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Existence and uniqueness result for a Navier problem involving Leray-Lions type operators in weighted Sobolev spaces

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Abstract. The Navier problem involving the p-biharmonic and the Leray-Lions operators with weights is considered in this paper. Using the theory of weighted Sobolev spaces and the Browder-Minty theorem to show the existence and uniqueness of weak solution to this problem. Firstly, we transform our problem into an equivalent operator equation; secondly, we use the Browder-Minty theorem to prove the existence and uniqueness of a weak solution to the problem concerned.

1. Introduction

In partial differential equations with weights, which have different types of singularities in the coefficients, it's natural to find solutions in weighted Sobolev spaces [13, 18]. Non-weighted Sobolev spaces $W^{k,t}(\mathcal{D})$, in general, appear as solution spaces for parabolic and elliptic partial differential equations.

There are a lot of examples of weight (see [18]). A well-established class of weights, introduced by B. Muckenhoupt [24] , is the class of A_p-weights (or Muckenhoupt class) . These classes have found many useful applications in harmonic analysis [2, 14, 15, 25, 27, 29].

Let $\mathcal{D}$ is a bounded open set in $\mathbb{R}^n$, ϕ, ϑ_1 and ϑ_2 are a weight functions. Our goal in this paper is to show the uniqueness and existence of a weak solution in the weighted Sobolev space $W_0^{1,t}(\mathcal{D}, \vartheta)$ (see Definition 2.4) for the Navier problem associated to the degenerate elliptic equation such that

$$\begin{cases} \Delta\Big[\phi(z)a(z,\Delta w)\Big] - \operatorname{div}\Big[\vartheta_1(z)\mathcal{K}(z,\nabla w)\Big] + \vartheta_2(z)|w|^{p-2}w = h(z) & \text{in } \mathcal{D}, \\ w(z) = \Delta w(z) = 0 & \text{on } \partial\mathcal{D}, \end{cases} \tag{1}$$

where the functions $a : \overline{\mathcal{D}} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\mathcal{K} : \mathcal{D} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are Carathéodory functions that satisfy the growth assumptions, monotonicity and ellipticity. Problem like (1) have been studied by many authors in the non-weighted case, such as : $a(z,\Delta w) = |\Delta w|^{p-2}\Delta w$, $\mathcal{K} \equiv 0$ and h depends on solution (see [5, 6]).

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The plan of this work is as follows. In Section 2, we give some basic results and some technical lemmas. In Section 3, we specify all the assumptions on $\mathcal{K}$, a and we present the notion of weak solution for the Problem (1). The main results will be proved in Section 4.

2. Preliminaries

To understand our findings, we must first review certain definitions and fundamental aspects which are used during this paper. Full presentations can be found in the monographs by A. Torchinsky [27] and J. Garcia-Cuerva et al. [16].

We will call a locally integrable function ϑ by a weight on $\mathbb{R}^n$ such that $\vartheta(z) > 0$ for a.e. $z \in \mathbb{R}^n$. Each weight ϑ gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ by integration. This measure will be denoted ϑ. Thus,

$$\vartheta(E) = \int_E \vartheta(z)dz \qquad \text{for measurable subset} \quad E \subset \mathbb{R}^n.$$

For $0 < t < \infty$, we denote by $L^t(\mathcal{D}, \vartheta)$ the space of measurable functions ϑ on $\mathcal{D}$ such that

$$\|h\|_{L^t(\mathcal{D},\vartheta)} = \left(\int_{\mathcal{D}} |h|^t \vartheta(z)dz \right)^{\frac{1}{t}} < \infty,$$

where h is a weight, and $\mathcal{D}$ be open in $\mathbb{R}^n$. It is widely known fact that the space $L^t(\mathcal{D}, \vartheta)$, endowed with this norm is a Banach space. We have also that the dual space of $L^t(\mathcal{D}, \vartheta)$ is the space $L^{t'}(\mathcal{D}, \vartheta^{1-t'})$.

Let us now specify the conditions on the weight ϑ that ensure that the functions in $L^t(\mathcal{D}, \vartheta)$ are locally integrable on $\mathcal{D}$.

