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Hypercyclic operators on Hilbert C*-modules

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Abstract. In this paper we characterize hypercyclic generalized bilateral weighted shift operators on the standard Hilbert module over the C^*-algebra of compact operators on the separable Hilbert space. Moreover, we give necessary and sufficient conditions for these operators to be chaotic and we provide concrete examples.

1. Introduction

Hypercyclicity and topological transitivity, as important linear dynamical properties of bounded linear operators, have been investigated in many research works; see [1, 3, 6, 8, 15] and their references. Specially, hypercyclic weighted shifts on $\ell^p(\mathbb{Z})$ were characterized in [10, 18], and then C.C. Chen and C.H. Chu, using aperiodic elements of locally compact groups, extended the results in [18] to weighted translations on Lebesgue spaces in the context of a second countable group [7].

Recently, in [12] we have for instance characterized hypercyclic weighted composition operators on the commutative C^*-algebra of continuous functions vanishing at infinity on a locally compact, non-compact Hausdorff space. Moreover, in [13] and [14] we have characterized hypercyclic elementary operators on the C^*-algebra of compact operators on a separable Hilbert space. The dynamics of some similar operators have been considered earlier such as conjugate operators, see [17], and left multiplication operators, see [5, 19, 20].

The main aim of this paper is to study the dynamics of generalized bilateral weighted shift operators on the standard Hilbert C^*-module over the C^*-algebra of compact operators on a separable Hilbert space, thus to generalize in this setting the results from [10, 18]. In Section 3 we characterize hypercyclic such operators and we also give necessary and sufficient conditions for these operators to be chaotic. In addition, we provide concrete examples.

Moreover, in Section 4 we provide an algebraic generalization of our results given in [12, 13] to the case of arbitrary non-unital C^*-algebras.

2. Preliminaries

If $\mathcal{X}$ is a Banach space, the set of all bounded linear operators from $\mathcal{X}$ into $\mathcal{X}$ is denoted by $B(\mathcal{X})$. Also, we denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

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Definition 2.1. *[11, Definition 2.1] Let $\mathcal{X}$ be a Banach space. A sequence $(T_n)_{n\in\mathbb{N}_0}$ of operators in $B(\mathcal{X})$ is called topologically transitive if for each non-empty open subsets U, V of $\mathcal{X}$, $T_n(U)\cap V \neq \emptyset$ for some $n \in \mathbb{N}$. If $T_n(U)\cap V \neq \emptyset$ holds from some n onwards, then $(T_n)_{n\in\mathbb{N}_0}$ is called topologically mixing.*

Definition 2.2. *[11, Definition 2.2] Let $\mathcal{X}$ be a Banach space. A sequence $(T_n)_{n\in\mathbb{N}_0}$ of operators in $B(\mathcal{X})$ is called hypercyclic if there is an element $x \in \mathcal{X}$ (called* hypercyclic vector*) such that the orbit $\{T_n x : n \in \mathbb{N}_0\}$ is dense in $\mathcal{X}$. The set of all hypercyclic vectors of a sequence $(T_n)_{n\in\mathbb{N}_0}$ is denoted by $HC((T_n)_{n\in\mathbb{N}_0})$. If $HC((T_n)_{n\in\mathbb{N}_0})$ is dense in $\mathcal{X}$, the sequence $(T_n)_{n\in\mathbb{N}_0}$ is called* densely hypercyclic. *An operator $T \in B(\mathcal{X})$ is called* hypercyclic *if the sequence $(T^n)_{n\in\mathbb{N}_0}$ is hypercyclic.*

Note that a sequence $(T_n)_{n\in\mathbb{N}_0}$ of operators in $B(\mathcal{X})$ is topologically transitive if and only if it is densely hypercyclic [9]. Also, a Banach space admits a hypercyclic operator if and only if it is separable and infinite-dimensional [1, 3].

Definition 2.3. *[11, Definition 2.3] Let $\mathcal{X}$ be a Banach space, and $(T_n)_{n\in\mathbb{N}_0}$ be a sequence of operators in $B(\mathcal{X})$. A vector $x \in \mathcal{X}$ is called a periodic element of $(T_n)_{n\in\mathbb{N}_0}$ if there exists a constant $N \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $T_{kN}x = x$. The set of all periodic elements of $(T_n)_{n\in\mathbb{N}_0}$ is denoted by $\mathcal{P}((T_n)_{n\in\mathbb{N}_0})$. The sequence $(T_n)_{n\in\mathbb{N}_0}$ is called chaotic if $(T_n)_{n\in\mathbb{N}_0}$ is topologically transitive and $\mathcal{P}((T_n)_{n\in\mathbb{N}_0})$ is dense in $\mathcal{X}$. An operator $T \in B(\mathcal{X})$ is called* chaotic *if the sequence $\{T^n\}_{n\in\mathbb{N}_0}$ is chaotic.*

3. Generalized weighted bilateral shift operators over C^*-algebras

Let H be a separable Hilbert space. The C^*-algebra of all bounded linear operators on H is denoted by $B(H)$ whereas we let $\mathcal{A} := B_0(H)$ be the C^*-algebra of all compact operators on H. For every self-adjoint $T, S \in B(H)$ we denote $T \leq S$ whenever $\langle (T-S)h, h\rangle \geq 0$ for all $h \in H$. Assume that $\{e_j\}_{j\in\mathbb{Z}}$ is an orthonormal basis for H, and for each $m \in \mathbb{N}$, P_m is the orthogonal projection onto $\mathrm{Span}\{e_{-m}, \dots, e_m\}$. Let $W := \{W_j\}_{j\in\mathbb{Z}}$ be a uniformly bounded sequence of invertible operators in $B(H)$ such that the sequence $\{W_j^{-1}\}_{j\in\mathbb{Z}}$ is also uniformly bounded in $B(H)$. Moreover, let U be a unitary operator on H. We define $T_{U,W}$ to be the operator on $\ell_2(\mathcal{A})$, the standard right Hilbert module over $\mathcal{A}$, given by

$$(T_{U,W}(x))_\xi := W_\xi \, x_{\xi-1} \, U$$

for all $\xi \in \mathbb{Z}$ and $x := (x_j)_{j\in\mathbb{Z}} \in \ell_2(\mathcal{A})$. It is easy to see that $T_{U,W}$ is a linear operator. Put $M := \sup_{j\in\mathbb{Z}} \|W_j\|$. Then, since for all $j \in \mathbb{Z}$, $M^2 U^* x_{j-1}^* x_{j-1} U - U^* x_{j-1}^* W_j^* W_j x_{j-1} U$ is a positive semidefinite operator on H, we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}} U^* x_{j-1}^* W_j^* W_j x_{j-1} U &\leq M^2 \sum_{j\in\mathbb{Z}} U^* x_{j-1}^* x_{j-1} U \\ &= M^2\, U^* (\sum_{j\in\mathbb{Z}} x_{j-1}^* x_{j-1}) U \\ &= M^2\, U^* \,\langle x, x\rangle\, U,\end{aligned}$$

so $\mathrm{Im} T_{U,W} \subseteq \ell_2(\mathcal{A})$ and $\|T_{U,W}\| \leq M$. Moreover, $T_{U,W}$ is invertible and its inverse $S_{U,W}$ is given by

$$(S_{U,W}(y))_\xi := W_{\xi+1}^{-1} y_{\xi+1} U^*$$

for all $y := (y_j)_j \in \ell_2(\mathcal{A})$ and $\xi \in \mathbb{Z}$. By some calculations we can see that

$$(T_{U,W}^n(x))_\xi = W_\xi W_{\xi-1} \dots W_{\xi-n+1} x_{\xi-n} U^n$$

and

$$(S_{U,W}^n(y))_\xi := W_{\xi+1}^{-1} W_{\xi+2}^{-1} \dots W_{\xi+n}^{-1} y_{\xi+n} U^{*n}$$

for all $n \in \mathbb{N}$, $\xi \in \mathbb{Z}$ and $x := (x_j)_j$, $y := (y_j)_j$ in $\ell_2(\mathcal{A})$.

