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Generalized quasi-Einstein warped products manifolds with respect to affine connections

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Abstract. In this paper, we study warped product on generalized quasi-Einstein manifolds with respect to affine connections. Initially, we deal with the elementary properties and existence of generalized quasi-Einstein warped products manifolds with respect to affine connections. Furthermore, it is proved that generalized quasi-Einstein manifold to be a quasi-Einstein manifold with respect to affine connections and we give three and four examples (both Riemannian and Lorentzian) of generalized quasi-Einstein manifolds to show the existence of such manifold. Finally, we construct two examples of warped product on generalized quasi-Einstein manifolds with respect to affine connections are also discussed.

1. Introduction

A Riemannian (or semi-Riemannian) manifold (M^n, g), $(n \geq 3)$ is named an Einstein manifold if the Ricci tensor $Ric(\neq 0)$ of type $(0,2)$ satisfies: $Ric = \frac{r}{n}g$, where r represents the scalar curvature of (M^n, g). Einstein manifolds form a natural subclass of several classes of (M^n, g) determined by a curvature restriction imposed on their Ricci tensor [3]. Also, Einstein manifolds play a key role in Riemannian geometry, general theory of relativity as well as in mathematical physics.

Approximately two decades ago, the idea of quasi-Einstein manifold was proposed and studied by Chaki and Maity [11]. An (M^n, g), $(n > 2)$ is said to be quasi-Einstein manifold $(QE)_n$ if its $Ric(\neq 0)$ satisfies

$$Ric(Z_1, Z_2) = ag(Z_1, Z_2) + bA(Z_1)A(Z_2), \tag{1}$$

where $a, b(\neq 0) \in \mathbb{R}$ and A is a non-zero 1-form such that

$$g(Z_1, \rho) = A(Z_1), \quad g(\rho, \rho) = A(\rho) = 1, \tag{2}$$

for all vector field Z_1 and a unit vector field ρ called the generator of $(QE)_n$. Also, the 1-form A is named the associated 1-form. From (1) it is clear that for $b = 0$, $(QE)_n$ reduces to an Einstein manifold.

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An $(M^n, g), (n \geq 3)$ is said to be generalized quasi-Einstein manifold $G(QE)_n$ [12] if its $Ric(\neq 0)$ satisfies

$$Ric(Z_1, Z_2) = ag(Z_1, Z_2) + bA(Z_1)A(Z_2) + c[A(Z_1)B(Z_2) + A(Z_2)B(Z_1)] \tag{3}$$

where $a, b(\neq 0), c(\neq 0) \in \mathbb{R}$ and $A(\neq 0), B(\neq 0)$ are 1-forms such that

$$g(Z_1, \rho) = A(Z_1), \quad g(Z_1, \sigma) = B(Z_1), \quad g(\rho, \rho) = 1, \quad g(\sigma, \sigma) = 1. \tag{4}$$

where ρ and σ are mutually orthogonal unit vector fields, i.e., $g(\rho, \sigma) = 0$ and are known as generators of $G(QE)_n$. $G(QE)_n$ has widely investigate the geometric properties and physical applications in general relativity [16, 17, 28] and also studied by several authors [6, 18, 25–27].

The concept of a semi-symmetric linear connection on a differentiable manifold was first introduced by Friedmann and Schouten in 1924 [1]. A generalization of the semi-symmetric connection in [19], Golab first defined a quarter-symmetric linear connection on a differentiable manifold in 1975. Many writers have examined the outcomes of warped products with affine connections, including Dey et al. [4, 20, 21], Pahan et al. [22, 23], Shenawy and Unal [24], among others.

An $(M^n, g), (n \geq 3)$ is said to be generalized quasi-constant sectional curvature [25] if its curvature tensor satisfies

$$\begin{aligned}\tilde{K}(Z_1, Z_2, Z_3, Z_4) = {} & a[g(Z_2, Z_3)g(Z_1, Z_4) - g(Z_1, Z_3)g(Z_2, Z_4)] \\ & + b[g(Z_1, Z_4)A(Z_2)A(Z_3) - g(Z_2, Z_4)A(Z_1)A(Z_3) \\ & + g(Z_2, Z_3)A(Z_1)A(Z_4) - g(Z_1, Z_3)A(Z_2)A(Z_4)] \\ & + c[g(Z_1, Z_4)B(Z_2)B(Z_3) - g(Z_2, Z_4)B(Z_1)B(Z_3) \\ & + g(Z_2, Z_3)B(Z_1)B(Z_4) - g(Z_1, Z_3)B(Z_2)B(Z_4)],\end{aligned} \tag{5}$$

where $a, b(\neq 0), c(\neq 0) \in \mathbb{R}$ and $A(\neq 0), B(\neq 0)$ are 1-forms.

2. Warped product manifolds admitting affine connection

The concept of a warped product introduced by Bishop et.al [15] in 1969 for the study of negative-curvature manifolds. Let $(\mathcal{B}, g_{\mathcal{B}})$ and $(\mathcal{F}, g_{\mathcal{F}})$ be two Riemannian manifolds with dim $\mathcal{B} = p > 0$, dim $\mathcal{F} = q > 0$ and $f : B \to (0, \infty)$, $f \in C^\infty(\mathcal{B})$. Consider the product manifold $\mathcal{B} \times \mathcal{F}$ with its projections $u : \mathcal{B} \times \mathcal{F} \to \mathcal{B}$ and $v : \mathcal{B} \times \mathcal{F} \to \mathcal{F}$. The warped product $\mathcal{B} \times_f \mathcal{F}$ is the manifold $\mathcal{B} \times \mathcal{F}$ with the Riemannian structure such that $\|Z_1\|^2 = \|u^*(Z_1)\|^2 + f^2(u(m))\|v^*(Z_1)\|^2$ for any vector field Z_1 on M. Thus we have

$$g_M = g_{\mathcal{B}} + f^2 g_{\mathcal{F}}, \tag{6}$$

where $\mathcal{B}$ is called the base of M and $\mathcal{F}$ the fiber. The function f is called the warping function of the warped product [5].

Since $\mathcal{B} \times_f \mathcal{F}$ is a warped product, then we have $D_{Z_1} Z_3 = D_{Z_3} Z_1 = (Z_1 lnf) Z_3$ for all vector fields Z_1, Z_3 on $\mathcal{B}$ and $\mathcal{F}$, respectively. Hence we find $R(Z_1 \wedge Z_3) = g(D_{Z_3} D_{Z_1} Z_1 - D_{Z_1} D_{Z_3} Z_1, Z_3) = \frac{1}{f}\{(D_{Z_1} Z_1) f - Z_1^2 f\}$. If we choose a local orthonormal basis $e_1, ..., e_n$ such that $e_1, ..., e_{n_1}$ are tangent to $\mathcal{B}$ and $e_{n_1+1}, ..., e_n$ are tangent to $\mathcal{F}$, then we have

$$\frac{\Delta f}{f} = \sum_{i=1}^{n} R(e_i \wedge e_j), \tag{7}$$

for each $j = n_1 + 1, ..., n$ [5].

Two lemmas from [5] are required for further work:

Lemma 2.1. *Let us assume that $M = \mathcal{B} \times_f \mathcal{F}$ is a warped product, and that K_M is the Riemannian curvature tensor. If we have the fields Z_1, Z_2, and Z_3 on $\mathcal{B}$ as well as P, Q, and Z_4 on $\mathcal{F}$, then:*
(1) $K_M(Z_1, Z_2)Z_3 = K_{\mathcal{B}}(Z_1, Z_2)Z_3$,

(2) $K_M(Z_1, Q)Z_2 = \frac{H^f(Z_1,Z_2)}{f} Q$, *where* H^f *is the Hessian of* f,
(3) $K_M(Z_1, Z_2)Q = K_M(Q, Z_4)Z_1 = 0$,
(4) $K_M(Z_1, Q)Z_4 = -(\frac{g(Q,Z_4)}{f})D_{Z_1}(gradf)$,
(5) $K_M(Q, Z_4)P = K_{\mathcal{F}}(Q, Z_4)P + (\frac{\|gradf\|^2}{f^2})g(Q, P)Z_4 - g(Z_4, P)Q$.

