of orientable combinatorial manifolds of dimensions $d = 4, 8, 16$ with $n = 9, 15, 27$ vertexes which are projective-like meaning they have non-trivial reduced homology groups isomorphic to $\mathbb{Z}$ in dimensions d and $d/2$.

In Section 6 we describe a method for constructing $d-$dimensional combinatorial self-dual manifolds $\mathbb{M}^d$ introduced in Section 5. As it turns out, to construct $\mathbb{M}^d$ in the ambient $[n]$, it is sufficient to construct a dual-upgrade of $(n-d-3)-$neighbourly sub-dual combinatorial sphere $\mathbb{S}^{d-1}$. Using results obtained in Section 3 and Dehn-Sommervile equations (50) we calculated $f-$vectors of all self-dual manifolds and $f-$vectors of the spheres they are obtained form.

1.2. Related papers and results

The paper is based on unpublished results of the first author's Ph.D. degree thesis [20], which was completed under the mentorship of the second author. The first author acknowledges kind remarks and useful suggestions of Wolfgang Kühnel who proposed, in a letter following the publication of [21], some new ideas for future research. F. Chapoton and L. Manivel have previously studied [22, Section 7] the relationship between the f-vectors of minimal triangulations of "projective planes" (over Hurwitz algebras) and the f-vectors of the corresponding spherical links. If the action of the symmetry group is vertex-transitive than all spherical links are isomorphic, allowing the authors to compute this f-vector, see [22, Figure 5].

Our approach, following into footsteps of [21], allows us to calculate (in Section 6.4) this f-vector without any assumption on the action of the symmetry group. Alexander Gaifullin in [6] also obtained this result (see [6, Table 1]) and constructed 634 vertex-transitive, and more than 10^{103} vertex non-transitive, combinatorial 16-manifolds like the octonionic projective plane. Gaifullin's conjecture is that all the constructed triangulations are PL homeomorphic to the octonionic projective plane.

Our approach, based on dual upgrades and elementary homology theory, is self-contained and may serve as an introduction to this attractive area of combinatorial topology.

2. Basic Definitions and the Main Theorem

The terminology used in this paper is mostly standard and the reader is referred to [1] for all undefined concepts. Recall that a simplicial complex $K \subseteq 2^{[n]}$ is any family of subsets of $[n] = \{1, 2, \ldots, n\}$ such that:

$$(\forall A \in K)(\forall B \subseteq [n])\, B \subseteq A \Rightarrow B \in K. \tag{4}$$

We however emphasize that the set of vertices $Vert(K) = \{v \in [n] \mid \{v\} \in K\}$ of K can in general be a proper subset of the ambient set $[n]$.

Definition 2.1. *The Alexander dual (or simply dual) of a complex $K \subseteq 2^{[n]}$ is the simplicial complex $K^\Lambda \subseteq 2^{[n]}$ where*

$$K^\Lambda = \{[n] \setminus A \mid A \notin K\}. \tag{5}$$

When we want to emphasize the ambient set $V \subseteq [n]$, the Alexander dual of the complex K is denoted by K^{Λ_V}. Theorem 4.1 shows that combinatorial properties of K^{Λ_V} is largely dependent on the cardinality of the ambient set V.

Definition 2.2. *Let $K \subseteq 2^V$ be a simplicial complex. We say that the complex K is:*

- ***sub-dual*** *in the ambient* V *if* $K \subseteq K^{\Lambda_V}$*;*
- ***self-dual*** *in the ambient* V *if* $K = K^{\Lambda_V}$*.*

There are simplicial complexes which are neither self or sub-dual.

Example 2.3. Let $\binom{[n]}{k}$ be the $(k-1)$-skeleton of the complex $\Delta_{n-1} = 2^{[n]}$. Then, by Definition 2.1, its Alexander dual in the ambient $[n]$ is

$$\binom{[n]}{k}^{\Lambda} = \{[n] \setminus A \mid |A| > k\} = \{A \mid |A| \leq n-k-1\} = \binom{[n]}{n-k-1}. \tag{6}$$

Therefore, (in the ambient $[n]$) the complex $\binom{[n]}{k}$ is sub-dual iff $2k+1 \leq n$ and self-dual iff $2k+1 = n$. Specially, if $k = 2$ we obtain the complex $\binom{[5]}{2} = K_5$, a complete graph on 5 vertexes shown on Figure 1.

If a given simplicial complex is sub-dual in the ambient $[n]$, all of its subcomplexes must also be sub-dual in the ambient $[n]$ because $K \subseteq L$ implies $L^{\Lambda} \subseteq K^{\Lambda}$. Therefore, using the Example 2.3, we obtain the following proposition.

Proposition 2.4. *A simplicial complex* K *of dimension* d *is always sub-dual in the ambient* $[n]$ *where* $n \geqslant 2d+3$*.*

The following theorem provides an efficient criterion for verifying sub and self-duality of a given simplicial complex.

Theorem 2.5. *Let* $K \subseteq 2^V$ *be a simplicial complex. In the ambient* V *the complex* K *is:*

(1) *sub-dual iff there is no simplex* $A \subseteq V$ *such that* $A \in K$ *and* $V \setminus A$ *belong to* K*;*

(2) *self-dual iff for arbitrary* $A \subseteq V$ *exactly one of the simplexes* A *or* $V \setminus A$ *belongs to* K *or equivalently*

$$(\forall A \subseteq V) A \in K \iff V \setminus A \notin K. \tag{7}$$

Proof:
(1)($\Rightarrow$) Let $K \subseteq K^{\Lambda}$ and let $A \subseteq V$ such that $A, V \setminus A \in K$. Then, because K is a subcomplex of K^{Λ}, we have $V \setminus A, A \in K^{\Lambda}$ which by Definition 2.1 implies that A and $V \setminus A$ do not belong to K contradicting our assumption.

($\Leftarrow$) Suppose there is no simplex $A \subseteq V$ such that A and $V \setminus A$ are in K. Then, for arbitrary $A \in K$, the simplex $V \setminus A$ is not in K which implies that $V \setminus (V \setminus A) = A$ is in K^{Λ}. Therefore $K \subseteq K^{\Lambda}$.

(2) Complex K will be self-dual if it is sub-dual. Therefore, for an arbitrary simplex $A \subseteq V$, at least one of the simplexes if A or $V \setminus A$ belongs to K. If both of them are in K then both of them will not be in $K^{\Lambda} = K$ which is not possible. □

Example 2.6. The complex $\Delta_{n-1} = 2^{[n]}$ is self-dual in the ambient $[n+1]$ and sub-dual in the ambient $[n+2]$.

Indeed, following Theorem 2.5, for arbitrary $A \subseteq [n+1]$, the set A does not contain the vertex $\{n+1\}$ iff $[n+1] \setminus A$ contains $\{n+1\}$ or equivalently, $A \in 2^{[n]}$ iff $[n+1] \setminus A \notin 2^{[n]}$ which confirms **(2)**. If, $\Delta_{n-1} = 2^{[n]} \subset 2^{[n+2]}$, then Δ_{n-1} is sub-complex of self-dual complex $2^{[n+1]}$ in the ambient $[n+2]$ ant therefore is sub-dual in the ambient $[n+2]$.