For each $J \in \mathbb{N}$, we denote $[J] := \{-J, -J+1, \dots, J-1, J\}$. In the following result, we give some equivalent condition for a sequence of powers of an operator $T_{U,W}$ to be densely hypercyclic on $\ell_2(\mathcal{A})$.

Proposition 3.1. *Let $(t_n)_n$ be an unbounded sequence of nonnegative integers. We denote $T_{U,W,n} := T_{U,W}^{t_n}$ for all $n \in \mathbb{N}$. Then, the followings are equivalent:*

1. *$(T_{U,W,n})_n$ is a densely hypercyclic sequence on $\ell_2(\mathcal{A})$.*
2. *For every $J, m \in \mathbb{N}$ there exist a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ and sequences $\{D_i^{(k)}\}_k$ and $\{G_i^{(k)}\}_k$ for all $i \in [J]$ of operators in $B_0(H)$ such that*

$$\lim_{k\to\infty} \|D_j^{(k)} - P_m\| = \lim_{k\to\infty} \|G_j^{(k)} - P_m\| = 0$$

and

$$\lim_{k\to\infty} \|W_{j+t_{n_k}} W_{j+t_{n_k}-1} \dots W_{j+1} D_j^{(k)}\|$$
$$= \lim_{k\to\infty} \|W_{j-t_{n_k}+1}^{-1} W_{j-t_{n_k}+2}^{-1} \dots W_j^{-1} G_j^{(k)}\| = 0$$

for all $j \in [J]$.

Proof. (1) $\Rightarrow$ (2): Let $(T_{U,W,n})_n$ be densely hypercyclic. Assume that $J, m \in \mathbb{N}$, and define $x = (x_j)_j \in \ell_2(\mathcal{A})$ by $x_j := P_m$ for all $j \in [J]$, and $x_j := 0$ for all $j \notin [J]$. Then, for each $k \in \mathbb{N}$, there exist an element $y^{(k)} \in \ell_2(\mathcal{A})$ and a term t_{n_k} such that $\|y^{(k)} - x\|_2 < \frac{1}{4^k}$ and $\|T_{U,W,n_k}(y^{(k)}) - x\|_2 < \frac{1}{4^k}$. We can assume that the sequence $(n_k)_k$ is strictly increasing, and $2J < t_{n_1} < t_{n_2} < \dots$. Hence,

$$\|W_j W_{j-1} \dots W_{j-t_{n_k}+1} y_{j-t_{n_k}} U^{t_{n_k}} - P_m\| < \frac{1}{4^k}$$

for all $j \in [J]$. However, since $t_{n_k} > 2J$ and $\|y^{(k)} - x\|_2 < \frac{1}{4^k}$, we have $\|y_{j-t_{n_k}}^{(k)}\| < \frac{1}{4^k}$ as $x_{j-t_{n_k}} = 0$ for all $j \in [J]$. Thus

$$\|W_{j-t_{n_k}+1}^{-1} \dots W_j^{-1} W_j \dots W_{j-t_{n_k}+1} y_{j-t_{n_k}}^{(k)} U^{t_{n_k}}\| = \|y_{j-t_{n_k}}^{(k)} U^{t_{n_k}}\| = \|y_{j-t_{n_k}}^{(k)}\| < \frac{1}{4^k}$$

for all $j \in [J]$. Similarly, since $\|T_{U,W}^{t_{n_k}}(y^{(k)}) - x\|_2 < \frac{1}{4^k}$, we have

$$\|W_{j+t_{n_k}} \dots W_{j+1} y_j^{(k)} U^{t_{n_k}}\| < \frac{1}{4^k},$$

so $\|W_{j+t_{n_k}} \dots W_{j+1} y_j^{(k)}\| < \frac{1}{4^k}$. Set

$$D_j^{(k)} := y_j^{(k)} \qquad \text{and} \qquad G_j^{(k)} := W_j W_{j-1} \dots W_{j-t_{n_k}+1} y_{j-t_{n_k}}^{(k)} U^{t_{n_k}}$$

for all $j \in [J]$. Then,

$$\|D_j^{(k)} - P_m\| < \frac{1}{4^k},\ \|G_j^{(k)} - P_m\| < \frac{1}{4^k},\ \|W_{j+t_{n_k}} \dots W_{j+1} D_j^{(k)}\| < \frac{1}{4^k}$$

and $\|W_{j-t_{n_k}+1}^{-1} \dots W_j^{-1} G_j^{(k)}\| < \frac{1}{4^k}$. Notice that since the coefficients of $y^{(k)}$ belong to $\mathcal{A} = B_0(H)$ which is an ideal of $B(H)$, by construction, $D_j^{(k)}$ and $G_j^{(k)}$ belong to $B_0(H)$ for all $j \in [J]$. This completes the proof.

(2) $\Rightarrow$ (1): Assume that the condition (2) holds. Choose two non-empty open subsets O_1 and O_2 of $\ell_2(\mathcal{A})$. Assume that $\mathcal{F}$ denotes the set of all elements $x = (x_j)_j \in \ell_2(\mathcal{A})$ such that for some $J, m \in \mathbb{N}$, $x_j = 0$ for all $j \notin [J]$ and $x_j = P_m x_j$ for all $j \in [J]$. Since $\mathcal{F}$ is dense in $\ell_2(\mathcal{A})$ [16, Proposition 2.2.1], we can find some $x = (x_j)_j \in O_1$ and $y = (y_j)_j \in O_2$ and sufficiently large J, m such that $x_j = y_j = 0$ for all $j \notin [J]$ and $x_j = P_m x_j$ and $y_j = P_m y_j$ for all $j \in [J]$. Choose the sequences $\{D_j^{(k)}\}_k$ and $\{G_j^{(k)}\}_k$ for $j \in [J]$ and the increasing sequence $\{n_k\}_k$ satisfying (ii) regarding these J, m. For each k, let u_k and v_k be sequences in $\ell_2(\mathcal{A})$ defined by $(u_k)_j := D_j^{(k)} x_j$ for $j \in [J]$, $(u_k)_j := 0$ for $j \notin [J]$, $(v_k)_j := G_j^{(k)} y_j$ for $j \in [J]$ and $(v_k)_j := 0$ for all $j \notin [J]$. Set

$$\eta_k := u_k + S_{U,W}^{t_{n_k}} v_k.$$

Since $\|D_j^{(k)} - P_m\| \to 0$ and $\|G_j^{(k)} - P_m\| \to 0$ as k tends to ∞, and $x_j = P_m x_j$ and $y_j = P_m y_j$ for $j \in [J]$, it would be routine to see that $u_k \to x$ and $v_k \to y$ as $k \to \infty$. Next, for each $j \in [J]$ we have