Proposition 2.1. *([19, 23]). Let $1 \le t < \infty$. If the weight ϑ is such that*

$$\vartheta^{\frac{-1}{t-1}} \in L^1_{loc}(\mathcal{D}) \qquad if \quad t > 1,$$

$$ess \sup_{z \in B} \frac{1}{\vartheta(z)} < +\infty \quad if \quad t = 1,$$

for every ball $B \subset \mathcal{D}$. Then,

$$L^t(\mathcal{D}, \vartheta) \subset L^1_{loc}(\mathcal{D}).$$

As a result, and subject to the conditions of the Proposition 2.1, the convergence in $L^t(\mathcal{D}, \vartheta)$ implies convergence in $L^1_{loc}(\mathcal{D})$. In addition, every function in $L^t(\mathcal{D}, \vartheta)$ has distributional derivatives. So it makes sense to talk about distributional derivatives of functions in $L^t(\mathcal{D}, \vartheta)$.

Definition 2.2. *Let $1 \le t < \infty$. A weight ϑ is said to be an A_t-weight, or ϑ belongs to the Muckenhoupt class, if there exists a positive constant $\zeta = \zeta(t, \vartheta)$ such that, for every ball $B \subset \mathbb{R}^n$*

$$\left(\frac{1}{|B|} \int_B \vartheta(z)dz \right) \left(\frac{1}{|B|} \int_B (\vartheta(z))^{\frac{-1}{t-1}} dz \right)^{t-1} \leqslant \zeta \qquad if \quad t > 1,$$

$$\left(\frac{1}{|B|} \int_B h(z)dz \right) ess \sup_{z \in B} \frac{1}{\vartheta(z)} \leqslant \zeta \qquad if \quad t = 1,$$

where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$.

The infimum over all such constants ζ is called the A_t constant of ϑ. We denote by A_t, $1 \leq t < \infty$, the set of all A_t weights.

If $1 \leq q \leq t < \infty$, then $A_1 \subset A_q \subset A_t$ and the A_q constant of f equals the A_t constant of f (we refer to [17, 18, 28] for more informations about A_p-weights).

Proposition 2.3. *([30]). Let $\vartheta \in A_t$ with $1 \leqslant t < \infty$ and let E be a measurable subset of a ball $B \subset \mathbb{R}^n$. Then*

$$\left(\frac{|E|}{|B|}\right)^t \leqslant C\frac{\vartheta(E)}{\vartheta(B)},$$

where C is the A_t constant of ϑ.

The weighted Sobolev space $W^{k,t}(\mathcal{D}, \vartheta)$ is defined as follows.

Definition 2.4. *Let $\mathcal{D} \subset \mathbb{R}^n$ be open, and let f be A_t-weights, $1 \leqslant t < \infty$. We define the weighted Sobolev space $W^{k,t}(\mathcal{D}, \vartheta)$ as the set of functions $w \in L^t(\mathcal{D}, \vartheta)$ with $D_k w \in L^t(\mathcal{D}, \vartheta)$, for $k = 1, ..., n$. The norm of w in $W^{k,t}(\mathcal{D}, \vartheta)$ is given by*

$$\|w\|_{W^{k,t}(\mathcal{D},\vartheta)} = \left(\int_{\mathcal{D}} |w(z)|^t \vartheta dz + \int_{\mathcal{D}} |\nabla w(z)|^t \vartheta dz\right)^{\frac{1}{t}}. \tag{2}$$

We also define $W_0^{1,t}(\mathcal{D}, \vartheta)$ as the closure of $C_0^\infty(\mathcal{D})$ in $W^{1,t}(\mathcal{D}, \vartheta)$ with respect to the norm (2).

Equipped by this norm, $W^{1,t}(\mathcal{D}, \vartheta)$ and $W_0^{1,t}(\mathcal{D}, \vartheta)$ are reflexive and separable Banach spaces ([19, Proposition 2.1.2.]. For more detail about the spaces $W^{1,t}(\mathcal{D}, \vartheta)$) see [18, 23]. The dual of space $W_0^{1,t}(\mathcal{D}, \vartheta)$ is the space defined as

$$\left[W_0^{1,t}(\mathcal{D}, \vartheta)\right]^* = \left\{h - \sum_{i=1}^{n} D_i h_i /, \frac{h_i}{\vartheta} \in L^{t'}(\mathcal{D}, \vartheta), i = 1, ..., n\right\}.$$

To show the main reasoning of this paper, we rely on the following results .