From Theorem 2.5 and Definition 2.1 we conclude that if a simplicial complex $K \subset 2^{[n]}$ is sub-dual, then it's Alexander dual K^{Λ} has to contain at least one of the simplexes $A, [n] \setminus A$ forming a partition of $[n]$ into disjoint subsets. The complexes with this property are also called 2-unavoidable, see [7] and [8] where these and more general r-unavoidable complexes are studied. In this setting, the self-dual simplicial complexes correspond to minimal 2−unavoidable complexes.

Here we are reminded of a simple but important concept of a link of a simplex in a given simplicial complex.

Definition 2.7. *Let $K \subseteq 2^{[n]}$ be a simplicial complex. The link of a simplex $A \in K$ is a simplicial complex $\mathrm{Lk}(A) \subset K$ given by*

$$\mathrm{Lk}(A) = \{B \in K \mid B \cap A = \emptyset, B \cup A \in K\}. \tag{8}$$

In case of a vertex $\{v\} \in [n]$ we will write $\mathrm{Lk}(v)$ instead of $\mathrm{Lk}(\{v\})$ for simplicity.

Following theorem, first proven in [21], provides a key insight into combinatorial structure of self-dual simplicial complexes and is fundamental for this paper.

Theorem 2.8. *(**The Structural Theorem**) Let $K \subset 2^{[n]}$ be a self-dual simplicial complex and $\{v\} \subset [n]$ any vertex. Then, simplicial complex $\mathrm{Lk}(v)$ is sub-dual in the ambient $[n] \setminus \{v\}$ and*

$$K = \mathrm{Lk}(v)^{\Lambda_{[n]\setminus\{v\}}} \cup \mathrm{C}\,\mathrm{Lk}(v) \tag{9}$$

where $\mathrm{Lk}(v)^{\Lambda_{[n]\setminus\{v\}}}$ is the Alexander dual of a simplicial complex $\mathrm{Lk}(v)$ in the ambient $[n] \setminus \{v\}$ and $\mathrm{C}\,\mathrm{Lk}(v) = \mathrm{Lk}(v) \cup \{A \cup \{v\} \mid A \in \mathrm{Lk}(v)\}$ is the cone of a simplicial complex $\mathrm{Lk}(v)$.

Conversely, if $K \subset 2^{[n-1]}$ is sub-dual simplicial complex in the ambient $[n-1]$, then

$$\Lambda K = K^{\Lambda_{[n-1]}} \cup \mathrm{C}K, \tag{10}$$

where $\mathrm{C}K = K * \left\{\emptyset, \{n\}\right\}$, is self-dual simplicial complex in the ambient $[n]$.

Proof: Let $K \subset 2^{[n]}$ be self-dual. We will prove the claim for the vertex n. First, simplicial complex $\mathrm{Lk}(n) \in 2^{[n-1]}$ is sub-dual in the ambient $[n-1]$. Indeed, if $\mathrm{Lk}(n)$ contains simplexes A and $[n-1] \setminus A$ then, by Definition 2.7, complex K will contain simplexes $A \cup \{n\}$ and $[n] \setminus (A \cup \{n\})$ contradicting the assumption that K is self-dual. Now, let $A \subset [n]$ be a simplex of the complex K. If $n \in A$, then $A \in \mathrm{C}\,\mathrm{Lk}(n)$. Let $n \notin A$. If $A \notin \mathrm{Lk}(n)^{\Lambda_{[n-1]}}$, then $[n-1] \setminus A \in \mathrm{Lk}(n)$ implying $([n-1] \setminus A) \cup \{n\} = [n] \setminus A \in K$ which cannot be true because $A \in K$ and K is self-dual. Therefore, $A \in \mathrm{Lk}(n)^{\Lambda_{[n-1]}}$ and we have proven that $K \subseteq \mathrm{Lk}(n)^{\Lambda_{[n-1]}} \cup \mathrm{C}\,\mathrm{Lk}(n)$.

If there exists $A \in \mathrm{Lk}(n)^{\Lambda_{[n-1]}} \subset 2^{[n-1]}$ such that $A \notin K$, then $[n-1] \setminus A \notin \mathrm{Lk}(n)$ and $[n] \setminus A \in K$. Because $n \notin A$ and $[n] \setminus A = ([n-1] \setminus A) \cup \{n\} \in K$ we have $[n-1] \setminus A \in \mathrm{Lk}(n)$ which is not possible. Therefore $\mathrm{Lk}(n)^{\Lambda_{[n-1]}} \subset K$ proving that $\mathrm{Lk}(v)^{\Lambda_{[n-1]}} \cup \mathrm{C}\,\mathrm{Lk}(v) \subseteq K$.

Now, let $K \subseteq 2^{[n-1]}$ be a sub-dual simplicial complex. Let us prove that $\Lambda K = K^{\Lambda_{[n-1]}} \cup \mathrm{C}K$ is self-dual in the ambient $[n]$. Let $A \subseteq [n]$ be arbitrary.

- Let $n \in A$. If $A \notin \Lambda K$ then $A \notin \mathrm{C}K$ implying $A \setminus \{n\} \notin K$ and thus we have $[n-1] \setminus (A \setminus \{n\}) = [n] \setminus A \in K^{\Lambda_{[n-1]}} \subseteq \Lambda K$. Therefore, $[n] \setminus A \in \Lambda K$. If $A \in \Lambda K$ then $A \in \mathrm{C}K$ implying $A \setminus \{n\} \in K$. If $[n] \setminus A = [n-1] \setminus (A \setminus \{n\})$ also belongs to ΛK, then $[n-1] \setminus (A \setminus \{n\}) \in K^{\Lambda_{[n-1]}}$ and this will happen if $[n-1] \setminus ([n-1] \setminus (A \setminus \{n\})) = A \setminus \{n\} \notin K$ which is not true. Therefore, $[n] \setminus A$ cannot be in ΛK.
- Let $n \notin A$ or equivalently $A \subseteq [n-1]$. If $A \notin \Lambda K$ then, A is a subset of $[n-1]$ such that $A \notin K^{\Lambda_{[n-1]}}$ implying its complement, $[n-1] \setminus A$ belongs to the Alexander dual of the complex $K^{\Lambda_{[n-1]}}$ which is equal to K. Thus we have $([n-1] \setminus A) \cup \{n\} = [n] \setminus A$ is a simplex of $\mathrm{C}K \subseteq \Lambda K$. If $A \in \Lambda K$, since $A \subset [n-1]$, we have $A \in K^{\Lambda_{[n-1]}}$ because K is sub-dual ($K \subseteq K^{\Lambda_{[n-1]}}$). Thus, $[n-1] \setminus A \notin K$. If $[n] \setminus A \in \Lambda K$, because $n \in [n] \setminus A$ we have $[n] \setminus A \in \mathrm{C}K$ and therefore $([n] \setminus A) \setminus \{n\} = [n-1] \setminus A \in K$ which is not true. Therefore, $[n] \setminus A \notin \Lambda K$.

Thus, we have shown that for any simplex $A \subseteq [n]$, $A \in \Lambda K$ iff $[n] \setminus A \notin \Lambda K$ proving that ΛK is self-dual in the ambient $[n]$. □

As a consequence of Theorem 2.8, we conclude that any self-dual simplicial complex, its topological and combinatorial properties, are fully determined by the topological and combinatorial properties of the link of any of it's vertices. For example, if we want to prove that two self-dual simplicial complexes are isomorphic, it is sufficient to prove that the link of a vertex in the first one is isomorphic to the link of a vertex in the second one simplifying the combinatorial classification of self-dual complexes.