Lemma 2.2. *Let us assume that* $M = \mathcal{B} \times_f \mathcal{F}$ *is a warped product, and that* Ric_M *is the Ricci tensor. If we have the fields* Z_1, Z_2, *and* Z_3 *on* $\mathcal{B}$ *as well as* P, Q, *and* Z_4 *on* $\mathcal{F}$, *then:*
(1) $Ric_M(Z_1, Z_2) = Ric_{\mathcal{B}}(Z_1, Z_2) - \frac{m}{f} H^f(Z_1, Z_2)$,
(2) $Ric_M(Z_1, Q) = 0$,
(3) $Ric_M(Q, Z_4) = Ric_{\mathcal{F}}(Q, Z_4) - g(Q, Z_4)(\frac{\Delta f}{f} + \frac{m-1}{f^2}\|gradf\|^2)$, *where* Δf *is the Laplacian of* f *on* $\mathcal{B}$

Furthermore, the condition is satisfies

$$scal_M = scal_{\mathcal{B}} + \frac{scal_{\mathcal{F}}}{f^2} - 2m\frac{\Delta f}{f} - m(m-1)\frac{|gradf|^2}{f^2}, \tag{8}$$

where $scal_{\mathcal{B}}$ and $scal_{\mathcal{F}}$ are scalar curvatures of $\mathcal{B}$ and $\mathcal{F}$, respectively.

Gebarowski investigated Einstein's warped product manifolds in his paper [2] and demonstrated the following three theorems about them:

Theorem 2.3. *Let* $dimI = 1$, $dim\mathcal{F} = n - 1 (n \geq 3)$, *and let* (M, g) *be a warped product of* $I \times_f \mathcal{F}$. *If* $\mathcal{F}$ *is an Einstein manifold with constant scalar curvature, as in the case of* $n = 3$, *and* f *is determined by one of the following formulas for any real number* β, *then* (M, g) *is an Einstein manifold.*

$$f^2(x) = \begin{cases} \frac{4}{\alpha} Rsinh^2 \frac{\sqrt{\alpha}(x+\beta)}{2}, & \text{if } \alpha > 0 \\ R(x+\beta)^2, & \text{if } \alpha = 0 \\ \frac{-4}{\alpha} Rsin^2 \frac{\sqrt{-\alpha}(x+\beta)}{2}, & \text{if } \alpha < 0 \end{cases}$$

$$f^2(x) = \begin{cases} e^{\alpha x}\beta, & \text{if } R > 0\ (\alpha \neq 0) \\ \frac{-4}{\alpha} Rcosh^2 \frac{\sqrt{\alpha}(x+\beta)}{2}, & \text{if } R = 0\ (\alpha > 0) \end{cases}$$

for $R < 0$, after integration $q''e^q + 2R = 0$ and $R = \frac{scal_{\mathcal{F}}}{(n-1)(n-2)}$.

Theorem 2.4. *Let* (M, g) *be the warped product of a complete connected s-dimensional Riemannian manifold* $\mathcal{F}$ *and a complete connected* $(1 < s < n)$ *Riemannian manifold* $\mathcal{B}$. $\mathcal{B}$ *is a sphere of radius* $\frac{1}{\sqrt{R}}$, *if* (M, g) *is a space with constant sectional curvature* $R > 0$.

Theorem 2.5. *Let* (M, g) *be a warped product* $\mathcal{B} \times_f \mathcal{F}$ *of a* $n-1$*-dimensional Riemannian manifold* $\mathcal{B}$ *and a one-dimensional Riemannian manifold I. If* (M, g) *is an Einstein manifold with scalar curvature* $scal_M > 0$ *and the Hessian of* f *is proportional to the metric tensor* $g_{\mathcal{B}}$, *then*
(1) $(\mathcal{B}, g_{\mathcal{B}})$ *is a* $(n-1)$*-dimensional sphere with radius* $= \left(\frac{(scal_{\mathcal{B}}}{(n-1)(n-2)}\right)^{\frac{-1}{2}}$
(2) (M, g) *denotes a space with constant sectional curvature* $R = \frac{scal_M}{n(n-1)}$.
We also investigate warped product manifolds with quarter-symmetric connections in this paper. Here, we look at propositions 3.1, 3.2, 3.3, and 3.4 of [14] and in this paper we denoted by 3.6, 3.7, 3.8 and 3.9, respectively, which will help us prove our results.

Proposition 2.6. *Let* $M = \mathcal{B} \times_f \mathcal{F}$ *be a warped product. Let Ric and* $\overline{Ric}$ *denote the Ricci tensors of M with respect to the Levi-Civita connection and a quarter-symmetric connection respectively. Let* $dim\mathcal{B} = n_1$, $dim\mathcal{F} = n_2$, $dimM = \overline{n} = n_1 + n_2$. *If* $Z_1, Z_2 \in \mathfrak{X}(\mathcal{B})$, $Q, Z_4 \in \mathfrak{X}(\mathcal{F})$ *and* $\rho \in \mathfrak{X}(\mathcal{B})$, *then*
(i) $\overline{Ric}(Z_1, Z_2) = \overline{Ric}_{\mathcal{B}}(Z_1, Z_2) + n_2[\frac{H^f_{\mathcal{B}}(Z_1,Z_2)}{f} + \mu_2\frac{\rho f}{f} g(Z_1, Z_2) + \mu_1\mu_2\Omega(\rho)g(Z_1, Z_2) + \mu_1 g(Z_2, D_{Z_1}\rho) - \mu_1^2\Omega(Z_1)\Omega(Z_2)]$

(ii) $\overline{Ric}(Z_1, V) = \overline{Ric}(Q, Z_1)$,
(iii) $\overline{Ric}(V, Z_4) = Ric_{\mathcal{F}}(Q, Z_4) + \{\mu_2 div_{\mathcal{B}}\rho + (n_2 - 1)\frac{|grad_{\mathcal{B}} f|^2_{\mathcal{B}}}{f^2}[(\overline{n} - 1)\mu_1\mu_2 - \mu_2^2]\Omega(\rho) + [(\overline{n} - 1)\mu_1 + (n_2 - 1)\mu_2]\frac{\rho f}{f} + \frac{\Delta_{\mathcal{B}} f}{f}\}g(Q, Z_4)$
where $div_{\mathcal{B}}\rho = \sum_{k=1}^{n_1} \epsilon_k \langle D_{W_k}\rho, W_k\rangle$ *and* W_k, $1 \le k \le n_1$, *is an orthonormal basis of* $\mathcal{B}$ *with* $\epsilon_k = g(W_k, W_k)$

Proposition 2.7. *Let* $M = \mathcal{B} \times_f \mathcal{F}$ *be a warped product,* $dim\mathcal{B} = n_1$, $dim\mathcal{F} = n_2$, $dimM = \overline{n} = n_1 + n_2$. *If* Z_1, $Z_2 \in \mathfrak{X}(\mathcal{B})$, Q, $Z_4 \in \mathfrak{X}(\mathcal{F})$ *and* $\rho \in \mathfrak{X}(\mathcal{B})$, *then*
(i) $\overline{Ric}(Z_1, Z_2) = \overline{Ric}_{\mathcal{B}}(Z_1, Z_2) + [(\overline{n} - 1)\mu_1\mu_2 - \mu_2^2]\Omega(\rho)g(Z_1, Z_2) + n_2\frac{H^f_{\mathcal{B}}(Z_1, Z_2)}{f} + \mu_2 g(Z_1, Z_2)div_{\mathcal{F}}\rho$,
(ii) $\overline{Ric}(Z_1, Q) = [(\overline{n} - 1)\mu_1 - \mu_2]\Omega(Q)\frac{Z_1 f}{f}$,
(iii) $\overline{Ric}(V, Z_1) = [\mu_2 - (\overline{n} - 1)\mu_1]\Omega(Q)\frac{Z_1 f}{f}$,
(iv) $\overline{Ric}(V, Z_4) = \overline{Ric}_{\mathcal{F}}(Q, Z_4) + g(Q, Z_4)\{(n_2 - 1)\frac{|grad_{\mathcal{B}} f|^2_{\mathcal{B}}}{f^2} + \frac{\Delta_{\mathcal{B}} f}{f} + [(\overline{n} - 1)\mu_1\mu_2 - \mu_2^2]\Omega(\rho) + \mu_2 div_{\mathcal{F}}\rho\} + [(\overline{n} - 1)\mu_1 - \mu_2]g(Z_4, D_Q\rho) + [\mu_2^2 + (1 - \overline{n})\mu_1^2]\Omega(Q)\Omega(Z_4)$

Proposition 2.8. *Let* $M = \mathcal{B} \times_f \mathcal{F}$ *be a warped product,* $dim\mathcal{B} = n_1$, $dim\mathcal{F} = n_2$, $dimM = \overline{n} = n_1 + n_2$. *If* $\rho \in \mathfrak{X}(\mathcal{B})$, *then*

$$\begin{aligned}\overline{scal}_M = \overline{scal}_{\mathcal{B}} + \frac{scal_{\mathcal{F}}}{f^2} + n_2(n - 1)\frac{|grad_{\mathcal{B}} f|^2_{\mathcal{B}}}{f^2} + n_2(\overline{n} - 1)(\mu_1 + \mu_2)\frac{\rho f}{f} + 2n_2\frac{\Delta_{\mathcal{B}} f}{f} \\ + [n_2(\overline{n} + n_1 - 1)\mu_1\mu_2 - n_2(\mu_1^2 + \mu_2^2)]\Omega(\rho) + n_2(\mu_1 + \mu_2)div_{\mathcal{B}}\rho.\end{aligned} \tag{9}$$

Proposition 2.9. *Let* $M = \mathcal{B} \times_f \mathcal{F}$ *be a warped product,* $dim\mathcal{B} = n_1$, $dim\mathcal{F} = n_2$, $dimM = \overline{n} = n_1 + n_2$. *If* $\rho \in \mathfrak{X}(F)$, *then*

$$\begin{aligned}\overline{scal}_M = \overline{scal}_{\mathcal{B}} + \frac{scal_F}{f^2}(\overline{n} - 1)(\mu_1 + \mu_2)div_{\mathcal{F}}\rho + [\overline{n}(\overline{n} - 1)\mu_1\mu_2 + (1 - \overline{n})(\mu_1^2 + \mu_2^2)]\Omega(\rho) \\ + n_2(n - 1)\frac{|grad_{\mathcal{B}} f|^2_{\mathcal{B}}}{f^2} + 2n_2\frac{\Delta_{\mathcal{B}} f}{f}\end{aligned} \tag{10}$$

In this section, we study generalized quasi-Einstein warped product manifolds and prove several results about them.