$$\begin{aligned}\|(S_{U,W}^{t_{n_k}}(v_k))_{j-t_{n_k}}\| &= \|W_{j+1-t_{n_k}}^{-1} \dots W_j^{-1} G_j^{(k)} y_j U^{-t_{n_k}}\| \\ &\leq \|W_{j+1-t_{n_k}}^{-1} \dots W_j^{-1} G_j^{(k)}\| \, \|y_j\| \to 0,\end{aligned}$$

as $k \to \infty$. On the other hand, for each $j \notin [J]$ we have $(S_{U,W}^{t_{n_k}}(v_k))_{j-t_{n_k}} = 0$. Thus, $S_{U,W}^{t_{n_k}}(v_k) \to 0$ as $k \to \infty$. Similarly, since

$$\| T_{U,W}^{t_{n_k}}(\mu_k)_{j+t_{n_k}} \| = \| W_{j+t_{n_k}} \dots W_j D_j^{(k)} x_j U^{t_{n_k}} \|$$

$$\leq \| W_{j+t_{n_k}} \dots W_j D_j^{(k)} \| \quad \| x_j \| \to 0$$

as $k \to \infty$ for all $j \in [J]$ and $T_{U,W}^{t_{n_k}}(\mu_k)_{j+t_{n_k}} = 0$ for $j \notin [J]$, we have that $T_{U,W}^{t_{n_k}}(u_k) \to 0$ as $k \to \infty$. It follows that $\eta_k \to x$ and $T_{U,W}^{t_{n_k}}(\eta_k) \to y$ as $k \to \infty$. Hence, the sequence $(T_{U,W,n})_n$ is topologically transitive, and thus it is densely hypercyclic on $\ell_2(\mathcal{A})$. □

Theorem 3.2. *Let $(t_n)_n$ be an unbounded sequence of nonnegative integers. Suppose that for every $j \in \mathbb{Z}$ there exist subsets $H_j^{(1)}$ and $H_j^{(2)}$ of H and a strictly increasing sequence $(n_k)_k \subseteq \mathbb{N}$ such that*

$$\lim_{k\to\infty} W_{j+t_{n_k}} W_{j+t_{n_k}-1} \dots W_{j+1} = 0 \quad \textit{pointwise on } H_j^{(1)}$$

and

$$\lim_{k\to\infty} W_{j-t_{n_k}+1}^{-1} W_{j-t_{n_k}+2}^{-1} \dots W_j^{-1} = 0 \quad \textit{pointwise on } H_j^{(2)}$$

for all $j \in \mathbb{Z}$. Then, the sequence $(T_{U,W,n})_n$ is densely hypercyclic on $\ell_2(\mathcal{A})$, where $T_{U,W,n} := T_{U,W}^{t_n}$ for all $n \in \mathbb{N}$.

Proof. Assume that $m, J \in \mathbb{N}$. Since for each $j \in \mathbb{Z}$, $H_j^{(1)}$ and $H_j^{(2)}$ are dense in H, we can find sequences $(f_{i,l}^{(j)})_i \subseteq H_j^{(1)}$ and $(g_{i,l}^{(j)})_i \subseteq H_j^{(2)}$ such that $f_{i,l}^{(j)} \to e_l$ and $g_{i,l}^{(j)} \to e_l$ as $i \to \infty$ for all $j \in [J]$ and $l \in [m]$. By the assumptions, one can construct a subsequence $(n_{k_i})_i$ such that

$$\|W_{j+t_{n_{k_i}}} W_{j+t_{n_{k_i}}-1} \dots W_{j+1} f_{i,l}^{(j)}\| < \frac{1}{2mi}$$

and

$$\|W_{j-t_{n_{k_i}}+1}^{-1} W_{j-t_{n_{k_i}}+2}^{-1} \dots W_j^{-1} g_{i,l}^{(j)}\| < \frac{1}{2mi}$$

for all $j \in [J]$ and $l \in [m]$. For each $j \in [J]$ define the operators $D_j^{(i)}$ and $G_j^{(i)}$ as

$$D_j^{(i)} e_l := \begin{cases} f_{i,l}^{(j)}, & \text{if } l \in [m] \\ 0, & \text{if } l \notin [m] \end{cases} \quad \text{and} \quad G_j^{(i)} e_l := \begin{cases} g_{i,l}^{(j)}, & \text{if } l \in [m] \\ 0, & \text{if } l \notin [m]. \end{cases}$$

By using the fact that strong convergence and uniform convergence coincide on finite dimensional subspaces, we can do the same as in the proof of [14, Proposition 2.7]. □

Example 3.3. *Let $H := L^2(\mathbb{R})$. Assume that $(w_j)_{j\in\mathbb{Z}} \subseteq L^\infty(\mathbb{R})$ such that each w_j is positive and invertible in $L^\infty(\mathbb{R})$, and also there exists an $M > 0$ such that $\|w_j\|_\infty, \|w_j^{-1}\|_\infty \leq M$ for all $j \in \mathbb{Z}$. Assume in addition that there exists an $\epsilon > 0$ such that $|w_j \chi_{[0,\infty)}| \leq 1 - \epsilon$ for all $j \geq 0$ and $|w_j \chi_{(-\infty,0)}| \geq 1 + \epsilon$ for all $j < 0$. Let $(r_j)_j$ be a sequence of positive* *numbers such that $r_j \geq C$ for all $j \in \mathbb{Z}$ and some $C > 0$. For each $j \in \mathbb{Z}$ let α_j to be the translation on $\mathbb{R}$ given by $\alpha_j(t) := t - r_j$. For each $j \in \mathbb{Z}$ assume that W_j is an operator on $L^2(\mathbb{R})$ defined by*

$$W_j(f) := w_j\,(f \circ \alpha_j)$$

for every $f \in L^2(\mathbb{R}) = H$. Then, each W_j is invertible in $B(H)$, and $\|W_j\|$, $\|W_j^{-1}\| \leq M$. By some calculations we have

$$W_{j+n}W_{j+n-1}\dots W_j f = w_{j+n}(w_{j+n-1} \circ \alpha_{j+n})\dots (w_j \circ \alpha_{j+1} \circ \dots \alpha_{j+n})(f \circ \alpha_j \circ \dots \circ \alpha_{j+n})$$

for all $f \in H$ and $j, n \in \mathbb{N}$. It follows that

$$\|W_{j+n}W_{j+n-1}\dots W_j f\| \leq \sup_{t \in \operatorname{supp} f} \Big((w_{j+n} \circ \alpha_{j+n}^{-1} \circ \dots \circ \alpha_j^{-1})(w_{j+n-1} \circ \alpha_{j+n-2}^{-1} \circ \dots \circ \alpha_j^{-1})\dots (w_j \circ \alpha_j^{-1})\Big)(t)\,\|f\|$$

for all $f \in H$ and $j, n \in \mathbb{N}$. Similarly, since for each j we have $W_j^{-1}(f) = (w_j^{-1} \circ \alpha_j^{-1})\,(f \circ \alpha_j^{-1})$ for all $f \in H$, we get that

$$W_{j-n+1}^{-1}W_{j-n+2}^{-1}\dots W_j^{-1} f = (w_{j-n+1}^{-1} \circ \alpha_{j-n+1}^{-1})\,(w_{j-n+2}^{-1} \circ \alpha_{j-n+2}^{-1} \circ \alpha_{j-n+1}^{-1})\dots (w_j^{-1} \circ \alpha_j^{-1} \circ \dots \circ \alpha_{j-n+1}^{-1})(f \circ \alpha_j^{-1} \circ \dots \circ \alpha_{j-n+1}^{-1})$$