Therefore, in the following two chapters, we will explore topological and combinatorial relations between a given sub-dual simplicial complex K and its self-dual upgrade, simplicial complex ΛK.

3. f-vectors of Dual Upgrades

For a given simplicial complex K in the ambient $[n]$, we define it's $f-$vecor $f(K) \in \mathbb{N}_0^{n+1}$ as:

$$f(K) = (f_0, f_1, \dots, f_n) \tag{11}$$

where $f(K)_i = f_i$ is the number of simplexes of the complex K of dimension $i-1$ (or cardinality i) for $i = 0, 1, \dots, n$. Then, the Euler-characteristic of the complex K is the alternating sum:

$$\chi(K) = \sum_{i=1}^{n} (-1)^{i+1} f_i. \tag{12}$$

Several elementary properties od the $f-$vektor and Euler characteristic are given in the following Lemma.

Lemma 3.1. *For arbitrary simplicial complexes K and L in the ambient $[n]$ we have*

$$f(K \cup L) = f(K) + f(L) - f(K \cap L), \quad \textit{and consequently} \quad \chi(K \cup L) = \chi(K) + \chi(L) - \chi(K \cap L). \tag{13}$$

If K is a simplicial complex and $L \subseteq K$ is arbitrary subset of simplexes then

$$f(K \setminus L) = f(K) - f(L). \tag{14}$$

Let $K \subseteq 2^{[n]}$ be a simplicial complex and $(f_0, f_1, \dots, f_n)$ its $f-$vektor. Form Definition 2.1, the Alexander dual of the complex K can also be expressed as $2^{[n]} \setminus \{[n] \setminus A \mid A \in K\}$. Thus we have:

$$f(K^\Lambda) = f(2^{[n]}) - f\Big(\{[n] \setminus A \mid A \in K\}\Big). \tag{15}$$

Notice that $i-$th coordinate of the vector $f(\{[n] \setminus A \mid A \in K\})$ is equal to the number of simplexes of the family $\{[n] \setminus A \mid A \in K\}$ of cardinality i and that is equal to the number of simplexes of the complex K of cardinality $n-i$ which by definition is f_{n-i}. Thus we have obtained part **(1)** of the following proposition.

Proposition 3.2. *For arbitrary simplicial complex $K \subseteq 2^{[n]}$ with $f-$vector $(f_0, f_1, \dots, f_n)$ we have:*

(1) $f(K^\Lambda)_i = \binom{n}{i} - f_{n-i}$ *for all* $i = 0, 1, \dots, n$;

(2) $\chi(K^\Lambda) = (-1)^{n+1}\Big(\chi(K) - f_0\Big) - f_n + 1$;

(3) *if* $\emptyset \neq K \subset 2^{[n]}$, *then* $\chi(K^\Lambda) = \begin{cases} \chi(K), & n \equiv 1 \mod 2, \\ -\chi(K) + 2, & n \equiv 0 \mod 2. \end{cases}$

Proof: For the part **(2)**, using **(1)**, we have:

$$\begin{aligned}\chi(K^{\Lambda}) &= \sum_{i=1}^{n} f(K^{\Lambda})_i = \sum_{i=1}^{n}(-1)^{i+1}\left(\binom{n}{i} - f_{n-i}\right) = -\sum_{i=1}^{n}\binom{n}{i}(-1)^i + \sum_{i=1}^{n}(-1)^i f_{n-i}\\ &= 1-(-1+1)^n + (-1)^{n+1}\sum_{i=1}^{n}(-1)^{n-i+1}f_{n-i} = 1 + (-1)^{n+1}\sum_{i=0}^{n-1}(-1)^{i+1}f_i\\ &= 1 + (-1)^{n+2}f_0 + (-1)^{n+1}\sum_{i=0}^{n}(-1)^{i+1}f_i - (-1)^{2n+2}f_n\\ &= (-1)^{n+1}\big(\chi(K) - f_0\big) - f_n + 1. \end{aligned} \tag{16}$$

Part **(3)** follows form part **(2)** because f−vektor of a simplicial complex $\emptyset \neq K \subset 2^{[n]}$ satisfies $f_0 = 1$ and $f_n = 0$ because $[n] \notin K$. □

Interesting consequence of previous proposition is that f−vector of self-dual simplicial complex $K \subset 2^{[n]}$ is fully determined by the f−vector of its $\lceil n/2 \rceil$−skeleton. Also, self-dual complexes in the ambient of even cardinality cannot have arbitrary Euler characteristic.

Corollary 3.3. *Let $K \subseteq 2^{[n]}$ be a self-dual simplicial complex with f−vector $(f_0, f_2, \dots, f_n)$. Then:*

(1) $f_i + f_{n-i} = \binom{n}{i}$ *for all* $i = 0, 1, \dots, n$;

(2) *if* $n = 2k$, *then* $\chi(K) = 2$;

(3) *if* $n = 2k + 1$, *then* $\chi(K)$ *is odd.*

Let us analyze f−vecots of self-dual upgrades. In light of the Theorem 2.8, our goal is to describe f−vector and consequently Euler characteristic of self-dual simplicial complex using f−vector of the link of its vertex.

Proposition 3.4. *Let $K \subseteq 2^{[n-1]}$ be arbitrary sub-dual simplicial complex with f−vector $f(K) = (f_0, f_1, \dots, f_{n-1})$. Then,*

(1) $f(\Lambda K)_i = \begin{cases} 1, & i = 0,\\ \binom{n-1}{i} - f_{n-1-i} + f_{i-1}, & i = 1, \dots, n-1,\\ 0, & i = n; \end{cases}$

(2) $\chi(\Lambda K) = \begin{cases} 2, & n \equiv 0 \mod 2,\\ -2\chi(K) + 3, & n \equiv 1 \mod 2. \end{cases}$

Proof: Let $\emptyset \neq K \subseteq 2^{[n-1]}$ be a sub-dual simplicial complex and $f(K) = (f_0, f_1, \dots, f_{n-1})$ it's f−vector. First note that complex CK (join if the complex K and $\{\emptyset, \{n\}\}$), has $f_i + f_{i-1}$ simplexes of cardinality i for all $i = 1, \dots, n$.

Because $\Lambda(K) = K^{\Lambda_{[n-1]}} \cup CK$ and $K^{\Lambda_{[n-1]}} \cap CK = K$, by Proposition 3.2 **(1)**, for all $i = 0, 1, \dots, n-1$ we obtain:

$$\begin{aligned} f(\Lambda K)_i &= f(K^{\Lambda_{[n-1]}})_i + f(CK)_i - f(K)_i\\ &= \binom{n-1}{i} - f_{n-1-i} + f_i + f_{i-1} - f_i\\ &= \binom{n-1}{i} - f_{n-1-i} + f_{i-1}. \end{aligned} \tag{17}$$

Since the simplicial complex ΛK is self-dual in the ambient $[n]$, it cannot contain the simplex $[n]$ implying $f(\Lambda K)_n = 0$. Similarly, $f(\Lambda K)_0 = 1$ because $\Lambda K = \emptyset$ is not self-dual.