Definition 2.5. *We denote by $\mathbb{H} = W_0^{1,p}(\mathcal{D}, \vartheta_1) \cap W^{2,t}\left(\mathcal{D}, \phi\right)$ with the norm*

$$\|w\|_{\mathbb{H}} = \|\Delta w\|_{L^t(\mathcal{D},\phi)} + \||\nabla w|\|_{L^p(\mathcal{D},\vartheta_1)}.$$

Theorem 2.6. *([13]). Let $\vartheta \in A_t$, $1 \leqslant t < \infty$, and let $\mathcal{D}$ be a bounded open set in $\mathbb{R}^n$. If $w_n \longrightarrow w$ in $L^t(\mathcal{D}, \vartheta)$, then there exist a subsequence (w_{n_m}) and $\psi \in L^t(\mathcal{D}, \vartheta)$ such that*

(i) *$w_{n_m}(z) \longrightarrow w(z)$, $n_m \longrightarrow \infty$, ϑ-a.e. on $\mathcal{D}$.*

(ii) *$|w_{n_m}(z)| \leqslant \psi(z)$, ϑ-a.e. on $\mathcal{D}$.*

Theorem 2.7. *([10]). Let $\vartheta \in A_t$, $1 < t < \infty$, and let $\mathcal{D}$ be a bounded open set in $\mathbb{R}^n$. There exist constants $M_{\mathcal{D}}$ and δ positive such that for all $\varphi \in W_0^{1,t}(\mathcal{D}, \vartheta)$ and all ν satisfying $1 \leqslant \nu \leqslant \frac{n}{n-1} + \delta$,*

$$\|\varphi\|_{L^{\nu t}(\mathcal{D},\vartheta)} \leqslant M_{\mathcal{D}} \|\nabla\varphi\|_{L^t(\mathcal{D},\vartheta)},$$

where $M_{\mathcal{D}}$ depends only on n, t, the A_t constant of ϑ and the diameter of $\mathcal{D}$.

Proposition 2.8. *([7]). Let $1 < p < \infty$.*

(i) *There exists a positive constant M_p such that for all $\eta, \mu \in \mathbb{R}^n$, we have*

$$\left||\mu|^{p-2}\mu - |\eta|^{p-2}\eta\right| \leq M_p|\mu - \eta|\big(|\mu| + |\eta|\big)^{p-2}.$$

(ii) *There exist two positive constants β_p and τ_p such that for every $z, y \in \mathbb{R}^n$, it holds that*

$$\beta_p\Big(|z| + |y|\Big)^{p-2}|z - y|^2 \leq \Big\langle |z|^{p-2}z - |y|^{p-2}y, z - y\Big\rangle \leq \tau_p\Big(|z| + |y|\Big)^{p-2}|z - y|^2.$$

Theorem 2.9. *([32]). Let $\mathbb{S} : \mathbb{H} \longrightarrow \mathbb{H}^*$ be a coercive, hemi-continuous and monotone operator on the real, separable, reflexive Banach space $\mathbb{H}$. Then the following statements are valid:*

1- *The equation $\mathbb{S}w = T$ has a solution w in $\mathbb{H}$, for all $T \in \mathbb{H}^*$.*

2- *If the operator $\mathbb{S}$ is strictly monotone, then equation $\mathbb{S}w = G$ has a unique solution $w \in \mathbb{H}$.*

3. Basic assumptions and concept of solutions

3.1. Basic assumptions

Let's give the specific conditions of the problem (1), we assume that the following assumptions: $\mathcal{D}$ be a bounded open subset of $\mathbb{R}^n$($n \geq 2$), $1 < t, p < \infty$, let ϕ, ϑ_1 and ϑ_2 are a weights functions, and let $\mathcal{K} : \mathcal{D} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, with $\mathcal{K}(z, \mu) = \Big(\mathcal{K}_1(z, \mu), ..., \mathcal{K}_n(z, \mu)\Big)$ and $a : \bar{\mathcal{D}} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfying the following assumptions:

(I) Be $\mathcal{K}_k$ (for $k = 1, ..., n$), and a are Carathéodory functions.