for all $f \in H$ and $j, n \in \mathbb{N}$. Hence,

$$\|W_{j-n+1}^{-1}W_{j-n+2}^{-1}\dots W_j^{-1} f\| \leq \sup_{t \in \operatorname{supp} f} \Big((w_{j-n+1}^{-1} \circ \alpha_{j-n+2} \circ \dots \circ \alpha_j)\,(w_{j-n+2}^{-1} \circ \alpha_{j-n+3} \circ \dots \circ \alpha_j)\dots w_j\Big)(t)\,\|f\|$$

for all $f \in H$ and $j, n \in \mathbb{N}$. It follows that for every $j \in \mathbb{Z}$, the sequences $(W_{j+n}\dots W_j)_n$ and $(W_{j-n+1}\dots W_j^{-1})_n$ converge pointwise on $C_c(\mathbb{R})$ which is dense in $L^2(\mathbb{R})$. Hence, the conditions in Theorem 3.2 are satisfied.

In fact, it sufficies to assume that there exist two strictly increasing sequences $\{n_k\}_k, \{n_i\}_i \in \mathbb{N}$ such that for each $j \in \{n_k\}_k \cup \{-n_i\}_i$ the operator W_j is constructed as above. If, for all $j \in \mathbb{Z} \setminus (\{n_k\}_k \cup \{-n_i\}_i)$, we have that $W_j(f) = w_j f$ for all $f \in H$ where w_j is a function on $\mathbb{R}$ satisfying that $\frac{1}{M} \leq |w_j| \leq 1$ whenever $j \geq 0$ and $M \geq |w_j| \geq 1$ whenever $j < 0$, then it is not hard to see that the conditions of Theorem 3.2 are still satisfied.

Proposition 3.4. *We have (ii) $\Rightarrow$ (i).*
(i) The operator $T_{U,W}$ is chaotic.
(ii) For every $J, m \in \mathbb{N}$ there exist a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ and a sequence $\{D_i^{(k)}\}_k$ for $i \in [J]$ of operators in $B_0(H)$ such that

$$\lim_{k\to\infty} \| D_i^{(k)} - P_m \| = 0$$

and

$$\lim_{k\to\infty} \sum_{l=1}^{\infty} \| W_{j+ln_k}W_{j+ln_k-1}\dots W_{j+1}D_j^{(k)} \| = \lim_{k\to\infty} \sum_{l=1}^{\infty} \| W_{j-ln_k+1}^{-1}W_{j-ln_k+2}^{-1}\dots W_j^{-1}D_j^{(k)} \| = 0,$$

for all $j \in [J]$, where the corresponding series are convergent for each k.

Proof. By Proposition 3.1. it suffices to show that $\mathcal{P}(T^n_{U,W})_n$ is dense in $\ell_2(\mathcal{A})$. Let O be an open subset of $\ell_2(\mathcal{A})$ and $x=(x_j)_{j\in\mathbb{Z}}\in O$. Then there exist some $J,m\in\mathbb{N}$ such that $y\in O$ with

$$y_j=\begin{cases} P_m x_j, & \text{for } j\in[J],\\ 0, & \text{else .}\end{cases}$$

Choose sequences $\{n_k\}_k$ and $\{D_i^{(k)}\}_k$ with $i\in[J]$ that satisfy the assumptions in (ii) with respect to J and m. For each $k\in\mathbb{N}$, set $Z^{(k)}=(Z_j^{(k)})_{j\in\mathbb{Z}}$ to be given by

$$Z_j^{(k)}=\begin{cases} D_j^{(k)}y_j, & \text{for } j\in[J],\\ 0, & \text{else ,}\end{cases}$$

and put

$$q_k=\sum_{l=0}^{\infty}T_{U,W}^{ln_k}(Z^{(k)})+\sum_{l=1}^{\infty}S_{U,W}^{ln_k}(Z^{(k)}).$$

Now, as in the proof of Proposition 3.1 part (2) implies (1), we observe that for each $j\in[J]$ and $l,k\in\mathbb{N}$ we have

$$\| T_{U,W}^{ln_k}(Z^{(k)})_{j-ln_k}\|\leq\| W_{j+ln_k}W_{j+ln_k-1}\dots W_{j+1}D_j^{(k)}\|\| y_j\|,$$

and

$$\| S_{U,W}^{ln_k}(Z^{(k)})_{j-ln_k}\|\leq\| W_{j-ln_k+1}^{-1}W_{j-ln_k+2}^{-1}\dots W_j^{-1}D_j^{(k)}\|\| y_j\|,$$

whereas for $j\notin[J]$ we have that

$$T_{U,W}^{ln_k}(Z^{(k)})_{j-ln_k}=S_{U,W}^{ln_k}(Z^{(k)})_{j-ln_k}=0.$$

So

$$\| q_k-y\|\leq\| D_{(0)}^{(k)}-P_m\|\| y_0\|$$

$$+\sum_{l=1}^{\infty}\sum_{j\in[J]}\| W_{j+ln_k}W_{j+ln_k-1}\dots W_{j+1}D_j^{(k)}\|\| y_j\|$$

$$+\sum_{l=1}^{\infty}\sum_{j\in[J]}\| W_{j-ln_k+1}^{-1}W_{j-ln_k+2}^{-1}\dots W_j^{-1}D_j^{(k)}\|\| y_j\|$$

$$\leq\| D_{(0)}^{(k)}-P_m\|\| y_0\|$$

$$+\sum_{j\in[J]}\| y\|(\sum_{l=1}^{\infty}\| W_{j+ln_k}W_{j+ln_k-1}\dots W_{j+1}D_j^{(k)}\|$$

$$+\sum_{l=1}^{\infty}\| W_{j-ln_k+1}^{-1}W_{j-ln_k+2}^{-1}\dots W_j^{-1}D_j^{(k)}\|)$$

for all $k\in\mathbb{N}$, which gives that $q_k\to y$ as $k\to\infty$.
Moreover, it is straightforward to check that $T_{U,W}^{ln_k}(q_k)=q_k$ for all l and k, hence $q_k\in\mathcal{P}(T^n_{U,W})_n$ for all k. □

Next, for each $n\in\mathbb{N}$, we set $C_{U,W}^{(n)}=\frac{1}{2}(T_{U,W}^n+S_{U,W}^n)$.