By Lemma 3.1, Euler characteristic of ΛK is equal to $\chi(K^{\Lambda_{[n-1]}}) + \chi(CK) - \chi(K)$. Because $f_n = 0$ and $f_0 = 1$, by Proposition 3.2, $\chi(K^{\Lambda_{[n-1]}}) = (-1)^n(\chi(K) - 1) + 1$. Concerning the cone CK, it's Euler characteristic is 1 because CK is triangulation of a contractible space. Thus we have:

$$\chi(\Lambda K) = (-1)^n\Big(\chi(K) - 1\Big) + 2 - \chi(K) \tag{18}$$

and property **(2)** as a consequence. □

Following Proposition 3.4, to find Euler characteristic of self-dual simplicial complex in the ambient of odd cardinality, it is sufficient to find Euler characteristic of the link of any of its vertices. Conversely, Euler characteristic of self-dual simplicial complex in an odd ambient determines the Euler characteristic of the link of any of its vertices.

Corollary 3.5. *For any self-dual simplicial complex $K \subseteq 2^{[2n+1]}$ and arbitrary $v \in [2n+1]$,*

$$\chi\Big(\mathrm{Lk}(v)\Big) = \frac{1}{2}\Big(3 - \chi(K)\Big). \tag{19}$$

Because every self-dual simplicial complex in the ambient $[2n+1]$ is a dual-upgrade of every sub-dual simplicial complex in the ambient $[2n]$, following Proposition 3.4, Euler characteristic of self-dual simplicial complex in an odd ambient can, in general, be an arbitrary odd number.

4. Homology and Cohomology of Dual Upgrades

In this section we make a simplified presentation of the relationship between the homology and cohomology of a given simplicial complex and its self-dual upgrade, described in Theorem 2.8. For a more detailed description, the reader is referred to [21].

Following theorem, known as *The Combinatorial Alexander Duality*, was originally introduced in [11]. For the proof, the reader is referred to [12].

Theorem 4.1. *Let K be a simplicial complex in the ambient $[n]$. Then*

$$H_k(K) \approx H^{n-3-k}(K^{\Lambda}) \tag{20}$$

where H_k and H^k represent the reduced homology and cohomology groups over integers.

By comparing Theorem 4.1 with original Alexander duality, we see that Alexander dual of a simplicial complex K provides sufficiently good combinatorial model of its complement within the Bier sphere $Bier(K) = K *_{\Delta} K^{\Lambda}$ (see [1]).

Let K be a simplicial complex in the ambient $[n]$, we my assume that K is sub-dual since, by Example 2.6, sub-duality can be achieved by enlarging the ambient $[n]$. Then, its dual upgrade ΛK is self-dual in the ambient $[n+1]$.

Corollary 4.2. *Let K be a sub-dual simplicial complex in the ambient $[n]$. Then, for its dual-upgrade ΛK we have:*

$$H_k\Big(\Lambda(K)\Big) \approx H^{n-k-2}\Big(\Lambda(K)\Big). \tag{21}$$

Since $K \subseteq K^\Lambda$, let us consider the long exact sequence for the pair (K^Λ, K):

$$\cdots \to H_k\Big(K^\Lambda\Big) \to H_k\Big(K^\Lambda, K\Big) \to H_{k-1}(K) \to H_{k-1}\Big(K^\Lambda\Big) \to \cdots \tag{22}$$

Groups $H_k(K^\Lambda, K)$ are isomorphic to $H_k(K^\Lambda/K)$ and the factor space K^Λ/K is homotopically equivalent to the space $K^\Lambda \cup CK$, which is precisely ΛK. Therefore, by replacing $H_k(K^\Lambda, K)$ with $H_k(\Lambda K)$ and trough composition with appropriate isomorphisms, we obtain the exact sequence:

$$\cdots \to H_k\Big(K^\Lambda\Big) \to H_k\Big(\Lambda K\Big) \to H_{k-1}(K) \to H_{k-1}\Big(K^\Lambda\Big) \to \cdots \tag{23}$$

Using Theorem 4.1 and The Universal Coefficient Theorem, homology groups $K_k(K^\Lambda)$ are easily determined. Therefore, homology of simplicial complex K, it's placement within K^Λ fully determine the homology of it's self-dual upgrade ΛK.

Following theorem demonstrates the usage of the sequence (23) for constructing specific self-dual simplicial complexes with prescribed homology groups satisfying Corollary 4.2.

Theorem 4.3. *Let K be a simplicial complex of dimension d in the ambient $[n]$ where $n \geq 2d + 3$. Then ΛK has the same homology and cohomology groups as the space $K^\Lambda \vee \Sigma K$ where $\vee$ is the wedge sum of spaces.*

Proof: By Proposition 2.4 the complex K is self dual in the ambient $[n]$.

Moreover, since the dimension of K is d, all groups $H_k(K)$ are trivial for $k > d$. Also, if $n \geq 2d + 3$, then by Theorem 4.1 and the Universal coefficient theorem, the only non trivial homology groups of K^Λ are in dimensions $n-3, n-4, \dots, n-d-3$ (note that $H_d(K)$ is torsion-free). Since $n-d-3 \geq 2d+3-d-3 = d$, we conclude that in the long exact sequence (23) for the pair (K^Λ, K) groups $H_k(K^\Lambda)$ and $H_{k-1}(K^\Lambda)$ are trivial or $H_k(K)$ and $H_{k-1}(K)$ are trivial. This implies that $H_k(K^\Lambda)$ is isomorphic to $H_{k-1}(K)$ if $k < d$ or $H_k(K^\Lambda)$ if $k \geq d$ which completes the proof. □

5. Existence of Self-dual Manifolds

In this chapter, using described mathematical apparatus, we analyze the existence and combinatorial properties of self-dual combinatorial manifolds.

Let $\mathbb{M}^d \subset 2^{[n]}$ be a connected $d-$dimensional combinatorial manifold which is self-dual in the ambient $[n]$. Then, link of a vertex n is a combinatorial $(d-1)-$dimensional sphere $\mathbb{S}^{d-1} \subset 2^{[n-1]}$ and by Theorem 2.8 simplicial complex $\mathbb{M}^d$ has a form

$$\mathbb{M}^d = (\mathbb{S}^{d-1})^\Lambda \cup C\mathbb{S}^{d-1}. \tag{24}$$

where $(\mathbb{S}^{d-1})^\Lambda$ is the Alexander dual of the sphere $\mathbb{S}^{d-1}$ in the ambient $[n-1]$ and $\mathbb{S}^{d-1}$ is sub-dual ie. $\mathbb{S}^{d-1} \subset (\mathbb{S}^{d-1})^\Lambda$. Since groups $H_k(\mathbb{S}^{d-1})$ are torsion-free, using The Universal Coefficients Theorem and The Combinatorial Alexander Duality (Theorem 4.1) we conclude that:

$$H_k\Big((\mathbb{S}^{d-1})^\Lambda\Big) = \begin{cases} \mathbb{O}, & k \neq n-d-3, \\ \mathbb{Z}, & k = n-d-3. \end{cases} \tag{25}$$

implying that $(\mathbb{S}^{d-1})^\Lambda$ has homology and co-homology groups as a sphere $\mathbb{S}^{n-d-3}$. It can be proven that $(\mathbb{S}^{d-1})^\Lambda$ is homotopically equivalent to a $(n-d-3)-$dimensional sphere. For convenience, let us label $(\mathbb{S}^{d-1})^\Lambda$ with $\mathbb{S}_*^{n-d-3}$. Thus, manifold $\mathbb{M}^d = \mathbb{S}_*^{n-d-3} \cup C\mathbb{S}^{d-1}$ is homotopically equivalent to the space $\mathbb{S}_*^{n-d-3}/\mathbb{S}^{d-1}$.