Theorem 2.10. *Let* (M, g) *be a warped product* $I \times_f \mathcal{F}$ *where* I *is an open interval in* $\mathbb{R}$, $dimI = 1$ *and* $dim\mathcal{F} = n - 1$, $n \ge 3$. *Then the following statements are equivalent.*
(i) If (M, g) *is a* $(GQ)_n$ *with respect to a quarter-symmetric connection then* $\mathcal{F}$ *is a* $(GQ)_n$ *for* $\rho = \frac{\partial}{\partial t}$ *with respect to the Levi-Civita connection.*
(ii) If (M, g) *is a* $(GQ)_n$ *with respect to a quarter-symmetric connection then the warping function* f *is a constant on* I *for* $\rho \in \mathfrak{X}(\mathcal{F})$, $\mu_2 \neq (n - 1)\mu_1$.

Proof. Suppose that $\rho \in \mathfrak{X}(\mathcal{B})$ and let g_I be the metric on I. Taking $f = e^{\frac{q}{2}}$ and using the Proposition 2.6, one obtains

$$\overline{Ric}_M\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) = (1 - n)\Big[\frac{1}{2}q'' + \frac{1}{4}q'^2 - \frac{1}{2}\mu_2 q' + \mu_1\mu_2 - \mu_1^2\Big]g_1\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big), \tag{11}$$

$$\overline{Ric}\Big(\frac{\partial}{\partial t}, Q\Big) = 0, \tag{12}$$

$$\begin{aligned}\overline{Ric}(Q, Z_4) = Ric_{\mathcal{F}}(Q, Z_4) + e^q\Big[\frac{n - 1}{4}(q')^2 + \frac{1}{2}\Big\{(n - 1)\mu_1 + (n - 2)\mu_2\Big\}q' \\ + \mu_2^2 + \frac{1}{2}q'' + (1 - n)\mu_1\mu_2\Big]g_{\mathcal{F}}(Q, Z_4),\end{aligned} \tag{13}$$

for all vector fields Q, Z_4 on $\mathcal{F}$.
Since M is $G(QE)_n$ with respect to quarter-symmetric connection, then form (3), we have

$$\overline{Ric}_M\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) = ag\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) + bA\Big(\frac{\partial}{\partial t}\Big)A\Big(\frac{\partial}{\partial t}\Big) + c\Big[A\Big(\frac{\partial}{\partial t}\Big)B\Big(\frac{\partial}{\partial t}\Big) + B\Big(\frac{\partial}{\partial t}\Big)A\Big(\frac{\partial}{\partial t}\Big)\Big] \tag{14}$$

and

$$\overline{Ric}_M(Q, Z_4) = ag(Q, Z_4) + bA(Q)A(Z_4) + c[A(Q)B(Z_4) + A(Z_4)B(Q)]. \tag{15}$$

Decomposing the vector fields P and P' separately into their components P_I, $P_{\mathcal{F}}$ and P'_I, $P'_{\mathcal{F}}$ on I and $\mathcal{F}$, respectively, we have $P = P_I + \eta_I P_{\mathcal{F}}$ and $P' = P'_I + \eta_2 P'_{\mathcal{F}}$. Since $dimI = 1$, taking $P_I = \frac{\partial}{\partial t}$ which gives $P = \frac{\partial}{\partial t} + \eta_1 P_{\mathcal{F}}$ and $P'_I = \frac{\partial}{\partial t}$ which gives $P' = \frac{\partial}{\partial t} + \eta_2 \frac{\partial}{\partial t} + P'_{\mathcal{F}}$, where η_1 and η_2 are functions on M. Thus, we have the following

$$\begin{aligned} A\Big(\frac{\partial}{\partial t}\Big) &= g\Big(\frac{\partial}{\partial t}, P\Big) = 1,\\ B\Big(\frac{\partial}{\partial t}\Big) &= g\Big(\frac{\partial}{\partial t}, P'\Big) = 1. \end{aligned} \tag{16}$$

Using equations (6) and (16), the equations (14) and (15) reduces to

$$\overline{Ric}_M\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) = a + b + 2c \tag{17}$$

and

$$\overline{Ric}_M(Q, Z_4) = ae^q g_{\mathcal{F}}(Q, Z_4) + bA(Q)A(Z_4) + c[A(Q)B(Z_4) + A(Z_4)B(Q)]. \tag{18}$$

Comparing the right hand side of the equations (11) and (17), one obtains

$$a + b + 2c = -\frac{n-1}{4}\Big[2q'' + (q')^2\Big]. \tag{19}$$

Similarly, comparing the right hand side of the equations (13) and (18) we get

$$\begin{aligned} Ric_{\mathcal{F}}(Q, Z_4) = e^q\Big[a - \Big\{&\frac{\overline{n}-1}{4}(q')^2 + \frac{1}{2}\Big((n-1)\mu_1 + (\overline{n}-2)\mu_2\Big)q'\mu_2^2 + \frac{1}{2}q'' + (1-n)\\ &\mu_1\mu_2\Big\}\Big]g_{\mathcal{F}}(Q, Z_4) + bA(Q)A(Z_4) + c[A(Q)B(Z_4) + A(Z_4)B(Q)], \end{aligned} \tag{20}$$

which gives that $\mathcal{F}$ is a $(GQ)_n$ with respect to connection for $\rho \in \mathfrak{X}(\mathcal{B})$ and use the Proposition 2.7, one gets

$$\overline{Ric}\Big(\frac{\partial}{\partial t}, Q\Big) = \frac{q'}{2}\Big[(n-1)\mu_1 - \mu_2\Big]\Omega(Q), \tag{21}$$

$$\overline{Ric}\Big(Q, \frac{\partial}{\partial t}\Big) = \frac{q'}{2}\Big[\mu_2 - (n-1)\mu_1\Big]\Omega(Q) \tag{22}$$

for any vector field $Q \in \mathfrak{X}(\mathcal{F})$. Since M is a $(GQ)_n$, we have

$$\begin{aligned} \overline{Ric}\Big(\frac{\partial}{\partial t}, Q\Big) &= \overline{Ric}\Big(Q, \frac{\partial}{\partial t}\Big)\\ &= ag\Big(Q, \frac{\partial}{\partial t}\Big) + bA(Q)A\Big(\frac{\partial}{\partial t}\Big) + c\Big[A(Q)B\Big(\frac{\partial}{\partial t}\Big) + B(Q)A\Big(\frac{\partial}{\partial t}\Big)\Big]. \end{aligned} \tag{23}$$

Now, $g\Big(Q, \frac{\partial}{\partial t}\Big) = 0$ as $\frac{\partial}{\partial t} \in \mathfrak{X}(B)$ and $Q \in \mathfrak{X}(\mathcal{F})$. Therefore, form (23), we get

$$\overline{Ric}\Big(\frac{\partial}{\partial t}, Q\Big) = \overline{Ric}\Big(Q, \frac{\partial}{\partial t}\Big) = bA(P)A\Big(\frac{\partial}{\partial t}\Big) + c\Big[A(Q)B\Big(\frac{\partial}{\partial t}\Big) + B(Q)A\Big(\frac{\partial}{\partial t}\Big)\Big]. \tag{24}$$

Hence, we have

$$bA(Q)A\Big(\frac{\partial}{\partial t}\Big) + c\Big[A(P)B\Big(\frac{\partial}{\partial t}\Big) + B(Q)A\Big(\frac{\partial}{\partial t}\Big)\Big] = \frac{q'}{2}\Big[(n-1)\mu_1 - \mu_2\Big]\Omega(Q) \tag{25}$$

$$bA(Q)A\Big(\frac{\partial}{\partial t}\Big) = c\Big[A(Q)B\Big(\frac{\partial}{\partial t}\Big) + B(Q)A\Big(\frac{\partial}{\partial t}\Big)\Big] + \frac{q'}{2}\Big[\mu_2 - (\overline{n}-1)\mu_1\Big]\Omega(Q). \tag{26}$$

From (24) and (25), we get

$$q' = 0, \tag{27}$$

when $\mu_2 - (n-1)\mu_1 \neq 0$. It follows that q is a constant on I. Then f is constant on I.