Proposition 3.5. *We have that (ii) implies (i).*
(i) The sequence $\{C_{U,W}^{(n)}\}_n$ is topologically transitive on $l_2(\mathcal{A})$.
(ii) For every $J, m \in \mathbb{N}$ there exist a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ and sequences of operators $\{D_i^{(k)}\}_k$, $\{G_i^{(k)}\}_k$ in $B_0(H)$ for $i \in [J]$ such that for all $j \in [J]$ we have that

$$\lim_{k\to\infty} \| D_j^{(k)} - P_m \| = \lim_{k\to\infty} \| G_j^{(k)} - P_m \| = 0,$$

$$\lim_{k\to\infty} \| W_{j+n_k} W_{j+n_k-1} \dots W_{j+1} D_j^{(k)} \|$$

$$= \lim_{k\to\infty} \| W_{j-n_k+1}^{-1} W_{j-n_k+2}^{-1} \dots W_j^{-1} D_j^{(k)} \|$$

$$= \lim_{k\to\infty} \| W_{j+n_k} W_{j+n_k-1} \dots W_{j+1} G_j^{(k)} \|$$

$$= \lim_{k\to\infty} \| W_{j-n_k+1}^{-1} W_{j-n_k+2}^{-1} \dots W_j^{-1} G_j^{(k)} \| = 0$$

and

$$\lim_{k\to\infty} \| W_{j+2n_k} W_{j+2n_k-1} \dots W_{j+1} G_j^{(k)} \|$$

$$= \lim_{k\to\infty} \| W_{j-2n_k+1}^{-1} W_{j-2n_k+2}^{-1} \dots W_j^{-1} G_j^{(k)} \| = 0$$

Proof. Let O_1 and O_2 be two non-empty open subset of $l_2(\mathcal{A})$. As in the proof of Proposition 3.1, part 2) $\Rightarrow$ 1), we can find some $J, m \in \mathbb{N}$, $x = (x_j)_j \in O_1$ and $y = (y_j)_j \in O_2$ such that $x_j = y_j = 0$ for all $j \neq [J]$ and $x_j = P_m x_j$, $y_j = P_m y_j$ for all $j \in [J]$. Choose the sequences $\{D_j^{(k)}\}_k$, $\{G_j^{(k)}\}_k$ for $j \in [J]$ and the strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ that satisfy the assumptions in *(ii)* with respect to these J, m. For each $k \in \mathbb{N}$, let $\mu_k, v_k \in l_2(\mathcal{A})$ be given by $(\mu_k)_j = D_j^{(k)} x_j$, $(v_k)_j = G_j^{(k)} y_j$ for $j \in [J]$ and $(\mu_k)_j = (v_k)_j = 0$ for $j \notin [J]$. Set

$$\eta_k = \mu_k + T_{U,W}^{n_k}(v_k) + S_{U,W}^{n_k}(v_k).$$

By the similar calculations as in the proof of Proposition 3.1, part 2) $\Rightarrow$ 1), we can show that the assumptions in *(ii)* imply that

$$\lim_{k\to\infty} T_{U,W}^{n_k}(v_k) = \lim_{k\to\infty} S_{U,W}^{n_k}(v_k) = 0,$$

$$\lim_{k\to\infty} T_{U,W}^{2n_k}(v_k) = \lim_{k\to\infty} S_{U,W}^{n_{2k}}(v_k) = 0,$$

$$\lim_{k\to\infty} \| \mu_k - x \| = \lim_{k\to\infty} \| v_k - y \| = 0,$$

$$\lim_{k\to\infty} T_{U,W}^{n_k}(\mu_k) = \lim_{k\to\infty} S_{U,W}^{n_k}(\mu_k) = 0.$$

It follows that $\eta_k \to x$ and $C_{U,W}^{(n_k)}(\eta_k) \to y$ as $k \to \infty$. □

Example 3.6. *Let $H = L^2(\mathbb{R})$. Given $m \in \mathbb{N}$, put for each $j, k \in \mathbb{N}$ the operator $D_j^{(k)}$ to be $D_j^{(k)} = G_j^{(k)} = \mathcal{L}_{\mathcal{X}_{[-k,k]}} P_m$ where $\mathcal{L}_{\mathcal{X}_{[-k,k]}}$ denotes the multiplication operator by $\mathcal{X}_{[-k,k]}$. Since the convergence in the operator norm and the pointwise convergence coincide on finite dimensional spaces, it follows that $\| D_j^{(k)} - P_m \| \to 0$ as $k \to \infty$ for all $j \in \mathbb{N}$. If we now for each $j \in \mathbb{N}$ let W_j be the operator from Example 3.3, it is not hard to see that the sufficient conditions of Proposition 3.4 and Proposition 3.5 are satisfied in this case.*

At the end of this section we give some necessary conditions for the set of periodic elements of $T_{U,W}$ to be dense in $l_2(\mathcal{A})$. For an operator $R \in B(H)$ we set

$$m(R) := sup\{C > 0 \mid \; \| Rh \| \geq C \| h \| \; \text{ for all } h \in H\}.$$

Further, for $J, m \in \mathbb{N}$ we let $\tilde{P}_{J,m} \in l_2(\mathcal{A})$ be given as

$$(\tilde{P}_{J,m})_j = \begin{cases} P_m, & \text{for } j \in [J], \\ 0, & \text{else }. \end{cases}$$

We have the following proposition.

Proposition 3.7. *Let $J, m \in \mathbb{N}$. We have that (i) implies (ii).*
(i) $\tilde{P}_{J,m}$ belongs to the closure of $\mathcal{P}(T^n_{U,W})_n$.
(ii) There exists a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that

$$\lim_{k\to\infty} m(W_{j+n_k} W_{j+n_k-1} \dots W_{j+1}) = 0$$

for all $j \in [J]$.

Proof. Let $J, m \in \mathbb{N}$ be given. For each $k \in \mathbb{N}$ there exists by the assumption some $x^{(k)} \in l_2(\mathcal{A})$ and some $n_k \in \mathbb{N}$ such that

$$\frac{1}{k^2} \geq \| x^{(k)} - \tilde{P}_{J,m} \|_2 \; \text{ and } T^{n_k}_{U,W}(x^{(k)}) = x^{(k)}.$$

Hence, for each $k \in \mathbb{N}$ and $j \in [J]$ we have that

$$\frac{1}{k^2} \geq \| x^{(k)}_j - P_m \| \geq \| x^{(k)}_j P_m - P_m \|,$$

whic gives that

$$\| x^{(k)}_j P_m \| \geq 1 - \frac{1}{k^2}$$

Thus, for each $k \in \mathbb{N}$ and $j \in [J]$ we can find some $h^{(k)}_j \in H$ with $h^{(k)}_j \neq 0$ such that

$$\| x^{(k)}_j P_m h^{(k)}_j \| \geq (1 - \frac{1}{k^2}) \| h^{(k)}_j \| .$$

Now, we also have that

$$\frac{1}{k^2} \geq \| T^{n_k}_{U,W}(x^{(k)}) - \tilde{P}_{J,m} \|_2$$

since $T^{n_k}_{U,W}(x^{(k)}) = x^{(k)}$ for each $k \in \mathbb{N}$. We may in fact assume that $J < n_1 < n_2 < \dots$. Hence, as $J < n_1 < n_2 < \dots$, we must have $\frac{1}{k^2} \geq \| (T^{n_k}_{U,W}(x^{(k)}))_j \|$ for all $j \in [J]$ which gives for all $j \in [J]$ and $k \in \mathbb{N}$ that

$$\| W_{j+n_k} W_{j+n_k-1} \dots W_{j+1} x^{(k)}_j \| \leq \frac{1}{k^2}.$$

Thus,

$$\frac{1}{k^2} \geq \| W_{j+n_k} W_{j+n_k-1} \dots W_{j+1} x^{(k)}_j P_m \|,$$

so

$$\frac{1}{k^2} \| h^{(k)}_j \| \geq \| W_{j+n_k} W_{j+n_k-1} \dots W_{j+1} x^{(k)}_j P_m \| \; \| h^{(k)}_j \|$$
$$\geq \| W_{j+n_k} W_{j+n_k-1} \dots W_{j+1} x^{(k)}_j P_m h^{(k)}_j \|$$

$$\geq m(W_{j+n_k}W_{j+n_k-1}\dots W_{j+1}) \parallel x_j^{(k)}P_m h_j^{(k)} \parallel$$

$$\geq (1-\frac{1}{k^2}) \parallel h_j^{(k)} \parallel m(W_{j+n_k}W_{j+n_k-1}\dots W_{j+1})$$

for all $j \in [J]$ and $k \in \mathbb{N}$. Since $h_j^{(k)} \neq 0$, we can divide on the both side of the inequality by $\parallel h_j^{(k)} \parallel$ and obtain that

$$\frac{1}{k^2-1} \geq m(W_{j+n_k}W_{j+n_k-1}\dots W_{j+1})$$

for all $j \in [J]$ and $k \in \mathbb{N}$. □

Similarly we can prove the following proposition.