From the long exact sequence (23) for the pair $(\mathbb{S}_*^{n-d-3}, \mathbb{S}^{d-1})$ we have

$$\cdots \to H_k(\mathbb{S}_*^{n-d-3}) \to H_k(\mathbb{M}^d) \to H_{k-1}(\mathbb{S}^{d-1}) \to H_{k-1}(\mathbb{S}_*^{n-d-3}) \to \cdots \tag{26}$$

Since $H_{n-d-3}(\mathbb{S}_*^{n-d-3}) = \mathbb{Z}$ and $H_{d-1}(\mathbb{S}^{d-1}) = \mathbb{Z}$ are only non-trivial reduced homology groups of $\mathbb{S}_*^{n-d-3}$ and $\mathbb{S}^{d-1}$, the combined rank of homology groups $H_k(\mathbb{M}^d)$ can be at most two.

Manifold $\mathbb{M}^d$ cannot be homeomorphic to a sphere because in that case $H_d(\mathbb{M}^d)$ is the only non-trivial homology group of $\mathbb{M}^d$ and then arbitrary placement of the group $H_{n-d-3}(\mathbb{S}_*^{n-d-3})$ produces an impossible exact sequence.

Now we focus on two distinct cases.

Let $\mathbb{M}$ be non-orientable. Then, $H_d(\mathbb{M}^d)$ is trivial and $H_{d-1}(\mathbb{M}^d)$ contains a $\mathbb{Z}_2$ summand forcing $H_{d-1}(\mathbb{S}_*^{n-d-3})$ to be non trivial or equivalently $n - d - 3 = d - 1$. Then, the sequence (26) has a form:

$$\mathbb{O} \to \mathbb{Z} \to \mathbb{Z} \to H_{d-1}(\mathbb{M}^d) \to \mathbb{O}. \tag{27}$$

For previous sequence to be exact, group $H_{d-1}(\mathbb{M}^d)$ has to be isomorphic to $\mathbb{Z}_2$.

Thus, non-orientable self-dual combinatorial manifold $\mathbb{M}^d$ exists in the ambient $[2d + 2]$ and has only one non-trivial homology group $\mathbb{Z}_2$ in dimension $d - 1$. Since the real projective plane $\mathbb{RP}^2$ is a unique non-orientable manifold with only one non trivial homology group $\mathbb{Z}_2$, we conclude that there can be only one self-dual non-orientable manifold on $n = 6$ vertices and its dimension is $d = 2$.

Let $\mathbb{M}^d$ be orientable. In this case, $H_d(\mathbb{M}^d)$ is isomorphic to $\mathbb{Z}$ implying $H_k(\mathbb{S}_*^{n-d-3})$ cannot be non-trivial for $k \geq d$. Also, $H_{d-1}(\mathbb{S}_*^{n-d-3})$ cannot be non-trivial because the sequence (26) will become

$$\mathbb{O} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to H_{d-1}(\mathbb{M}^d) \to \mathbb{O} \tag{28}$$

and such exact sequence does not exist regardless of the group $H_{d-1}(\mathbb{M}^d)$. Thus, non trivial group $H_k(\mathbb{S}_*^{n-d-3})$ is in dimension $k < d - 1$ and the sequence (26) has a form

$$\mathbb{O} \to H_d(\mathbb{M}^d) \to \mathbb{Z} \to \mathbb{O} \to \cdots \to \mathbb{O} \to \mathbb{Z} \to H_{n-d-3}(\mathbb{M}^d) \to \mathbb{O} \tag{29}$$

implying that reduced homology groups of manifold $\mathbb{M}^d$ are

$$H_k(\mathbb{M}^d) = \begin{cases} \mathbb{O}, & k \neq d, n-d-3, \\ \mathbb{Z}, & k = d, n-d-3. \end{cases} \tag{30}$$

Note that, in this form, groups $H_k(\mathbb{M}^d)$ satisfy Corollary 4.2 arising form The Combinatorial Alexander duality. However, non-reduced homology and co-homology groups of orientable manifolds (labeled here H'_k and H'^k) satisfy the Poincaré Duality:

$$H'^k(\mathbb{M}) \approx H'_{d-k}(\mathbb{M}) \quad \text{for} \quad k = 0, 1, \ldots, d. \tag{31}$$

Using the Universal coefficient theorem, from Poincaré duality we conclude that groups $H_{n-d-3}(\mathbb{M}^d)$ and $H_{d-(n-d-3)}(\mathbb{M}^d)$ have to be isomorphic and this will be the case when $d - (n - d - 3) = n - d - 3$.

Thus, orientable self-dual combinatorial manifold $\mathbb{M}^d$ is of an even dimension, exists in the ambient $[3d/2 + 3]$ and has only two non-trivial homology groups $\mathbb{Z}$ in dimensions d and $d/2$.

Following theorem vas proven by, U. Brehm, W. Kühnel in [9].

Theorem 5.1. *If a combinatorial manyfold of dimension d which is not homeomorphic to a sphere has a triangulation on n vertexes then:*

$$n \geq 3\lceil d/2 \rceil + 3 \tag{32}$$

where equality holds only in dimensions $d = 2, 4, 8, 16$.

Since for our d−dimensional self-dual orientable manifolds $\mathbb{M}^d \subset 2^{[n]}$ we have:

$$n = 3d/2 + 3 \tag{33}$$

we conclude that their dimension has to be equal to $d = 4, 8, 16$.

Thus, we have proven that there is only four homological types of self-dual combinatorial manifolds and all of them are "projective-like".

Theorem 5.2. *Let $\mathbb{M}^d \subseteq 2^{[n]}$ be a self-dual combinatorial manifold.*

- *If $\mathbb{M}^d$ is non-orientable, then $d = 2$, $n = 6$ and $\mathbb{M}^2$ has only one non trivial homology group $H_1(\mathbb{M}^2) \approx \mathbb{Z}_2$.*
- *If $\mathbb{M}^d$ is orientable, then $d = 4, 8, 16$, $n = 3d/2 + 3$, and non trivial homology groups of $\mathbb{M}^d$ are $H_k(\mathbb{M}^d) \approx \mathbb{Z}$ for $k = d, d/2$ and Euler characteristic of $\mathbb{M}^d$ is 3.*

6. Construction of Self-dual combinatorial Manifolds

In this section we turn our attention on construction of self-dual manifolds $\mathbb{M}^d \subseteq 2^{[n]}$ where $n = 3d/2 + 3$ described in Theorem 5.2.