Now, we consider the warped product $M = \mathcal{B} \times_f I$ with $dim\mathcal{B} = n-1$, $dimI = 1$, $n \geq 3$. Under this assumption, we prove the following theorem. □

Theorem 2.11. *Let (M, g) be a warped product $\mathcal{B} \times_f I$, where $dimI = 1$ and $dim\mathcal{B} = n-1$, $n \geq 3$, then*
(i) if $P \in \mathfrak{X}(\mathcal{B})$ is parallel on $\mathcal{B}$ with respect to the Levi-Civita connection on $\mathcal{B}$, f is a constant on $\mathcal{B}$ and (M, g) is a $(GQ)_n$ with respect to a quarter-symmetric connection, then,

$$a = [(n-1)\mu_1\mu_2 - \mu_2^2]\Omega(\rho).$$

(ii) f is a constant on $\mathcal{B}$ if (M, g) is a $(GQ)_n$ with respect to a quarter-symmetric connection for $\rho \in \mathfrak{X}(I)$, and $\mu_2 \neq (n-1)\mu_1$.
(iii) M is a $(GQ)_n$ with respect to a quarter-symmetric connection if f is a constant on $\mathcal{B}$ and $\mathcal{B}$ is a $(GQ)_n$ with respect to the Levi-Civita connection for $\rho \in \mathfrak{X}(I)$.

Proof. Let (M, g) is a $(GQ)_n$ with respect to a quarter-symmetric connection. Then we have

$$\overline{Ric}(Z_1, Z_2) = ag(Z_1, Z_2) + bA(Z_1)A(Z_2) + c[A(Z_1)B(Z_2) + A(Z_2)B(Z_1)]. \tag{28}$$

Decomposing the vector fields P and Q separately into their components $P_{\mathcal{B}}$ and P_I on $\mathcal{B}$ and I, respectively, we have

$$P = P_I + P_{\mathcal{B}} \quad and \quad Q = Q_I + Q_{\mathcal{B}}. \tag{29}$$

Since $dimI = 1$, we can take $P_I = \eta_1 \frac{\partial}{\partial t}$ and $Q_I + \eta_2 \frac{\partial}{\partial t}$ which gives $P = P_{\mathcal{B}} + \eta_1 \frac{\partial}{\partial t}$ and $Q = Q_{\mathcal{B}} + \eta_2 \frac{\partial}{\partial t}$ where η_1, η_2 is a function on M. From (28), (29) and Proposition 2.6, one gets

$$\begin{aligned}\overline{Ric}^{\mathcal{B}}(Z_1, Z_2) &= ag_{\mathcal{B}}(Z_1, Z_2) + bg_B(Z_1, P_B)g_B(Z_2, P_B) + c[g_B(Z_1, P_B)g_B(Z_2, Q_B) \\ &\quad + g_B(Z_2, P_B)g_B(Z_1, Q_B)] - \Big[\frac{H^f(Z_1, Z_2)}{f} + \mu_2\frac{\rho f}{f}g(Z_1, Z_2) \\ &\quad + \mu_1\mu_2\Omega(\rho)g(Z_1, Z_2) + \mu_1 g(Z_2, D_{Z_1}\rho) - \mu_1^2\Omega(Z_1)\Omega(Z_2)\Big].\end{aligned} \tag{30}$$

Now, contraction of (28) over Z_1 and Z_2, gives

$$\begin{aligned}\overline{scal}^{\mathcal{B}} &= a(n-1) + bg_{\mathcal{B}}(P_{\mathcal{B}}, P_{\mathcal{B}}) + c[g_{\mathcal{B}}(Z_1, P_{\mathcal{B}})g_{\mathcal{B}}(Z_2, Q_{\mathcal{B}}) + g_{\mathcal{B}}(Z_1, Q_{\mathcal{B}})g_{\mathcal{B}}(Z_2, P_{\mathcal{B}})] \\ &\quad - \Big[\frac{\Delta_B}{f} + \mu_2(n-1)\frac{\rho f}{f} + [(n-1)\mu_1\mu_2 - \mu_1^2]\Omega(\rho) + \mu_1\sum_{i=1}^{n-1} g(e_i, D_{e_i}\rho)\Big].\end{aligned} \tag{31}$$

Again, contraction of (28) over Z_1 and Z_2, yields

$$\overline{scal}^{M} = an + bg_{\mathcal{B}}(P_{\mathcal{B}}, P_{\mathcal{B}}) + c[g_{\mathcal{B}}(Z_1, P_{\mathcal{B}})g_{\mathcal{B}}(Z_2, Q_{\mathcal{B}}) + g_{\mathcal{B}}(Z_1, Q_{\mathcal{B}})g_{\mathcal{B}}(Z_2, P_{\mathcal{B}})]. \tag{32}$$

Making use of (32) in (31), one gets

$$\begin{aligned}\overline{scal}^{\mathcal{B}} = \overline{scal}^{M} - a - \frac{\Delta_{\mathcal{B}} f}{f} - \mu_2(n-1)\frac{\rho f}{f} - [(n-1)\mu_1\mu_2 - \mu_1^2]\Omega(\rho) \\ - \mu_1 \sum_{i=1}^{n-1} g(e_i, D_{e_i}\rho)\Big]\end{aligned} \tag{33}$$

On the other hand form Proposition 2.8, one obtains

$$\begin{aligned}\overline{scal}^{M} = \overline{scal}^{\mathcal{B}} + (n-1)(\mu_1+\mu_2)\frac{\rho f}{f} + 2\frac{\Delta_{\mathcal{B}} f}{f} + [2(n-1)\mu_1\mu_2 - (\mu_1^2+\mu_2^2)]\Omega(\rho) \\ + (\mu_1+\mu_2)\sum_{i=1}^{n-1} g(e_i, D_{e_i}\rho)\Big].\end{aligned} \tag{34}$$

From (33) and (34), we obtain

$$\begin{aligned}& a + \frac{\Delta_{\mathcal{B}} f}{f} + \mu_2(\overline{n}-1)\frac{\rho f}{f} + [(n-1)\mu_1\mu_2 - \mu_1^2]\Omega(\rho) + \mu_1\sum_{i=1}^{\overline{n}-1} g(e_i, D_{e_i}\rho)\Big] \\ &= (n-1)(\mu_1+\mu_2)\frac{\rho f}{f} + 2\frac{\Delta_{\mathcal{B}} f}{f} + [2(\overline{n}-1)\mu_1\mu_2 - (\mu_1^2+\mu_2^2)]\Omega(\rho) \\ &+ (\mu_1+\mu_2)\sum_{i=1}^{\overline{n}-1} g(e_i, D_{e_i}\rho)\Big]\end{aligned} \tag{35}$$

Since f is a constant on $\mathcal{B}$ and $\rho \in \mathfrak{X}(\mathcal{B})$ is parallel, then one gets

$$a = [(\overline{n}-1)\mu_1\mu_2 - \mu_2^2]\Omega(\rho).$$

(ii) Let $\rho \in \mathfrak{X}(I)$. By the use of Proposition 2.7, we obtain

$$\overline{Ric}(Z_1, \rho) = [(n-1)\mu_1\mu_2 - \mu_2^2]\Omega(\rho)\frac{Z_1 f}{f} \tag{36}$$

and

$$\overline{Ric}(\rho, Z_1) = [\mu_2 - (n-1)\mu_1]\Omega(\rho)\frac{Z_1 f}{f}. \tag{37}$$

Since M is a $(GQ)_n$, we have

$$\overline{Ric}(Z_1, \rho) = \overline{Ric}(\rho, Z_1) = ag(Z_1, \rho) + bA(Z_1)A(\rho) + c[A(Z_1)B(\rho) + A(\rho)B(Z_1)].$$

Again, we have $g(Z_1, \rho) = 0$ for $Z_1 \in \mathfrak{X}(\mathcal{B})$ and $\rho \in \mathfrak{X}(I)$. Thus, we obtain

$$Z_1 f = 0,$$

where $\mu_2 \neq (n-1)\mu_1$. Which implies that f is constant on $\mathcal{B}$.