Proposition 3.8. *Let $J, m \in \mathbb{N}$. We have that (i) implies (ii).*
(i) $\tilde{P}_{J,m}$ belongs to the closure of $\mathcal{P}(S^n_{U,W})_n$.
(ii) There exists a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that

$$\lim_{k\to\infty} m(W^{-1}_{j-n_k+1}W^{-1}_{j-n_k+2}\dots W^{-1}_{j}) = 0$$

for all $j \in [J]$.

4. Hypercyclic operators on C^*-algebras

Let $\mathcal{A}$ be a non-unital C^*-algebra such that $\mathcal{A}$ is a closed two-sided ideal in a unital C^*-algebra $\mathcal{A}_1$. Let Φ be an isometric $*$-isomorphism of $\mathcal{A}_1$ such that $\Phi(\mathcal{A}) = \mathcal{A}$. Assume that there exists a net $\{p_\alpha\}_\alpha \subseteq \mathcal{A}$ consisting of self-adjoint elements with $\parallel p_\alpha \parallel \leq 1$ for all α and such that $\{p_\alpha^2\}_\alpha$ is an approximate unit for $\mathcal{A}$. Suppose in addition that for all α there exists some $N_\alpha \in \mathbb{N}$ such that $\Phi^n(p_\alpha)\cdot p_\alpha = 0$ for all $n \geq N_\alpha$ (which gives that $0 = (\Phi^n(p_\alpha)\cdot p_\alpha)^* = p_\alpha \cdot \Phi^n(p_\alpha)$ since Φ is a $*$-isomorphism). Let $b \in G(\mathcal{A}_1)$ and $T_{\Phi,b}$ be the operator on $\mathcal{A}_1$ defined by $T_{\Phi,b}(a) = b\cdot\Phi(a)$ for all $a \in \mathcal{A}_1$. Then $T_{\Phi,b}$ is a bounded linear operator on $\mathcal{A}_1$ and since $\mathcal{A}$ is an ideal in $\mathcal{A}_1$, it follows that $T_{\Phi,b}(\mathcal{A}) \subseteq \mathcal{A}$ because $\Phi(\mathcal{A}) = \mathcal{A}$. The inverse of $T_{\Phi,b}$, which we will denote by $S_{\Phi,b}$, is given as $S_{\Phi,b}(a) = \Phi^{-1}(b^{-1})\cdot\Phi^{-1}(a)$ for all $a \in \mathcal{A}_1$. Again, since $\Phi^{-1}(\mathcal{A}) = \mathcal{A}$ and $\mathcal{A}$ is a two-sided ideal in $\mathcal{A}_1$, we have that $S_{\Phi,b}(\mathcal{A}) \subseteq \mathcal{A}$, hence $T_{\Phi,b}(\mathcal{A}) = \mathcal{A} = S_{\Phi,b}(\mathcal{A})$.

By some calculations one can check that for all $a \in \mathcal{A}$ and $n \in \mathbb{N}$ we have

$$T^n_{\Phi,b}(a) = b\cdot\Phi(b)\dots\Phi^{n-1}(b)\Phi^n(a),$$

$$S^n_{\Phi,b}(a) = \Phi^{-1}(b^{-1})\Phi^{-2}(b^{-1})\dots\Phi^{-n}(b^{-1})\cdot\Phi^{-n}(a).$$

Proposition 4.1. *The following statements are equivalent.*
(i) $T_{\Phi,b}$ is hypercyclic on $\mathcal{A}$.
(ii) For every p_α there exists a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ and sequences $\{q_k\}_k$, $\{d_k\}_k$ in $\mathcal{A}$ such that

$$\lim_{k\to\infty} \parallel q_k - p_\alpha^2 \parallel = \parallel d_k - p_\alpha^2 \parallel = 0$$

and

$$\lim_{k\to\infty} \parallel \Phi^{-n_k}(b)\Phi^{-n_k+1}(b)\dots\Phi^{-1}(b)q_k \parallel$$

$$= \lim_{k\to\infty} \parallel \Phi^{n_k-1}(b^{-1})\Phi^{n_k-2}(b^{-1})\dots\Phi(b^{-1})b^{-1}d_k \parallel = 0$$

Proof. We prove first *i*) ⇒ *ii*).

Let p_α be given. Since $T_{\Phi,b}$ is hypercyclic, there exists some $n_1 \geq N_\alpha$ and some $a_1 \in \mathcal{A}$ such that $\| a_1 - p_\alpha \| < \frac{1}{4}$ and $\| b \cdot \Phi(b) \ldots \Phi^{n_1-1}(b)\Phi^{n_1}(a_1) - p_\alpha \| < \frac{1}{4}$. Since $0 = p_\alpha \Phi^{n_1}(p_\alpha) = \Phi^{n_1}(p_\alpha) \cdot p_\alpha$, we get

$$\begin{aligned} \| \Phi^{n_1}(a_1) \cdot p_\alpha \| &= \| (\Phi^{n_1}(a_1) - \Phi^{n_1}(p_\alpha)) \cdot p_\alpha \| \\ &\leq \| \Phi^{n_1}(a_1 - p_\alpha) \| \\ &= \| a_1 - p_\alpha \| \leq \frac{1}{4}, \text{ so} \end{aligned}$$

$$\| \Phi^{n_1}(a_1) \cdot p_\alpha \| \leq \| a_1 - p_\alpha \| \leq \frac{1}{4}. \tag{1}$$

Moreover,

$$\| (a_1 - p_\alpha)p_\alpha \| \leq \| a_1 - p_\alpha \| \leq \frac{1}{4}. \tag{2}$$

Similarly, $0 = \Phi^{-n_1}(p_\alpha) \cdot p_\alpha = p_\alpha \cdot \Phi^{-n_1}(p_\alpha)$, so we get

$$\begin{aligned} &\| \Phi^{-n_1}(b)\Phi^{-n_1+1}(b) \ldots \Phi^{-1}(b)a_1 p_\alpha \| \\ &= \| \Phi^{-n_1}(b\Phi(b) \ldots \Phi^{n_1-1}(b)\Phi^{n_1}(a_1) - p_\alpha)p_\alpha \| \\ &\leq \| \Phi^{-n_1}(b\Phi(b) \ldots \Phi^{n_1-1}(b)\Phi^{n_1}(a_1) - p_\alpha) \| \\ &= \| b\Phi(b) \ldots \Phi^{n_1-1}(b)\Phi^{n_1}(a_1) - p_\alpha \| \leq \frac{1}{4}, \text{ so} \end{aligned}$$

$$\| \Phi^{-n_1}(b)\Phi^{-n_1+1}(b) \ldots \Phi^{-1}(b)a_1 p_\alpha \| \leq \frac{1}{4}. \tag{3}$$