(II) There exists a positive function $h \in L^\infty(\mathcal{D})$, and a positive constant M_0 and $\tau \in L^{p'}(\mathcal{D}, \vartheta_1)$ $\Big($with $\frac{1}{p} + \frac{1}{p'} = 1\Big)$, such that :

$$|\mathcal{K}(z, \mu)| \leq \tau(z) + h(z)|\mu|^{p-1},$$

and

$$|a(z, \eta)| \leq M_0(1 + |\eta|^{t-1})$$

$\Big($with $\frac{1}{t} + \frac{1}{t'} = 1\Big)$.

(III) There exists a constant $\alpha > 0$ such that :

$$\Big\langle \mathcal{K}(z, \mu) - \mathcal{K}(z, \mu^{'}), \mu - \mu^{'}\Big\rangle \geqslant \alpha|\mu - \mu^{'}|^p,$$

and

$$\Big(a(z, \mu) - a(z, \mu^{'})\Big)\Big(\mu - \mu^{'}\Big) \geqslant 0,$$

whenever $\mu, \mu' \in \mathbb{R}^n$ with $\mu \neq \mu^{'}$.

(IV) There exists a constant $\beta > 0$ such that :

$$\Big\langle \mathcal{K}(z, \mu), \mu\Big\rangle \geqslant \beta|\mu|^p,$$

and

$$a(z, \mu) \cdot \mu \geqslant |\mu|^t.$$

3.2. Concept of solutions

The definition of a weak solution for Problem (1) can be said the following.

Definition 3.1. *One says $w \in \mathbb{H}$ is a weak solution to Problem* (1), *provided that*

$$\int_{\mathcal{D}} a(z,\Delta w)\Delta v\phi dz + \int_{\mathcal{D}} \langle \mathcal{K}(z,\nabla w),\nabla v\rangle\, \vartheta_1\, dz + \int_{\mathcal{D}} |w|^{p-2} w\, v\, \vartheta_2\, dz = \int_{\mathcal{D}} hv dz,$$

for all $v \in \mathbb{H}$.

Remark 3.2. *We seek to establish a relationship between ϑ_1, ϑ_2 , in order to ensure the existence and uniqueness of solution for our Problem* (1). *At first we notice, for all ϑ_1, $\vartheta_2 \in A_p$ we have :*

- *If $\frac{\vartheta_2}{\vartheta_1} \in L^r(\mathcal{D},\vartheta_1)$ where $r = \frac{p}{p-p'}$ and $1 < p' < p < \infty$, then, by Hölder inequality we obtain*

$$\|w\|_{L^{p'}(\mathcal{D},\vartheta_2)} \leqslant M_{p,p'} \|w\|_{L^p(\mathcal{D},\vartheta_1)},$$

where $M_{p,p'} = \|\frac{\vartheta_2}{\vartheta_1}\|^{1/p'}_{L^r(\mathcal{D},\vartheta_1)}$.

4. Main result

4.1. Result on the existence and uniqueness

The main result of this article is given in the next theorem.

Theorem 4.1. *Let $\vartheta_i \in A_p (i = 1, 2)$ and $\phi \in A_t$, $1 < p,t < \infty$ and assume that the assumptions **(I) – (IV)** hold. If $\frac{h}{\vartheta_1} \in L^{p'}(\mathcal{D},\vartheta_1)$ and $\frac{\vartheta_2}{\vartheta_1} \in L^{\frac{p}{p-p'}}(\mathcal{D},\vartheta_1)$ Then the problem* (1) *has exactly one solution $w \in \mathbb{H}$.*

4.2. Proof of Theorem 4.1

The essential one of our proof is to reduce the (1) to an operator problem $\mathcal{A}w = \mathcal{G}$ and apply the Theorem 2.9.