Since combinatorial manifold $\mathbb{M}^d \subseteq 2^{[n]}$ is a pure simplicial complex, if $A \subseteq [n]$ is a simplex of dimension $d + 1$, then $A \notin \mathbb{M}^d$ which by Theorem 2.5 **(2)** implies that $[n] \setminus A \in K$. Thus, the complex $\mathbb{M}^d$ contains every simplex $A \subseteq [n]$ where $|A| \leq n - d - 2$ implying that:

$$\binom{[n]}{\leq n - d - 2} \subset \mathbb{M}^d. \tag{34}$$

Therefore, self-dual manifold $\mathbb{M}^d \subseteq 2^{[n]}$ is $(n - d - 2)$−neighbourly. Also, if $A \subseteq [n]$ is a simplex of dimension $d - 1$, since $\mathbb{M}^d$ is a combinatorial manifold, simplex A is a face of exactly two simplexes of a simplicial complex $\mathbb{M}^d$. If $f(\mathbb{M}^d) = (f_0, f_1, \ldots, f_n)$ is the f−vector of the simplicial complex $\mathbb{M}^d$, by Corollary 3.3 and Theorem 5.2, numbers f_i satisfy the following equations:

$$\begin{aligned}
&\textstyle\sum_{i=1}^{n}(-1)^{i+1} f_i = 3;\\
&2f_d = (d + 1)f_{d+1};\\
&f_i = \tbinom{n}{i}, \text{ for all } i = 0, 1, \ldots, n - d - 2;\\
&f_i + f_{n-i} = \tbinom{n}{i}, \text{ for all } i = n - d - 1, \ldots, d + 1;\\
&f_i = 0, \text{ for all } i = d + 2, \ldots, n.
\end{aligned} \tag{35}$$

By Theorem 2.8, self-dual simplicial complex $\mathbb{M}^d$ is a dual-upgrade of a sub-dual combinatorial sphere $\mathbb{S}^{d-1} \subseteq 2^{[n-1]}$ i.e.

$$\mathbb{M}^d = (\mathbb{S}^{d-1})^{\wedge} \cup C\mathbb{S}^{d-1} \tag{36}$$

and $\mathbb{S}^{d-1}$ is the link of the vertex $n \in [n]$. By (34), complex $\mathbb{M}^d$ contains every simplex $A \subseteq [n]$ where $|A| \leq n - d - 2$ and therefore the link of a vertex n i.e. the sphere $\mathbb{S}^{d-1}$ contains every simplex $A \subseteq [n - 1]$ such that $|A| \leq n - d - 3$ implying that

$$\binom{[n-1]}{\leq n - d - 3} \subset \mathbb{S}^{d-1} \tag{37}$$

meaning the sphere $\mathbb{S}^{d-1} \subseteq 2^{[n-1]}$ is $(n - d - 3)$−neighbourly simplicial complex. Therefore if $f(\mathbb{S}^{d-1}) = (s_0, s_1, \ldots, s_{n-1})$, by Proposition 3.4 and from equation (37), coordinates s_i, $i = 0, 1 \ldots, n - 1$ satisfy the following system of equations

$$\begin{aligned}
&\textstyle\sum_{i=1}^{n}(-1)^{i+1}s_i = 0;\\
&2s_{d-1} = ds_d,\ (\mathbb{S}^{d-1} \text{ is a combinatorial manifold});\\
&s_i = \tbinom{n-1}{i},\ \text{for all } i = 0,1,\dots,n-d-3;\\
&\tbinom{n-1}{i} - s_{n-1-i} + s_{i-1} = f_i,\ \text{for all } i = 1,\dots,n;\\
&s_i = 0,\ \text{for all } i = d+1,\dots,n-1.
\end{aligned} \tag{38}$$

Let $d = 4, 8, 16$ and $n = 3d/2 + 3$. let $\mathbb{S}^{d-1} \subseteq 2^{[n-1]}$ be a sub-dual combinatorial sphere satisfying (37). Then, its dual upgrade $\Lambda\mathbb{S}^{d-1} = (\mathbb{S}^{d-1})^{\Lambda} \cup C\mathbb{S}^{d-1} \subseteq 2^{[n]}$ is a self-dual simplicial complex of dimension d. By The Combinatorial Alexander duality 4.1, and The Universal Coefficient Theorem, groups $H_k((\mathbb{S}^{d-1})^{\Lambda})$ are trivial for $k \neq n-d-3$ and isomorphic to $\mathbb{Z}$ for $k = n-d-3$. Thus, the long exact sequence (23) for the pair $((\mathbb{S}^{d-1})^{\Lambda}, \mathbb{S}^{d-1})$ is of the form

$$\mathbb{O} \to H_d(\Lambda\mathbb{S}^{d-1}) \to \mathbb{Z} \to \mathbb{O} \to \cdots \to \mathbb{O} \to \mathbb{Z} \to H_{n-d-3}(\Lambda\mathbb{S}^{d-1}) \to \mathbb{O}. \tag{39}$$

Therefore, there are only two non-trivial reduced homology groups $H_k(\Lambda\mathbb{S}^{d-1}) \approx \mathbb{Z}$ for $k = d, d/2$. Implying that $\Lambda\mathbb{S}^{d-1}$ has homology groups prescribed in Theorem 5.2. To complete the construction, only thing left to do is to prove that $\Lambda\mathbb{S}^{d-1}$ is a combinatorial manifold. This can be done in two ways:

- prove that $(\mathbb{S}^{d-1})^{\Lambda}$ (in the ambient $[n-1]$) is a combinatorial manyfold with boundary $\mathbb{S}^{d-1}$ or
- prove that $\mathrm{Lk}v \subset \Lambda\mathbb{S}^{d-1}$ is a combinatorial sphere for all $v \in [n]$.

Since in this case $\mathrm{Lk}(v) \subseteq 2^{[n]\setminus v}$ is a $(d-1)-$dimensional simplicial complex with $n - 1 = 3d/2 + 3 - 1 < 3\lceil(d-1)/2\rceil + 3$ (for $d = 4, 8, 16$), in order to prove that it is a combinatorial sphere, by Theorem 5.1, it is sufficient to prove that $\mathrm{Lk}(v) \subset 2^{[n]\setminus v}$ is a combinatorial manifold.

At this state, $\mathrm{Lk}(v) \subset \Lambda\mathbb{S}^{d-1}$ has interesting properties. Since $\Lambda\mathbb{S}^{d-1}$ is $d-$dimensional, similarly to (34), we have

$$\binom{[n]}{\leq n-d-2} \subseteq \Lambda\mathbb{S}^{d-1} \tag{40}$$

and therefore, complex $\mathrm{Lk}(v)$ is $(d-1)-$dimensional and $\binom{[n]\setminus\{v\}}{\leq n-d-3} \subseteq \mathrm{Lk}(v)$ implying $H_k(\mathrm{Lk}(v)) \approx \mathbb{O}$ for $k \leq n-d-3$ and $k \geq d$. Since $n-d-3 \geq 2$, we conclude that $\mathrm{Lk}(v)$ is simply-connected. Furthermore, by Theorem 2.8, the complex $\Lambda\mathbb{S}^{d-1}$ also has a form

$$\Lambda\mathbb{S}^{d-1} = \mathrm{Lk}(v)^{\Lambda_{[n]\setminus\{v\}}} \cup C\mathrm{Lk}(v). \tag{41}$$

Following The Universal Coefficient Theorem, cohomology groups $H^k(\mathrm{Lk}(v))$ are trivial for $k \leq n-d-3$ and $k > d$ therefore, by The Combinatorial Alexander Duality 4.1, groups $H_k(\mathrm{Lk}(v)^{\Lambda})$ are trivial for $k < n-d-4$ and $k \geq d-1$. Then, the long exact sequence (23) for the pair $(\mathrm{Lk}(v)^{\Lambda}, \mathrm{Lk}(v))$, has a form:

$$\begin{aligned}
\cdots \to \mathbb{O} \to \mathbb{Z} \to &H_{d-1}\big(\mathrm{Lk}(v)\big) \to \mathbb{O} \to \cdots\\
\cdots \to \mathbb{O} \to H_k\big(\mathrm{Lk}(v)\big) \to &H_k\big(\mathrm{Lk}(v)^{\Lambda}\big) \to \mathbb{O} \to \cdots\\
\cdots \to \mathbb{O} \to H_{n-d-3}\big(\mathrm{Lk}(v)^{\Lambda}\big) \to &\mathbb{Z} \to \mathbb{O} \to \cdots
\end{aligned} \tag{42}$$

Thus, $H_{d-1}(\mathrm{Lk}(v)) \approx \mathbb{Z}$ and for $k = d/2 + 1, \dots, d-2$ groups $H_k(\mathrm{Lk}(v)) \approx \mathbb{Z}$ are isomorphic to $H_k(\mathrm{Lk}(v)^{\Lambda})$ which is by The Combinatorial Alexander Duality 4.1 isomorphic to $H^{n-k-4}(\mathrm{Lk}(v))$.