Finally, we have

$$\begin{aligned} &\| b\Phi(b) \ldots \Phi^{n_1-1}(b)\Phi^{n_1}(a_1)p_\alpha - p_\alpha^2 \| \\ &= \| (b\Phi(b) \ldots \Phi^{n_1-1}(b)\Phi^{n_1}(a_1) - p_\alpha) \cdot p_\alpha \| \\ &\leq \| b\Phi(b) \ldots \Phi^{n_1-1}(b)\Phi^{n_1}(a_1) - p_\alpha \| \leq \frac{1}{4}, \text{ so} \end{aligned}$$

$$\| b\Phi(b) \ldots \Phi^{n_1-1}(b)\Phi^{n_1}(a_1)p_\alpha - p_\alpha^2 \| \leq \frac{1}{4}. \tag{4}$$

By (1) we also get that

$$\begin{aligned} &\| \Phi^{n_1-1}(b^{-1})\Phi^{n_1-2}(b^{-1}) \ldots \Phi(b^{-1})b^{-1} b\Phi(b) \ldots \Phi^{n_1-1}(b)\Phi^{n_1}(a_1)p_\alpha \| \\ &= \| \Phi^{n_1}(a_1)p_\alpha \| \leq \frac{1}{4}. \end{aligned}$$

Put $q_1 = a_1 p_\alpha$ and $d_1 = b\Phi(b) \ldots \Phi^{n_1-1}(b)\Phi^{n_1}(a_1)p_\alpha$. Then $\| q_1 - p_\alpha^2 \| < \frac{1}{4}$, $\| d_1 - p_\alpha^2 \| < \frac{1}{4}$, $\| \Phi^{-n_1}(b)\Phi^{-n_1+1}(b) \ldots \Phi^{-1}(b)q_1 \| \leq \frac{1}{4}$ and

$$\| \Phi^{n_1-1}(b^{-1})\Phi^{n_1-2}(b^{-1}) \ldots \Phi^{-1}(b^{-1})b^{-1}d_1 \| \leq \frac{1}{4}.$$

Next, since $T_{\Phi,b}$ is hypercyclic, we can find a hypercyclic vector a_2 and some $n_2 > n_1$ such that $\| a_2 - p_\alpha \| < \frac{1}{4^2}$ and $\| T^{n_2}_{\Phi,b}(a_2) - p_\alpha \| < \frac{1}{4^2}$ and continue as above to find q_2 and d_2 in $\mathcal{A}$ such that $\| q_2 - p^2_\alpha \| < \frac{1}{4^2}$, $\| d_2 - p^2_\alpha \| < \frac{1}{4^2}$ and

$$\| \Phi^{-n_2}(b)\Phi^{-n_2+1}(b)\dots\Phi^{-1}(b)q_2 \| \leq \frac{1}{4^2},$$

$$\| \Phi^{n_2-1}(b^{-1})\Phi^{n_2-2}(b^{-1})\dots\Phi(b^{-1})d_2 \| \leq \frac{1}{4}.$$

Proceeding inductively, we can construct the sequences $\{n_k\}_k, \{q_k\}_k$ and $\{d_k\}_k$ with the properties in *ii*), so *i*) $\Rightarrow$ *ii*).

Now we prove the opposite implication.

Let O_1 and O_2 be two open non-empty subsets of $\mathcal{A}$. Since $\{p^2_\alpha\}$ is an approximate unit in $\mathcal{A}$, we can find some $x \in O_1$, $y \in O_2$ such that $x = p^2_\alpha x$ and $y = p^2_\alpha y$ for so sufficiently large α. Choose the sequences $\{n_k\}_k, \{q_k\}_k, \{d_k\}_k$ satisfying the conditions of (*ii*) with respect to p_α. For each $k \in \mathbb{N}$, set $x_k = q_k x + S^{n_k}_{\Phi,b}(d_k y)$.

We have that

$$\begin{aligned}
\| S^{n_k}_{\Phi,b}(d_k y) \| &= \| \Phi^{-1}(b^{-1})\dots\Phi^{-n_k}(b^{-1})\Phi^{-n_k}(d_k y) \| \\
&= \| \Phi^{n_k}(\Phi^{-1}(b^{-1})\dots\Phi^{-n_k}(b^{-1})\Phi^{-n_k}(d_k y)) \| \\
&= \| \Phi^{n_k-1}(b^{-1})\dots\Phi(b^{-1})\cdot b^{-1}d_k y \| \\
&\leq \| \Phi^{n_k-1}(b^{-1})\dots\Phi^{-1}(b^{-1})b^{-1}d_k \| \| y \| \to 0 \text{ as } k \to \infty.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\| T^{n_k}_{\Phi,b}(q_k y) \| &= \| b\Phi(b)\dots\Phi^{n_k-1}(b)\Phi^{n_k}(q_k y) \| \\
&= \| \Phi^{-n_k}(b\Phi(b)\dots\Phi^{n_k-1}(b)\Phi^{n_k}(q_k y)) \| \\
&\leq \| \Phi^{-n_k}(b)\Phi^{-n_k+1}(b)\dots\Phi^{-1}(b)q_k \| \| y \| \to 0 \text{ as } k \to \infty.
\end{aligned}$$

It follows that $x_k \to x$ and $T^{n_k}_{\Phi,b}(x_k) \to y$, as $k \to \infty$, so $T_{\Phi,b}$ is topologically transitive, thus hypercyclic on $\mathcal{A}$. $\square$

Remark 4.2. *We notice that the assumption that for all α there exists some N_α such that $\Phi^n(p_\alpha)p_\alpha = 0$ for all $n \geq N_\alpha$ is in fact not needed for the proof of the implication* (*ii*) *implies* (*i*) *in Proposition 4.1.*

Example 4.3. *Let H be a separable Hilbert space and U be a unitary operator on H satisfying the condition (2) from [13] with respect to an orthonormal basis $\{e_j\}_{j\in\mathbb{Z}}$. Set Φ to be the $*$-isomorphism on $B(H)$ given by $\Phi(F) = U^*FU$. Then, by the condition (2) from [13], given $m \in \mathbb{N}$ there exists an $N_m \in \mathbb{N}$ such that $P_m U^n P_m = 0$ for $n \geq N_m$ (where P_m is the orthogonal projection onto $Span\{e_{-m}, \dots, e_m\}$ as in [13].) Moreover, $\{P_m\}_{m\in\mathbb{N}}$ is an approximate unit for $B_0(H)$ by [16, Proposition 2.2.1]. Hence, for all $n \geq N_m$ we have $\Phi^n(P_m)P_m = U^{*n}P_m U^n P_m = 0$. Here $\mathcal{A}_1 = B(H)$ and $\mathcal{A} = B_0(H)$. By some calculations we see that the conditions in part* (*ii*) *in Proposition 4.1 are the same as the conditions (3) and (4) in [13]. The operator $T_{U,W}$ from [13] is actually the operator $T_{\Phi,WU}$ (because $WFU = WU(U^*FU)$ for all $F \in B_0(H)$). For concrete examples satisfying these conditions we refer to examples from [13] and [14]. In fact, in [14] it has been proved that these conditions are equivalent to the condition that the operator W satisfies hypercyclicity criterion on H. For more details about this criterion, see [4].*