We define

$$\mathcal{F} : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{R}$$

and

$$\mathcal{G} : \mathbb{H} \longrightarrow \mathbb{R},$$

where $\mathcal{F}$ and $\mathcal{G}$ are defined below.
Then $w \in H$ is a weak solution of (1) if and only if

$$\mathcal{F}(w,v) = \mathcal{G}(v), \qquad \text{for all } v \in \mathbb{H}.$$

The proof of Theorem 4.1 is divided into several nots.

4.2.1. Equivalent operator equation

In this subsection, we prove that the Problem (1) is equivalent to an operator equation $\mathcal{A}w = \mathcal{G}$.

Using Hölder inequality, Theorem 2.7, we obtain

$$\begin{aligned} |\mathcal{G}(v)| &\leq \int_{\mathcal{D}} \frac{|h|}{\vartheta_1} |v|\vartheta_1\, dz \\ &\leq \|h/\vartheta_1\|_{L^{p'}(\mathcal{D},\vartheta_1)} \|v\|_{L^p(\mathcal{D},\vartheta_1)} \\ &\leq M_{\mathcal{D}} \|h/\vartheta_1\|_{L^{p'}(\mathcal{D},\vartheta_1)} \|v\|_{\mathbb{H}}. \end{aligned}$$

Since $f/\vartheta_1 \in L^{p'}(\mathcal{D}, \vartheta_1)$, then $\mathcal{G} \in \mathbb{H}^*$.
The operator **F** is broken down into the from

$$\mathcal{F}(w, v) = \mathcal{F}_1(w, v) + \mathcal{F}_3(w, v) + \mathcal{F}_4(w, v),$$

where $\mathcal{F}_i : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{R}$, for $i = 1, 2, 3$, are defined as

$$\mathcal{F}_1(w, v) = \int_{\mathcal{D}} \langle \mathcal{K}(z, \nabla w), \nabla v \rangle \vartheta_1 dz \quad , \quad \mathcal{F}_2(w, v) = \int_{\mathcal{D}} a(z, \Delta w) \Delta v \, \phi dz$$
$$\textit{and} \qquad \mathcal{F}_3(w, v) = \int_{\mathcal{D}} |w|^{p-2} w \, v \, \vartheta_2 \, dz.$$

Then, we have

$$|\mathcal{F}(w, v)| \leq |\mathcal{F}_1(w, v)| + |\mathcal{F}_2(w, v)| + |\mathcal{F}_3(w, v)|. \tag{3}$$

On the other hand, we get by using **(II)**, Hölder inequality and Theorem 2.7,

$$\begin{aligned} |\mathcal{F}_1(w, v)| &\leq \int_{\mathcal{D}} |\mathcal{K}(z, \nabla w)| |\nabla v| \vartheta_1 dz \\ &\leq \int_{\mathcal{D}} \Big(\tau + h |\nabla w|^{p-1} \Big) |\nabla v| \vartheta_1 dz \\ &\leq \|\tau\|_{L^{p'}(\mathcal{D}, \vartheta_1)} \|\nabla v\|_{L^p(\mathcal{D}, \vartheta_1)} + \|h\|_{L^\infty(\mathcal{D})} \|\nabla w\|^{p-1}_{L^p(\mathcal{D}, \vartheta_1)} \|\nabla v\|_{L^p(\mathcal{D}, \vartheta_1)} \\ &\leq \Big(\|\tau\|_{L^{p'}(\mathcal{D}, \vartheta_1)} + \|h\|_{L^\infty(\mathcal{D})} \|w\|^{p-1}_{\mathbb{H}} \Big) \|v\|_{\mathbb{H}}, \end{aligned}$$

and

Analogously, using **(II)**, Hölder inequality, Remark 3.2 (ii) and Theorem 2.7, we obtain

$$\begin{aligned} |\mathcal{F}_2(w, v)| &\leq \int_{\mathcal{D}} |a(z, \Delta w)| |\Delta v| \phi dz \\ &\leq M_0 \int_{\mathcal{D}} (1 + |\Delta w|^{t-1}) |\Delta v| \phi dz \\ &\leq M_0 \Big[\|\Delta w\|_{L^t(\mathcal{D}, \phi)} + (\phi(D))^{1/t'} \Big] \|\Delta v\|_{L^t(\mathcal{D}, \phi)} \\ &\leq M_0 \Big[\|w\|_{\mathbb{H}} + (\phi(D))^{1/t'} \Big] \|v\|_{\mathbb{H}}. \end{aligned}$$