(iii) Suppose that $\mathcal{B}$ is a $(GQ)_n$ with respect to the Levi-Civita connection. Then we have

$$\overline{Ric}^{\mathcal{B}}(Z_1, Z_2) = ag(Z_1, Z_2) + bA(Z_1)A(Z_2) + c[A(Z_1)B(Z_2) + A(Z_2)B(Z_1)], \tag{38}$$

for every vector fields Z_1, Z_2 tangent to $\mathcal{B}$. From Proposition 2.7, we obtain

$$\overline{Ric}^M(Z_1, Z_2) = \overline{Ric}^{\mathcal{B}}(Z_1, Z_2) + [(n-1)\mu_1\mu_2 - \mu_2^2]\Omega(\rho)g(Z_1, Z_2) + \frac{H^f(Z_1, Z_2)}{f},$$

for every vector fields $\rho \in \mathfrak{X}(I)$. Since f is a constant, $H^f(Z_1, Z_2) = 0 \ \forall \ Z_1, Z_2 \in \mathfrak{X}(\mathcal{B})$. Then the above equation reduces to

$$\overline{Ric}^M(Z_1, Z_2) = \overline{Ric}^{\mathcal{B}}(Z_1, Z_2) + [(n-1)\mu_1\mu_2 - \mu_2^2]\Omega(\rho)g(Z_1, Z_2). \tag{39}$$

Using (38) and (39), one obtains

$$\begin{aligned}\overline{Ric}^M(Z_1, Z_2) = (a + [(n-1)\mu_1\mu_2 - \mu_2^2]\Omega(\rho))g(Z_1, Z_2) + bA(Z_1)A(Z_2) \\ + c[A(Z_1)B(Z_2) + A(Z_2)B(Z_1)].\end{aligned} \tag{40}$$

This implies that M is a $(GQ)_n$ with respect to a quarter-symmetric connection. □

Theorem 2.12. *Consider the warped product manifold (M, g) of $I \times_f \mathcal{B}$. If the two generators P and Q in a $(GQ)_n$ are parallel to I with respect to a quarter-symmetric connection, then M is a $(QE)_n$ with respect to a quarter-symmetric connection.*

Proof. Let the generator P is a parallel vector field, then $\overline{K}(Z_1, Z_2)P = 0$. Thus

$$\overline{Ric}(Z_1, P) = 0. \tag{41}$$

Consider

$$P = P_{\mathcal{B}} + f^2 P_I \quad and \quad Q = Q_{\mathcal{B}} + f^2 Q_I. \tag{42}$$

From (3), we have

$$\overline{Ric}(Z_1, Z_2) = ag(Z_1, Z_2) + bA(Z_1)A(Z_2) + c[A(Z_1)B(Z_2) + A(Z_2)B(Z_1)]. \tag{43}$$

Putting $Z_2 = P$ and using (42) in (43), one gets

$$\begin{aligned}\overline{Ric}(Z_1, P) &= ag(Z_1, P) + bA(Z_1)A(P) + c[A(Z_1)B(P) + A(P)B(Z_1)] \\ &= \{a + b(f^4 + 1)\}g_I(Z_1, P_I)f^2 + c(f^4 + 1)g_I(Z_1, Q_I)f^2\end{aligned} \tag{44}$$

From (13), we have

$$\begin{aligned}\overline{Ric}_M(Z_1, Z_2) = Ric_I(Z_1, Z_2) + e^q\Big[\frac{n-1}{4}(q')^2 + \frac{1}{2}\Big\{(n-1)\mu_1 + (n-2)\mu_2\Big\}q' \\ + \mu_2^2 + \frac{1}{2}q'' + (1-n)\mu_1\mu_2\Big]g_I(Z_1, Z_2),\end{aligned} \tag{45}$$

for vector fields Z_1, Z_2 on I.
Since P is parallel to I, then from above relation

$$\begin{aligned}\overline{Ric}_M(Z_1, Z_2) &= e^q\Big[\frac{n-1}{4}(q')^2 + \frac{1}{2}\Big\{(n-1)\mu_1 + (n-2)\mu_2\Big\}q' + \mu_2^2 + \frac{1}{2}q'' \\ &+ (1-n)\mu_1\mu_2\Big]g_I(Z_1, P_B + f^2 P_I) \\ &= f^2 e^q\Big[\frac{n-1}{4}(q')^2 + \frac{1}{2}\Big\{(n-1)\mu_1 + (n-2)\mu_2\Big\}q' \\ &+ \mu_2^2 + \frac{1}{2}q'' + (1-n)\mu_1\mu_2\Big]g_I(Z_1, Z_2).\end{aligned} \tag{46}$$

Comparing (44) and (46), one obtains

$$c = 0. \tag{47}$$

Making use of (47) in (3), one gets

$$Ric(Z_1, Z_2) = ag(Z_1, Z_2) + bA(Z_1)A(Z_2),$$

i.e., $(QE)_n$ with respect to quarter symmetric connection. Similarly, if Q is parallel to I, one also gets

$$c = 0.$$

So the manifold also becomes $(QE)_n$ with respect to quarter symmetric connection.

Theorem 2.13. *Let (M, g) be a warped product $\mathcal{B} \times_f \mathcal{F}$ of a complete connected r -dimensional $(1 < k < n)$ Riemannian manifold $\mathcal{B}$ and $(n-k)$-dimensional Riemannian manifold $\mathcal{F}$.*
(i) $\mathcal{B}$ is a two-dimensional Einstein manifold if (M, g) is a space with generalized quasi-constant sectional curvature, the Hessian of f is proportional to the metric tensor $g_{\mathcal{B}}$, and the associated vector fields W and W' are the general vector field on M or W, $W' \in \mathfrak{X}(\mathcal{B})$.
(ii) $\mathcal{B}$ is a two-dimensional Einstein manifold if (M, g) is a space of generalized quasi-constant sectional curvature with the associated vector fields W, $W' \in \mathfrak{X}(\mathcal{F})$.

Let M is a generalized quasi-constant sectional curvature space. Then, using (5) we can write

$$\begin{aligned}\tilde{K}(Z_1, Z_2, Z_3, Z_4) = {} & a[g(Z_2, Z_3)g(Z_1, Z_4) - g(Z_1, Z_3)g(Z_2, Z_4)] + b[g(Z_1, Z_4)A(Z_2)A(Z_3) \\ & - g(Z_2, Z_4)A(Z_1)A(Z_3) + g(Z_2, Z_3)A(Z_1)A(Z_4) - g(Z_1, Z_3)A(Z_2)A(Z_4)] \\ & + c[g(Z_1, Z_4)B(Z_2)B(Z_3) - g(Z_2, Z_4)B(Z_1)B(Z_3) \\ & + g(Z_2, Z_3)B(Z_1)B(Z_4) - g(Z_1, Z_3)B(Z_2)B(Z_4)],\end{aligned} \tag{48}$$

for all Z_1, Z_2, Z_3, Z_4 on $\mathcal{B}$.

Decomposing the vector fields W and W' uniquely into its components $W_{\mathcal{B}}$, $W_{\mathcal{F}}$ and $W'_{\mathcal{B}}$, $W'_{\mathcal{F}}$ on $\mathcal{B}$ and $\mathcal{F}$, respectively, we can write $W = W_{\mathcal{B}} + W_{\mathcal{F}}$ and $W' = W'_{\mathcal{B}} + W'_{\mathcal{F}}$. Then we can write

$$\begin{aligned} g(Z_1, W) = g(Z_1, W_{\mathcal{B}}) = g_{\mathcal{B}}(Z_1, W_{\mathcal{B}}) = A(Z_1) \\ g(Z_1, W') = g(Z_1, W'_{\mathcal{B}}) = g_{\mathcal{B}}(Z_1, W'_{\mathcal{B}}) = B(Z_1). \end{aligned} \tag{49}$$

Making use of (6) and (49) in (48) and by use of Lemma 2.1 and then putting $Z_1 = Z_4 = e_i$, where e_i is an orthonormal basis, one obtains

$$\begin{aligned} Ric_{\mathcal{B}}(Z_2, Z_3) = {} & [a(k-1) + bg_{\mathcal{B}}(W_{\mathcal{B}}, W_{\mathcal{B}})]g_{\mathcal{B}}(Z_2, Z_3) + b(k-2)A(Z_2)A(Z_3) \\ & + c(k-1)[A(Z_2)B(Z_3) + A(Z_3)B(Z_2)]. \end{aligned} \tag{50}$$

This shows that $\mathcal{B}$ is a generalized quasi-Einstein manifold. Again, putting $Z_2 = Z = e_i$, where e_i is an orthonormal basis, one obtains

$$scal_{\mathcal{B}} = (k-1)[ak + 2bg_{\mathcal{B}}(W_{\mathcal{B}}, W_{\mathcal{B}})]. \tag{51}$$

In view of (7) and (51), we infer that

$$\frac{\Delta f}{f} = \frac{ak + bg_{\mathcal{B}}(W_{\mathcal{B}}, W_{\mathcal{B}})}{2}. \tag{52}$$

However, since the metric tensor g_B is proportional to the Hesssian of f, we can write as

$$H^f(Z_1, Z_2) = \frac{\Delta f}{k} g_{\mathcal{B}}(Z_1, Z_2). \tag{53}$$

Using (51) and (52) in (53) we get

$$H^f(Z_1, Z_2) + Rfg_{\mathcal{B}}(Z_1, Z_2) = 0,$$

where $R = \frac{(k-1)(bg_{\mathcal{B}}(W_{\mathcal{B}}W_{\mathcal{B}}))-sacl_{\mathcal{B}}}{2k(k-1)}$ holds on $\mathcal{B}$. According to OBATA's theorem [10], in $(k+1)$-dimensional Euclidean space, $\mathcal{B}$ is isometric to the sphere of radius $\frac{1}{\sqrt{R}}$. Since $\mathcal{B}$ is a result of this, we know that it is an Einstein manifold. Therefore, $k = 2$ because $b \neq 0$, $c \neq 0$. As a result, $\mathcal{B}$ is a two-dimensional Einstein manifold.