Example 4.4. *Let $H = L^2(\mathbb{R})$. For each $j, k, m \in \mathbb{N}$ we let $D^{(k)}_j, P_m$ be the operators on H as in Example 3.6 and for each $J \in \mathbb{N}$ we let $\tilde{P}_{J,m}$ be the orthogonal projection on $l_2(B_0(H))$ induced by P_m and $[J]$, as defined on page 10 in Section 3. Let $\mathcal{K}(l_2(B_0(H)))$ denote the C^*-algebra of compact operators on $l_2(B_0(H))$ in the sense of [16, Section 2.2]. Then it is not hard to see that $\{\tilde{P}_{J,m}\}_{J,m\in\mathbb{N}}$ is an approximate unit for $\mathcal{K}(l_2(B_0(H)))$. For $j \in \mathbb{Z}$ we let W_j be the operator on H from Example 3.3. Let $T_{U,W}$ be the operator on $l_2(B_0(H))$ defined in Section 3, where $W = \{W_j\}_{j\in\mathbb{Z}}$. If $U = I$,* *then $T_{I,W}$ is a bounded, adjointable operator on $l_2(B_0(H))$ which is linear with respect to the C^*-algebra $B_0(H)$. (Recall that we consider $l_2(B_0(H))$ as the right Hilbert C^*-module). For each $k \in \mathbb{N}$, set $\tilde{D}_k$ to be the operator on $l_2(B_0(H))$ given by $\tilde{D}_k(\{x_j\}_{j\in\mathbb{Z}}) = \{D_j^{(k)}x_j\}_{j\in\mathbb{Z}}$ for all $\{x_j\}_{j\in\mathbb{Z}} \in l_2(B_0(H))$. Since $\| D_j^{(k)} \| \leq 1$ for all $j \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have that $\tilde{D}_k$ is a bounded $B_0(H)$-linear, adjointable operator on $l_2(B_0(H))$ for all $k \in \mathbb{N}$. By the similar arguments as in Example 3.6 we can deduce that*

$$\lim_{k\to\infty} \| \tilde{D}_k\tilde{P}_{J,m} - \tilde{P}_{J,m} \| = 0$$

for all $J, m \in \mathbb{N}$. Moreover, for all $k, J, m \in \mathbb{N}$ we have that

$$\lim_{n\to\infty} \| T_{I,W}^{n}\tilde{D}_k P_{J,m} \| = \lim_{n\to\infty} \| T_{I,W}^{-n}\tilde{D}_k\tilde{P}_{J,m} \| = 0$$

Hence, for all $J, m \in \mathbb{N}$ we can construct a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that

$$0 = \lim_{k\to\infty} \| T_{I,W}^{n_k}\tilde{D}_k\tilde{P}_{J,m} \| = \lim_{k\to\infty} \| T_{I,W}^{-n_k}\tilde{D}_k\tilde{P}_{J,m} \| = 0$$

Let now $\mathcal{A} = \mathcal{K}(l_2(B_0(H)))$ and $\mathcal{A}_1$ be the C^*-algebra of all bounded $B_0(H)$-linear, adjointable operators on $l_2(B_0(H))$. If $\tilde{U}$ is a unitary operator on $l_2(B_0(H))$, we let Φ be the $*$-isomorphism on $\mathcal{A}_1$ given by $\Phi(F) = \tilde{U}^*F\tilde{U}$ for all $F \in \mathcal{A}_1$. Put then $b = T_{I,W}\tilde{U} \in G(\mathcal{A}_1)$. By the same arguments as in Example 4.3 we can deduce that the conditions of Proposition 4.1 are satisfied in this case.

Example 4.5. *Let X be a locally compact Hausdorff space, $\mathcal{A} = C_0(X)$, $\mathcal{A}_1 = C_b(X)$ and Φ be given by $\Phi(f) = f \circ \alpha$ for all $f \in C_b(X)$ where α is a homeomorphism of X. Put*

$$S = \{f \in C_c(X) \mid 0 \leq f \leq 1 \text{ and } f_{|_K} = 1 \text{ for some compact } K \subset X\}.$$

If $\tilde{S} = \{f^2 \mid f \in S\}$, then $\tilde{S}$ is an approximate unit for $C_0(X)$. Suppose that α is aperiodic, *that is for each compact subset K of X, there exists a constant $N > 0$ such that for each $n \geq N$, we have $K \cap \alpha^n(K) = \emptyset$. By some calculations it is not hard to see that in this case the conditions in Proposition 4.1 are equivalent to the condition that for every compact subset K of Ω there exists a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$, such that*

$$0 = \lim_{k\to\infty}(\sup_{t\in K}\prod_{j=0}^{n_k-1}(b \circ \alpha^{j-n_k})(t)) = \lim_{k\to\infty}(\sup_{t\in K}\prod_{j=0}^{n_k-1}(b \circ \alpha^{j})^{-1}(t)),$$

For the concrete examples satisfying these conditions, we refer to examples in [12].

If $a \in \mathcal{A}_1$, in the sequel we shall denote by L_a the left multiplier by a.

Corollary 4.6. *If there exist dense subsets Ω_1 and Ω_2 of $\mathcal{A}$ and a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that*

$$L_{\Phi^{-n_k}(b)\Phi^{-n_k+1}(b)\ldots\Phi^{-1}(b)} \overset{k\to\infty}{\longrightarrow} 0$$

pointwise on Ω_1 and

$$L_{\Phi^{n_k-1}(b^{-1})\Phi^{n_k-2}(b^{-1})\ldots\Phi(b^{-1})b^{-1}} \overset{k\to\infty}{\longrightarrow} 0$$

pointwise on Ω_2, then $T_{\Phi,b}$ is hypercyclic on $\mathcal{A}$.

Proof. Let p_α be given. Since Ω_1 and Ω_2 are dense in $\mathcal{A}$, there exist some $q_1 \in \Omega_1$ and $d_1 \in \Omega_2$ such that

$$\| q_1 - p_\alpha^2 \| < \frac{1}{4} \text{ and } \| d_1 - p_\alpha^2 \| < \frac{1}{4}.$$

By the assumption we can find some n_{k_1} such that

$$\| \Phi^{-n_k}(b)\Phi^{-n_k+1}(b)\ldots\Phi^{-1}(b)q_1 \| < \frac{1}{4}$$

and

$$\| \Phi^{n_k-1}(b^{-1})\Phi^{n_k-2}(b^{-1})\ldots\Phi(b^{-1})b^{-1}d_1 \| < \frac{1}{4}$$

for all $k \geq k_1$. Then we find some $q_2 \in \Omega_1, d_2 \in \Omega_2$ such that

$$\| q_2 - p_\alpha^2 \| < \frac{1}{4^2} \text{ and } \| d_2 - p_\alpha^2 \| < \frac{1}{4^2}.$$

By the assumption, we can find some $k_2 \geq k_1$ such that

$$\| \Phi^{-n_k}(b)\Phi^{-n_k+1}(b)\ldots\Phi^{-1}(b)q_2 \| < \frac{1}{4^2}$$

and

$$\| \Phi^{n_k-1}(b^{-1})\Phi^{n_k-2}(b^{-1})\ldots\Phi(b^{-1})b^{-1}d_2 \| < \frac{1}{4^2},$$

for all $k \geq k_2$. Proceeding inductively, we can construct the strictly increasing sequence $\{n_{k_i}\}_i$ and the sequences $\{q_i\}_i$ in $\{d_i\}_i$ in $\mathcal{A}$ satisfying the conditions of Proposition 4.1. □

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