Next, we get

$$\begin{aligned} |\mathcal{F}_3(w, v)| &\leq \int_{\mathcal{D}} |w|^{p-1} |v| \vartheta_2 dz \\ &\leq \Big(\int_{\mathcal{D}} |w|^p \vartheta_2 dz \Big)^{1/p'} \Big(\int_{\mathcal{D}} |v|^p \vartheta_2 dz \Big)^{1/p} \\ &= \|w\|^{p-1}_{L^p(\mathcal{D}, \vartheta_2)} \|v\|_{L^p(\mathcal{D}, \vartheta_2)} \\ &\leq M^{p-1}_{\mathcal{D}} M_{\mathcal{D}} \|\nabla w\|^{p-1}_{L^p(\mathcal{D}, \vartheta_2)} \|\nabla v\|_{L^p(\mathcal{D}, \vartheta_2)} \\ &\leq M^p_{\mathcal{D}} M^p_{p,p'} \|w\|^{p-1}_{\mathbb{H}} \|v\|_{\mathbb{H}}. \end{aligned}$$

Hence, in (3) we obtain, for all $w, v \in H$

$$|\mathcal{F}(w, v)| \leq \Big[\|\tau\|_{L^{p'}(\mathcal{D}, \vartheta_1)} + \|h\|_{L^\infty(\mathcal{D})} \|w\|^{p-1}_{H} + M^p_{\mathcal{D}} M^p_{p,p'} \|w\|^{p-1}_{\mathbb{H}} + M_0 \Big[\|w\|_{\mathbb{H}} + (\phi(D))^{1/t'} \Big] \Big] \|v\|_{\mathbb{H}}.$$

Then for each $w \in \mathbb{H}$, $\mathcal{F}(w, .)$ is linear and continuous . Thus, there exists a linear and continuous operator on H denoted by $\mathcal{A}$ such that

$$\langle \mathcal{A} w, v \rangle = \mathcal{F}(w, v), \quad \text{for all } w, v \in \mathbb{H}.$$

Moreover, we have

$$\|\mathcal{A}w\|_* \leq \quad \Big[\|\tau\|_{L^{p'}(\mathcal{D},\vartheta_1)} + \|h\|_{L^\infty(\mathcal{D})}\|w\|_H^{p-1} + \ M_{\mathcal{D}}^p M_{p,p'}^p \|w\|_{\mathbb{H}}^{p-1} + M_0\Big[\|w\|_{\mathbb{H}} + (\phi(D))^{1/t'}\Big]\Big],$$

where

$$\|\mathcal{A}w\|_* := \sup\Big\{|\langle \mathcal{A}w, v\rangle| = |\mathcal{F}(w,v)| : v \in H, \|v\|_{\mathbb{H}} = 1\Big\},$$

is the norm in $\mathbb{H}^*$. This gives us the operator

$$\begin{aligned} \mathcal{A} : \mathbb{H} &\longrightarrow \mathbb{H}^* \\ w &\longmapsto \mathcal{A}w. \end{aligned}$$

It is therefore possible that the equation of the problem (1) is equivalent to the equation of the operator

$$\mathcal{A}w = \mathcal{G}, \quad w \in \mathbb{H}.$$

4.2.2. Monotonicity and Coercivity of the operator $\mathcal{A}$

⋆ Now, we show that $\mathcal{A}$ is strictly monotone. Indeed.
Let $v_1, v_2 \in \mathbb{H}$ with $v_1 \neq v_2$. We have