Suppose that the associated vector fields $W, W' \in \mathfrak{X}(\mathcal{B})$ then in view of (6) and (48) and then putting $Z_1 = Z_4 = e_i$, where e_i is an orthonormal basis, one obtains

$$\begin{aligned} scal_{\mathcal{B}}(Z_2, Z_3) &= [a(k-1) + b]g_{\mathcal{B}}(Z_2, Z_3) \\ &= b(k-2)g_{\mathcal{B}}(Z_2, W)g_{\mathcal{B}}(Z_3, W) + c(k-1)[g_{\mathcal{B}}(Z_2, W)g_{\mathcal{B}}(Z_3, W') \\ &+ g_{\mathcal{B}}(Z_2, W')g_{\mathcal{B}}(Z_3, W)], \end{aligned} \tag{54}$$

which shows that $\mathcal{B}$ is a $G(QE)_n$. Putting $Z_2 = Z_3 = e_i$ in (54), where e_i is an orthonormal basis, one obtains

$$scal_{\mathcal{B}} = (k-1)[ak + 2b]. \tag{55}$$

In view of (6) and (48) (for $W, W' \in \mathfrak{X}(\mathcal{B})$), one obtains

$$\frac{\Delta f}{f} = \frac{ak + b}{2}. \tag{56}$$

However, since the metric tensor $g_{\mathcal{B}}$ is proportional to the Hesssian of f, we can write as

$$H^f(Z_1, Z_2) = \frac{\Delta f}{k} g_{\mathcal{B}}(Z_1, Z_2). \tag{57}$$

Using (55) and (56) in (57) we get

$$H^f(Z_1, Z_2) + Rfg_{\mathcal{B}}(Z_1, Z_2) = 0,$$

where $R = \frac{(k-1)b - sacl_{\mathcal{B}}}{2k(k-1)}$ holds on $\mathcal{B}$. According to OBATA's theorem [10], in $(k+1)$-dimensional Euclidean space, $\mathcal{B}$ is isometric to the sphere of radius $\frac{1}{\sqrt{R}}$. Since $\mathcal{B}$ is a result of this, we know that it is an Einstein manifold. Therefore, $k = 2$ because $b \neq 0$, $c \neq 0$. As a result, $\mathcal{B}$ is a two-dimensional Einstein manifold. Suppose that the associated vector fields $W, W' \in \mathfrak{X}(\mathcal{F})$, then the relation (48) reduces to

$$\tilde{K}(Z_1, Z_2, Z_3, Z_4) = a[g(Z_2, Z_3)g(Z_1, Z_4) - g(Z_1, Z_3)g(Z_2, Z_4)]. \tag{58}$$

Making use of (6) in (58), one gets

$$\tilde{K}(Z_1, Z_2, Z_3, Z_4) = a[g_{\mathcal{B}}(Z_2, Z_3)g_{\mathcal{B}}(Z_1, Z_4) - g_{\mathcal{B}}(Z_1, Z_3)g_{\mathcal{B}}(Z_2, Z_4)]. \tag{59}$$

Contraction of (59) over Z_1 and Z_4, one gets

$$Ric_{\mathcal{B}}(Z_2, Z_3) = a(k-1)g_{\mathcal{B}}(Z_2, Z_3), \tag{60}$$

which shows that $\mathcal{B}$ is an Einstein manifold with scalar curvature $scal_{\mathcal{B}} = ak(k-1)$. This complete the proofs. □

Theorem 2.14. *Let (M, g) be a warped product $\mathcal{B} \times_f I$ of a complete connected $(n-1)$-dimensional Riemannian manifold $\mathcal{B}$ and one-dimensional Riemannian manifold I. $(\mathcal{B}, g_{\mathcal{B}})$ is a $(n-1)$-dimensional sphere with radius $rd = \frac{n-1}{\sqrt{scal_{\mathcal{B}} + a}}$ if (M, g) is a $G(QE)_n$ with constant associated scalars a, b, c and d $P, P' \in \mathfrak{X}(M)$ and if the Hessian of f is proportional to the metric tensor $g_{\mathcal{B}}$.*

Proof. Suppose that M is a warped product manifold. Then by use of Lemma 2.2 we can write

$$Ric_{\mathcal{B}}(Z_1, Z_2) = Ric_M(Z_1, Z_2) + \frac{1}{f}H^f(Z_1, Z_2), \tag{61}$$

for all Z_1, Z_2 on $\mathcal{B}$. Since M is a $G(QE)_n$, we have

$$Ric_M(Z_1, Z_2) = ag(Z_1, Z_2) = bA(Z_1)A(Z_2) + c[A(Z_1)B(Z_2) + A(Z_2)B(Z_1)]. \tag{62}$$

Decomposing the vector fields P and P' uniquely into its components P_I, P_F and P'_I, P'_F on $\mathcal{B}$ and I, respectively, we can write

$$P = P_{\mathcal{B}} + P_I \quad P' = P'_{\mathcal{B}} + P'_I. \tag{63}$$

In view of (6),(62) and (63) the relation (61) can be write as

$$\begin{aligned} Ric_{\mathcal{B}}(Z_1, Z_2) = {} & ag_{\mathcal{B}}(Z_1, Z_2) + bg_{\mathcal{B}}(Z_1, P_{\mathcal{B}})g_{\mathcal{B}}(Z_2, P_{\mathcal{B}}) + c[g_{\mathcal{B}}(Z_1, P_{\mathcal{B}})g_{\mathcal{B}}(Z_2, P'_{\mathcal{B}}) \\ & + g_{\mathcal{B}}(Z_1, P'_{\mathcal{B}})g_{\mathcal{B}}(Z_2, P_{\mathcal{B}})] + \frac{1}{f}H^f(Z_1, Z_2). \end{aligned} \tag{64}$$

Contraction above relation over Z_1 and Z_2, one gets

$$scal_{\mathcal{B}} = a(n-1) + bg_{\mathcal{B}}(P_{\mathcal{B}}, P_{\mathcal{B}}) + \frac{\Delta f}{f}. \tag{65}$$

Again Contraction of (61) over Z_1 and Z_2, one gets

$$scal_{\mathcal{B}} = an + bg_{\mathcal{B}}(P_{\mathcal{B}}, P_{\mathcal{B}}). \tag{66}$$

Making use of (66) in (65), one gets

$$scal_{\mathcal{B}} = scal_M - a + \frac{\Delta f}{f}$$

In view of Lemma 2.2, we know that

$$-\frac{scal_M}{n} = \frac{\Delta f}{f}. \tag{67}$$

The above two relations gives us $scal_{\mathcal{B}} = \frac{n-1}{n}scal_M - a$. However, since the metric tensor $g_{\mathcal{B}}$ is proportional to the Hesssian of f, we can write as

$$H^f(Z_1, Z_2) = \frac{\Delta f}{n-1}g_{\mathcal{B}}(Z_1, Z_2).$$

As the consequence of (67) we have $\frac{\Delta f}{n-1} = -\frac{1}{n(n-1)}scal_M f$, that is,

$$H^f(Z_1, Z_2) + \frac{scal_{\mathcal{B}} + a}{(n-1)^2}fg_{\mathcal{B}}(Z_1, Z_2) = 0.$$

Thus, B is isometric to the $(n-1)$-dimensional sphere of radius $rd = \frac{n-1}{\sqrt{scal_b + a}}$. □

3. Examples of 3 and 4-dimensional $G(QE)_n$

Example 3.1. *We define a Riemannian metric g in 3-dimensional space* $\mathbb{R}^3$ *by the relation*

$$ds^2 = g_{ij}dx^i dx^j = (x^3)^{4/3}[(dx^1)^2 + (dx^2)^2] + (dx^3)^2 \tag{68}$$

where x^1, x^2, x^3 *are non-zero finite. The covariant and contravariant components of the metric tensor are*

$$g_{11} = g_{22} = (x^3)^{4/3}, \quad g_{33} = 1, \quad g_{ij} = 0 \quad \forall \quad i \neq j \tag{69}$$

and

$$g^{11} = g^{22} = \frac{1}{(x^3)^{4/3}}, \quad g^{33} = 1, \quad g^{ij} = 0 \quad \forall \quad i \neq j. \tag{70}$$