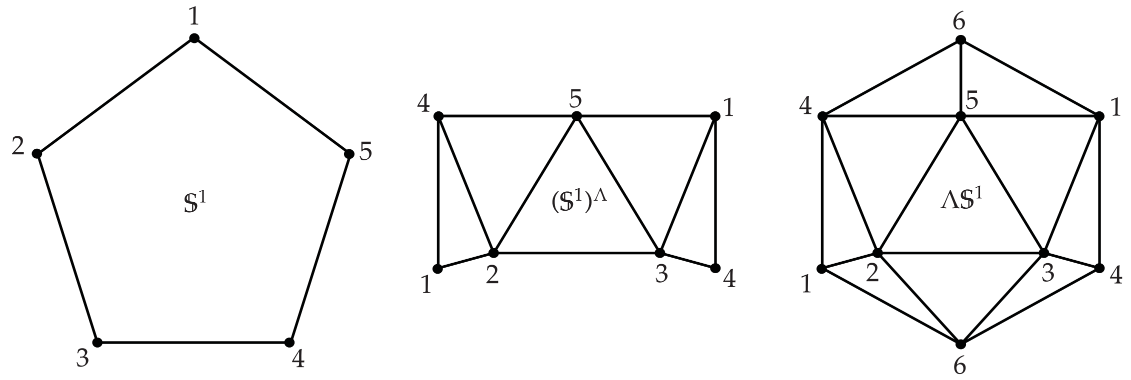

Figure 2: Pentagonal cycle, its dual, the Möebious band and its dual-upgrade, the hemi-icosahedron.

6.1. *The Real Projective Plane* $\mathbb{R}P^2$

Let us construct a self-dual manifold $\mathbb{M}^2 \in 2^{[6]}$. By Theorem 2.8, it will be a dual upgrade of a circle $\mathbb{S}^1 \subseteq 2^{[5]}$. Let $\mathbb{S}^1$ be the pentagonal cycle shown in Figure 2 (left). Since $\mathbb{S}^1$ is one-dimensional, by Proposition 2.4 it is sub-dual in the ambient [5], so its dual upgrade $\Lambda\mathbb{S}^1$ is a simplicial complex in the ambient [6]. Since minimal non simplices of $\mathbb{S}^1$ are its diagonals, the maximal simplexes of K^Λ are their complements. Therefore, $(\mathbb{S}^1)^\Lambda$ is a triangulation of the Möebious band with boundary K, as shown in Figure 2 (center).

By adding the cone $C\mathbb{S}^1$ along the boundary of $(\mathbb{S}^1)^\Lambda$, we obtain the hemi-icosahedron, a minimal triangulation of the real projective plane with 6 vertices shown on Figure 2 (right).

6.2. *The Complex Projective Plane* $\mathbb{R}P^2$

Let $\mathbb{M}^4 \subseteq 2^{[9]}$ be a self-dual combinatorial manifold and $f(\mathbb{M}^4) = (f_0, f_1, \dots, f_9)$ its $f-$vector. From equations (35) we have

$$\begin{aligned} f_6 = f_7 = f_8 = f_9 &= 0 \\ f_1 = 9, f_2 = 36, f_3 &= 84, \\ 2f_4 &= 5f_5, \\ 57 - f_4 + f_5 &= 3, \\ f_4 + f_5 &= 126. \end{aligned} \tag{43}$$

Previous system has only one solution $f_i = 0$ for $i > 5$ and:

$$(f_1, f_2, f_3, f_4, f_5) = (9, 36, 84, 90, 36). \tag{44}$$

Now, to construct $\mathbb{M}^4$ we need to construct a dual-upgrade of a sub-dual sphere $\mathbb{S}^3 \subseteq 2^{[8]}$. By (37) we have $\binom{[8]}{\leq 2} \subseteq \mathbb{S}^3$ and by (38), if $f(\mathbb{S}^3) = (s_1, s_2, \dots, s_8)$ is the $f-$vector of the sphere $\mathbb{S}^3$ we have:

$$\begin{aligned} s_5 = s_6 = s_7 = s_8 &= 0, \\ s_1 = 8, s_2 &= 28, \\ 2s_3 &= 4s_4, \\ -20 + s_3 - s_4 &= 0. \end{aligned} \tag{45}$$

This system also has a unique solution so the sphere $\mathbb{S}^3$ has $f-$vector such that $s_i = 0$ for $i \geq 5$ and

$$(s_1, s_2, s_3, s_4) = (8, 28, 40, 20). \tag{46}$$

Among 39 combinatorially distinct spheres on 8 vertices, there are four with prescribed f−vector and only one of them, the Brückner's sphere (see [16]) is sub-dual and suitable for further construction.

Let $\mathbb{S}^3$ be the Brückner's sphere. Since, minimal triangulation $\mathbb{R}P^2_9$ of the complex projective plane described in [15] has a Brückner sphere as a link of all of its vertexes, by Theorem 2.8 we conclude that the complex $\Lambda\mathbb{S}^3$ is isomorphic to $\mathbb{C}P^2_9$ and thus a combinatorial manifold. Therefore, there's a combinatorially unique self-dual 4−dimensional combinatorial manifold on 9 vertices, the complex projective plane.

6.3. *The Quaternionic Projective Plane* $\mathbb{H}P^2$

Let $\mathbb{M}^8 \subseteq 2^{[15]}$ be a self-dual combinatorial manifold and $f(\mathbb{M}^8) = (f_0, f_1, \dots, f_{15})$ its f−vector. From equations (35) we have

$$\begin{aligned} f_{10} = f_{11} = f_{12} = f_{13} = f_{14} = f_{15} = 0,\\ f_1 = 15, f_2 = 105, f_3 = 455, f_4 = 1365, f_5 = 3003,\\ 2f_8 = 9f_9,\\ 2003 - f_6 + f_7 - f_8 + f_9 = 3,\\ f_6 + f_9 = 5005,\\ f_7 + f_8 = 6435. \end{aligned} \tag{47}$$

Previous system also has only one solution $f_i = 0$ for $i > 10$ and:

$$(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9) = (15, 105, 455, 1365, 3003, 4515, 4230, 2205, 490). \tag{48}$$

In order to obtain $\mathbb{M}^8$, we need to construct a dual-upgrade of a sub-dual sphere $\mathbb{S}^7 \subset 2^{[14]}$. By (37) we have $\binom{[14]}{\leq 4} \subset \mathbb{S}^7$ and if $f(\mathbb{S}^7) = (s_1, s_2, \dots, s_{14})$ is the f−vector of the sphere $\mathbb{S}^7$, following (38) we have:

$$\begin{aligned} s_9 = s_{10} = s_{11} = s_{12} = s_{13} = s_{14} = 0,\\ s_1 = 14, s_2 = 91, s_3 = 364, s_4 = 1001,\\ 2s_7 = 8s_8,\\ -714 + s_5 - s_6 + s_7 - s_8 = 0,\\ 3003 + s_5 - s_8 = 4515. \end{aligned} \tag{49}$$