The only non-vanishing components of the Christoffel symbols are

$$\begin{Bmatrix} 1 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 23 \end{Bmatrix} = \frac{2}{3x^3}, \quad \begin{Bmatrix} 3 \\ 11 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 22 \end{Bmatrix} = \frac{-2}{3}(x^3)^{\frac{1}{3}}. \tag{71}$$

The non-zero derivatives of (71), we have

$$\frac{\partial}{\partial x^3}\begin{Bmatrix} 1 \\ 13 \end{Bmatrix} = \frac{\partial}{\partial x^3}\begin{Bmatrix} 2 \\ 23 \end{Bmatrix} = \frac{-2}{3(x^3)^2}, \quad \frac{\partial}{\partial x^3}\begin{Bmatrix} 3 \\ 11 \end{Bmatrix} = \frac{\partial}{\partial x^3}\begin{Bmatrix} 3 \\ 22 \end{Bmatrix} = \frac{-2}{9(x^3)^{\frac{2}{3}}}. \tag{72}$$

For the Riemannian curvature tensor,

$$K^l_{ijk} = \underbrace{\begin{vmatrix} \frac{\partial}{\partial x^j} & \frac{\partial}{\partial x^k} \\ \begin{Bmatrix} l \\ ij \end{Bmatrix} & \begin{Bmatrix} l \\ ik \end{Bmatrix} \end{vmatrix}}_{=I} + \underbrace{\begin{vmatrix} \begin{Bmatrix} m \\ ik \end{Bmatrix} & \begin{Bmatrix} m \\ ij \end{Bmatrix} \\ \begin{Bmatrix} l \\ mk \end{Bmatrix} & \begin{Bmatrix} l \\ mj \end{Bmatrix} \end{vmatrix}}_{=II}.$$

The non-zero components of (I) are:

$$K^1_{331} = \frac{\partial}{\partial x^3}\begin{Bmatrix} 1 \\ 31 \end{Bmatrix} = \frac{-2}{3(x^3)^2},$$
$$K^2_{332} = \frac{\partial}{\partial x^3}\begin{Bmatrix} 2 \\ 32 \end{Bmatrix} = \frac{-2}{3(x^3)^2},$$

and the non-zero components of (II) are:

$$K^1_{331} = \begin{Bmatrix} m \\ 31 \end{Bmatrix}\begin{Bmatrix} 1 \\ m3 \end{Bmatrix} - \begin{Bmatrix} m \\ 33 \end{Bmatrix}\begin{Bmatrix} 1 \\ m1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 31 \end{Bmatrix}\begin{Bmatrix} 1 \\ 13 \end{Bmatrix} - \begin{Bmatrix} 1 \\ 33 \end{Bmatrix}\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \frac{4}{9(x^3)^2},$$

$$K^2_{332} = \begin{Bmatrix} m \\ 32 \end{Bmatrix}\begin{Bmatrix} 2 \\ m3 \end{Bmatrix} - \begin{Bmatrix} m \\ 33 \end{Bmatrix}\begin{Bmatrix} 2 \\ m2 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 32 \end{Bmatrix}\begin{Bmatrix} 2 \\ 23 \end{Bmatrix} - \begin{Bmatrix} 2 \\ 33 \end{Bmatrix}\begin{Bmatrix} 2 \\ 22 \end{Bmatrix} = \frac{4}{9(x^3)^2},$$

$$K^1_{221} = \begin{Bmatrix} m \\ 21 \end{Bmatrix}\begin{Bmatrix} 1 \\ m2 \end{Bmatrix} - \begin{Bmatrix} m \\ 22 \end{Bmatrix}\begin{Bmatrix} 1 \\ m1 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 21 \end{Bmatrix}\begin{Bmatrix} 1 \\ 32 \end{Bmatrix} - \begin{Bmatrix} 3 \\ 22 \end{Bmatrix}\begin{Bmatrix} 1 \\ 31 \end{Bmatrix} = \frac{4}{9(x^3)^{\frac{2}{3}}},$$

Adding components corresponding (I) and (II), we have

$$K^1_{221} = \frac{4}{9(x^3)^{\frac{2}{3}}}, \; K^1_{331} = \frac{-2}{9(x^3)^2} = K^2_{332}.$$

Thus, the non-zero components of curvature tensor, up to symmetry are,

$$\overline{K}_{1331} = \overline{K}_{2332} = \frac{-2}{9(x^3)^{\frac{2}{3}}}, \quad \overline{K}_{1221} = \frac{4}{9}(x^3)^{\frac{2}{3}},$$

and the Ricci tensor

$$Ric_{11} = g^{jh}\overline{K}_{1j1h} = g^{22}\overline{K}_{1212} + g^{33}\overline{K}_{1313} = \frac{2}{9(x^3)^{\frac{2}{3}}},$$

$$Ric_{22} = g^{jh}\overline{K}_{2j2h} = g^{11}\overline{K}_{2121} + g^{33}\overline{K}_{2323} = \frac{2}{9(x^3)^{\frac{2}{3}}},$$

$$Ric_{33} = g^{jh}\overline{K}_{3j3h} = g^{11}\overline{K}_{3131} + g^{22}\overline{K}_{3232} = \frac{-4}{9(x^3)^2},$$

Let us consider the associated scalars a, b, c and the 1-forms are defined by

$$a = \frac{-4}{9(x^3)^2}, \ b = \frac{6(x^3)^{\frac{4}{3}}}{9}, c = \frac{1}{9(x^3)^2},$$

$$A_i(x) = \begin{cases} \frac{1}{x^3}, & \text{if } i=1 \\ (x^3)^{\frac{2}{3}}, & \text{if } i=2 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad B_i(x) = \begin{cases} (x^3)^{\frac{2}{3}}, & \text{if } i=2 \\ 0, & \text{otherwise} \end{cases}$$

where generators are unit vector fields, then from (3), we have

$$Ric_{11} = ag_{11} + bA_1A_1 + 2cA_1B_1, \tag{73}$$

$$Ric_{22} = ag_{22} + bA_2A_2 + 2cA_2B_2, \tag{74}$$

$$Ric_{33} = ag_{33} + bA_3A_3 + 2cA_3B_3, \tag{75}$$

$$\begin{aligned} R.H.S. \quad of \quad (73) &= ag_{11} + bA_1A_1 + 2cA_1B_1 \\ &= \frac{-4}{9(x^3)^{\frac{2}{3}}} + \frac{6}{9(x^3)^{\frac{2}{3}}} \\ &= \frac{2}{9(x^3)^{\frac{2}{3}}} \\ &= L.H.S. \quad of \quad (73) \end{aligned}$$

By similar argument it can be shown that (74) and (75) are also true.
Hence $(\mathbb{R}^3, g)$ *is a* $G(QE)_3$.

Example 3.2. *Lorentzian manifold* $(\mathbb{R}^3, g)$ *endowed with the metric given by*

$$ds^2 = g_{ij}dx^idx^j = -(x^3)^{4/3}[(dx^1)^2 + (dx^2)^2] + (dx^3)^2, \tag{76}$$

where x^1, x^2, x^3 *are non-zero finite, then* $(\mathbb{R}^3, g)$ *is a* $G(QE)_3$.

Example 3.3. *We define a Riemannian metric g in 4-dimensional space* $\mathbb{R}^4$ *by the relation*

$$ds^2 = g_{ij}dx^idx^j = (1 + 2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2] \tag{77}$$

where x^1, x^2, x^3, x^4 *are non-zero finite and* $p = e^{x^1}k^{-2}$. *Then the covariant and contravariant components of the metric are*

$$g_{11} = g_{22} = g_{33} = g_{44} = (1 + 2p), \quad g_{ij} = 0 \quad \forall \quad i \neq j \tag{78}$$

and

$$g^{11} = g^{22} = g^{33} = g^{44} = \frac{1}{1+2p}, \quad g^{ij} = 0 \quad \forall \quad i \neq j. \tag{79}$$

The only non-vanishing components of the Christoffel symbols are

$$\begin{aligned} &\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 14 \end{Bmatrix} = \frac{\partial}{\partial x^1} \begin{Bmatrix} 4 \\ 14 \end{Bmatrix} = \frac{p}{1+2p}, \\ &\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = \frac{-p}{1+2p}. \end{aligned} \tag{80}$$

The non-zero derivatives of (80), we have

$$\begin{aligned} &\frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \frac{p}{(1+2p)^2}, \\ &\frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = \frac{-p}{1+2p)^2}. \end{aligned} \tag{81}$$

For the Riemannian curvature tensor,

$$K^l_{ijk} = \underbrace{\begin{vmatrix} \frac{\partial}{\partial x^j} & \frac{\partial}{\partial x^k} \\ \begin{Bmatrix} l \\ ij \end{Bmatrix} & \begin{Bmatrix} l \\ ik \end{Bmatrix} \end{vmatrix}}_{=I} + \underbrace{\begin{vmatrix} \begin{Bmatrix} m \\ ik \end{Bmatrix} & \begin{Bmatrix} m \\ ij \end{Bmatrix} \\ \begin{Bmatrix} l \\ mk \end{Bmatrix} & \begin{Bmatrix} l \\ mj \end{Bmatrix} \end{vmatrix}}_{=II}.$$

The non-zero components of (I) are:

$$\begin{aligned} K^1_{212} &= \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \frac{-p}{(1+2p)^2}, \\ K^1_{313} &= \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \frac{-p}{(1+2p)^2}, \\ K^1_{414} &= \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = \frac{-p}{(1+2p)^2} \end{aligned}$$

and the non-zero components of (II) are:

$$\begin{aligned} K^2_{332} &= \begin{Bmatrix} m \\ 32 \end{Bmatrix}\begin{Bmatrix} 2 \\ m3 \end{Bmatrix} - \begin{Bmatrix} m \\ 33 \end{Bmatrix}\begin{Bmatrix} 2 \\ m2 \end{Bmatrix} = -\begin{Bmatrix} 1 \\ 33 \end{Bmatrix}\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{p^2}{(1+2p)^2}, \\ K^2_{442} &= \begin{Bmatrix} m \\ 42 \end{Bmatrix}\begin{Bmatrix} 2 \\ m4 \end{Bmatrix} - \begin{Bmatrix} m \\ 44 \end{Bmatrix}\begin{Bmatrix} 2 \\ m2 \end{Bmatrix} = -\begin{Bmatrix} 1 \\ 44 \end{Bmatrix}\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{p^2}{(1+2p)^2}, \\ K^3_{443} &= \begin{Bmatrix} m \\ 43 \end{Bmatrix}\begin{Bmatrix} 3 \\ m4 \end{Bmatrix} - \begin{Bmatrix} m \\ 44 \end{Bmatrix}\begin{Bmatrix} 3 \\ m3 \end{Bmatrix} = -\begin{Bmatrix} 1 \\ 44 \end{Bmatrix}\begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \frac{p^2}{(1+2p)^2}. \end{aligned}$$

Adding components corresponding (I) and (II), we have

$$K^1_{221} = K^1_{331} = K^1_{441} = \frac{p}{(1+2p)^2},$$

$$K^2_{332} = K^2_{442} = K^3_{443} = \frac{p^2}{(1+2p)^2}.$$

Thus, the non-zero components of curvature tensor, up to symmetry are given by

$$\overline{K}_{1221} = \overline{K}_{1331} = \overline{K}_{1441} = \frac{p}{1+2p},$$

$$\overline{K}_{2332} = \overline{K}_{2442} = \overline{K}_{3443} = \frac{p^2}{1+2p}$$

and the Ricci tensor are given by

$$Ric_{11} = g^{jh}\overline{K}_{1j1h} = g^{22}\overline{K}_{1212} + g^{33}\overline{K}_{1313} + g^{44}\overline{K}_{1414} = \frac{3p}{(1+2p)^2},$$

$$Ric_{22} = g^{jh}\overline{K}_{2j2h} = g^{11}\overline{K}_{2121} + g^{33}\overline{K}_{2323} + g^{44}\overline{K}_{2424} = \frac{p}{(1+2p)},$$

$$Ric_{33} = g^{jh}\overline{K}_{3j3h} = g^{11}\overline{K}_{3131} + g^{22}\overline{K}_{3232} + g^{44}\overline{K}_{3434} = \frac{p}{(1+2p)},$$

$$Ric_{44} = g^{jh}\overline{K}_{4j4h} = g^{11}\overline{K}_{4141} + g^{22}\overline{K}_{4242} + g^{33}\overline{K}_{4343} = \frac{p}{(1+2p)}.$$

The scalar curvature r is given by

$$r = g^{11}Ric_{11} + g^{22}Ric_{22} + g^{33}Ric_{33} + g^{44}Ric_{44} = \frac{6p(1+p)}{(1+2p)^3}.$$

Let us consider the associated scalars a, b, c and the 1-forms are defined by

$$a = \frac{3p}{(1+2p)^3}, \quad b = 2p, \quad c = \frac{-p}{(1+2p)^2}$$

$$A_i(x) = \begin{cases} \frac{1}{1+2p}, & \text{if } i=1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad B_i(x) = \begin{cases} 1, & \text{if } i=1 \\ -1, & \text{if } i=2 \\ 0, & \text{otherwise} \end{cases}$$

where generators are unit vector fields, then from (3), we have

$$Ric_{11} = ag_{11} + bA_1A_1 + 2cA_1B_1, \tag{82}$$

$$Ric_{22} = ag_{22} + bA_2A_2 + 2cA_2B_2, \tag{83}$$

$$Ric_{33} = ag_{33} + bA_3A_3 + 2cA_3B_3, \tag{84}$$

$$Ric_{44} = ag_{44} + bA_4A_4 + 2cA_4B_4, \tag{85}$$

$$\begin{aligned} R.H.S. \quad of \quad (82) &= ag_{11} + bA_1A_1 + 2cA_1B_1 \\ &= \frac{3p}{(1+2p)^2} + \frac{2p}{(1+2p)^2} - \frac{2p}{(1+2p)^2} \\ &= \frac{3p}{(1+2p)^2} \\ &= L.H.S. \quad of \quad (82) \end{aligned}$$

By similar argument it can be shown that (83) *to* (85) *are also true.*
Hence $(\mathbb{R}^4, g)$ *is a* $G(QE)_4$.

Example 3.4. *Lorentzian manifold* $(\mathbb{R}^4, g)$ *endowed with the metric given by*

$$ds^2 = g_{ij}dx^idx^j = -(1+2p)(dx^1)^2 + (1+2p)[(dx^2)^2 + (dx^3)^2 + (dx^4)^2]$$

where x^1, x^2, x^3 *and* x^4 *are non-zero finite, then* $(\mathbb{R}^4, g)$ *is a* $G(QE)_4$.

4. Example of generalized quasi-Einstein warped product manifold

In this section, we will have look at examples 3.1 and 3.3, which is a three and four dimensional examples of a generalized quasi-Einstein manifold.

Example 4.1. *Let us assume that the Riemannian manifold denoted by* (R^3, g) *is endowed with the metric*

$$ds^2 = g_{ij}dx^i dx^j = (x^3)^{4/3}[(dx^1)^2 + (dx^2)^2] + (dx^3)^2,$$

where x^1, x^2, x^3 *are non-zero finite. In order to define the warped product on* $G(QE)_3$, *we consider the warping function* $f : \mathbb{R}_{\neq 0} \to (0, \infty)$ *by* $f(x^3) = (x^3)^{\frac{2}{3}}$ *and notice that* $f = (x^3)^{\frac{2}{3}} > 0$ *is a smooth function. This allows us to define the warped product. The line element that is defined on* $\mathbb{R}_{\neq 0} \times R^2$ *and has the form* $B \times_f F$, *where* $B = \mathbb{R}_{\neq 0}$ *is the base and* $F = R^2$ *is the fibre.*

So, we can write $ds^2_M = ds^2_B + f^2 ds^2_F$, *i.e.,*

$$ds^2 = g_{ij}dx^i dx^j = (dx^2)^2 + (dx^3)^2 + \{(x^3)^{2/3}\}[(dx^1)^2 + (dx^2)^2],$$

which represents an example of a Riemannian warped product on $G(QE)_3$.

Example 4.2. *Let us assume that the Riemannian manifold denoted by* (R^4, g) *is endowed with the metric*

$$ds^2 = g_{ij}dx^i dx^j = (1 + 2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2],$$

where x^1, x^2, x^3 *and* x^4 *are non-zero finite. In order to define the warped product on* $G(QE)_3$, *we consider the warping function* $f : R^3 \to (0, \infty)$ *by* $f(x^1, x^2, x^3) = \sqrt{1 + 2p}$ *and notice that* $f > 0$ *is a smooth function. This allows us to define the warped product. The line element that is defined on* $R^3 \times R$ *and has the form* $B \times_f F$, *where* $B = R^3$ *is the base and* $F = R$ *is the fibre.*

So, we can write $ds^2_M = ds^2_B + f^2 ds^2_F$, *i.e.,*

$$ds^2 = g_{ij}dx^i dx^j = (1 + 2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + \sqrt{1 + 2p}(dx^4)^2,$$

which also represents an example of a Riemannian warped product on $G(QE)_4$.

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