Previous system does not have a unique solution. In order to determine $f(\mathbb{S}^7)$, we will use Dehn-Sommerville-equations (see [19]) which for every d−dimensional orientable combinatorial manifold $\mathbb{M}^d$ with f−vector $f(\mathbb{M}^d) = (f_1, \dots, f_{d+1})$ and every $k = 0, 1, \dots, d+1$ (where $f_0 = \chi(\mathbb{M}^d)$) state that:

$$\sum_{i=k}^{d+1} (-1)^{i+1} \binom{i}{k} f_i = (-1)^{d+1} f_k. \tag{50}$$

To solve (49), it is sufficient to add only one equation, for example for $k = 1$:

$$3080 - 5s_5 + 6s_6 - 7s_7 + 8s_8 = 14. \tag{51}$$

With this equation, the system (49) has a unique solution $s_i = 0$ for $i \geq 9$ and

$$(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8) = (14, 91, 364, 1001, 1806, 1974, 1176, 294). \tag{52}$$

Since there is no suitable catalog of available 7−dimensional spheres on 14 vertices, the required sphere will have to be constructed. As it turns out, it is sufficient to start with any sub-dual 7−dimensional sphere on 14 vertices, say for example a Bier sphere (see [1]), and use Pachner moves (see [18]) which preserve sub-duality to obtain the sphere with desired f−vector. This procedure is easily programable on almost every programming language and using computer assisted calculations, many non-isomorphic suitable spheres where obtained, including one isomorphic to the vertex-link of the vertex-transitive minimal triangulation of the quaternionic projective plane $\mathbb{H}P^2_{15}$ described in [13].

6.4. The Octanionic Projective Plane $\mathbb{O}P^2$

Let $\mathbb{M}^{16} \subseteq 2^{[27]}$ be a self-dual combinatorial manifold and $f(\mathbb{M}^8) = (f_0, f_1, \dots, f_{27})$ its $f-$vector. From equations (35) we have

$$\begin{aligned}
f_{18} = f_{19} = f_{20} = f_{21} = f_{22} = f_{23} = f_{134} = f_{25} = f_{26} = f_{27} = 0,\\
f_1 = 27, f_2 = 351, f_3 = 2925, f_4 = 17550, f_5 = 80730,\\
f_6 = 296010, f_7 = 888030, f_8 = 2220075, f_9 = 4686825,\\
2f_{16} = 17f_{17},\\
3124551 - f_{10} + f_{11} - f_{12} + f_{13} - f_{14} + f_{15} - f_{16} + f_{17} = 3,\\
f_{10} + f_{17} = 8436285,\\
f_{11} + f_{16} = 13037895,\\
f_{12} + f_{15} = 17383860,\\
f_{13} + f_{14} = 20058300.
\end{aligned} \tag{53}$$

The system (53) does not have a unique solution so we again use Dehn-Sommerville-equations (50). As it turns out, it is sufficient to use equations for $k = 2$ and $k = 4$:

$$\begin{aligned}
&-121482504 + 45f_{10} - 55f_{11} + 66f_{12} - 78f_{13}+\\
&\qquad +91f_{14} - 105f_{15} + 120f_{16} - 136f_{17} = -351, &(54)\\
&-462161700 + 210f_{10} - 330f_{11} + 495f_{12} - 715f_{13}+\\
&\qquad 1001f_{14} - 1365f_{15} + 1820f_{16} - 2380f_{17} = -17550 &(55)
\end{aligned}$$

which, together with (53) have a unique solution $f_i = 0$ for $i \geq 18$ and

$$\begin{aligned}
(f_1, \dots, f_{17}) = (&27, 351, 2925, 17550, 80730, 296010, 888030, 2220075, 4686825, 8335899,\\
&12184614, 14074164, 12301200, 7757100, 3309696, 853281, 100386)
\end{aligned} \tag{56}$$

Previous vector, recognized as $f-$vector of Cayley-like manifold with 27 vertexes, was first derived by Wolfgang Kühnel in [10].

To construct $\mathbb{M}^{16}$, we need a sub-dual sphere $\mathbb{S}^{15} \subset 2^{[26]}$. Then, manifold $\mathbb{M}^{16}$ will be its dual upgrade. By (37), $\binom{[26]}{\leq 8} \subseteq \mathbb{S}^{15}$ and if $f(\mathbb{S}^{15}) = (s_1, s_2, \dots, s_{26})$ from (38) we have:

$$\begin{aligned}
s_{17} = s_{18} = s_{19} = s_{20} = s_{21} = s_{22} = s_{23} = s_{24} = s_{25} = s_{26} = 0,\\
s_1 = 26, s_2 = 325, s_3 = 2600, s_4 = 14950, s_5 = 65780,\\
s_6 = 230230, s_7 = 657800, s_8 = 1562275,\\
2s_{15} = 16s_{16},\\
-1081574 + s_9 - s_{10} + s_{11} - s_{12} + s_{13} - s_{14} + s_{15} - s_{16} = 0,\\
5311735 + s_9 - s_{16} = 8335899,\\
7726160 + s_{10} - s_{15} = 12184614,\\
9657700 + s_{11} - s_{14} = 14074164.
\end{aligned} \tag{57}$$

Since (57) does not have a unique solution, to obtain $f(\mathbb{S}^{15})$ we use equations $k = 1, 3, 5$ from (50):

$$\begin{aligned}
&8998704 - 9s_9 + 10s_{10} - 11s_{11} + 12s_{12}-\\
&\qquad 13s_{13} + 14s_{14} - 15s_{15} + 16s_{16} = 26, &(58)\\
&68468400 - 84s_9 + 120s_{10} - 165s_{11} + 220s_{12}-\\
&\qquad -286s_{13} + 364s_{14} - 455s_{15} + 560s_{16} = 2600, &(59)\\
&74989200 - 126s_9 + 252s_{10} - 462s_{11} + 792s_{12}-\\
&\qquad -1287s_{13} + 2002s_{14} - 3003s_{15} + 4368s_{16} = 65780. &(60)
\end{aligned}$$

Therefore, the sphere $\mathbb{S}^{15}$ has f−vector such that $s_i = 0$ for $i \geq 17$ and

$$(s_1, \ldots, s_{16}) = (26, 325, 2600, 14950, 65780, 230230, 657800, 1562275, 3087370, \\ 4964102, 6255184, 5922800, 4022200, 1838720, 505648, 63206). \quad (61)$$

Using the same program as in construction of quaternionic projective plane, several sub-dual combinatorial spheres with described f−vector where obtained, and it is shown that dual-upgrade of at least one of them is a combinatorial manifold. Description of the program and computer obtained results will be given in a subsequent publication.

Alexander Gaifullin had shown in [6] that there are 634 vertex-transitive and more than 10^{103} non-vertex-transitive non isomorphic combinatorial manifolds with 27 vertexes and all of them are self-dual implying that there are is at least as much sub-dual 8−neighbourly combinatorial spheres on 26 vertexes. By Theorem 2.8 and previous computations, all of them have the same f−vector.

Corollary 6.1. *If $\mathbb{M}^d \subseteq 2^{[n]}$ is self-dual combinatorial manifold, then for all $v, w \in [n]$*

$$f\Big(\mathrm{Lk}(v)\Big) = f\Big(\mathrm{Lk}(v)\Big). \quad (62)$$

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