$$\begin{aligned} \Big\langle \mathcal{A}v_1 - \mathcal{A}v_2, v_1 - v_2\Big\rangle &= \mathcal{F}(v_1, v_1 - v_2) - \mathcal{F}(v_2, v_1 - v_2) \\ &= \int_{\mathcal{D}} \Big\langle \mathcal{K}(z, \nabla v_1), \nabla(v_1 - v_2)\Big\rangle \vartheta_1 dz - \int_{\mathcal{D}} \Big\langle \mathcal{K}(z, \nabla v_2), \nabla(v_1 - v_2)\Big\rangle \vartheta_1 dz \\ &\quad + \int_{\mathcal{D}} a(z, \Delta v_1)\Delta(v_1 - v_2)\phi dz - \int_{\mathcal{D}} a(z, \Delta v_2)\Delta(v_1 - v_2)\phi dz \\ &\quad + \int_{\mathcal{D}} |v_1|^{p-2} v_1 (v_1 - v_2)\vartheta_2 dz - \int_{\mathcal{D}} |v_2|^{p-2} v_2 (v_1 - v_2)\vartheta_2 dz \\ &= \int_{\mathcal{D}} \Big\langle \mathcal{K}(z, \nabla v_1) - \mathcal{K}(z, \nabla v_2), \nabla(v_1 - v_2)\Big\rangle \vartheta_1 dz \\ &\quad + \int_{\mathcal{D}} \Big(a(z, \Delta v_1) - a(z, \Delta v_2)\Big)\Delta\Big(v_1 - v_2\Big)\phi dz \\ &\quad + \int_{\mathcal{D}} \Big(|v_1|^{p-2} v_1 - |v_2|^{p-2} v_2\Big)\Big(v_1 - v_2\Big)\vartheta_2 dz. \end{aligned}$$

Thanks to **(III)** and Proposition 2.8 (ii), we obtain

$$\begin{aligned} \Big\langle \mathcal{A}v_1 - \mathcal{A}v_2, v_1 - v_2\Big\rangle &\geq \alpha \int_{\mathcal{D}} |\nabla(v_1 - v_2)|^p \ \vartheta_1 \ dz + \ \beta_p \int_{\mathcal{D}} \Big(|v_1| + |v_2|\Big)^{p-2} |v_1 - v_2|^2 \ \vartheta_2 \ dz \\ &\geq \alpha \int_{\mathcal{D}} |\nabla(v_1 - v_2)|^p \vartheta_1 dz \\ &\geq \alpha \|\nabla(v_1 - v_2)\|^p_{L^p(\mathcal{D},\vartheta_1)}. \end{aligned}$$

Therefore, $\mathcal{A}$ is strictly monotone .

⋆ In this not, we prove that the operator $\mathcal{A}$ is coercive. let $w \in \mathbb{H}$, we have

$$\begin{aligned} \langle \mathcal{A}w, w\rangle &= \mathcal{F}(w, w) \\ &= \mathcal{F}_1(w, w) + \mathcal{F}_3(w, w) + \mathcal{F}_4(w, w) \\ &= \int_{\mathcal{D}} \langle \mathcal{K}(z, \nabla w), \nabla w\rangle \vartheta_1 dz + \int_{\mathcal{D}} a(z, \Delta w)\Delta w \ \phi dz + \int_{\mathcal{D}} |w|^p \vartheta_2 dz. \end{aligned}$$

Moreover, from **(IV)** and Theorem 2.7(with $\nu = 1$), we obtain

$$\begin{aligned} \langle \mathcal{A}w, w\rangle &\geq \int_{\mathcal{D}} |\Delta w|^t \phi dz + \beta_1 \int_{\mathcal{D}} |\nabla w|^p \vartheta_1 dz + \int_{\mathcal{D}} |w|^p \vartheta_2 dz \\ &\geq \int_{\mathcal{D}} |\Delta w|^t \phi dz + min(\beta_1, 1)\left[\int_{\mathcal{D}} |\nabla w|^p \vartheta_1 dz + \int_{\mathcal{D}} |w|^p \vartheta_2 dz\right] \\ &\geq min(\beta_1, 1)\|w\|^p_{\mathbb{H}}. \end{aligned}$$

Hence, we obtain

$$\frac{\langle \mathcal{A}w, w\rangle}{\|w\|_{\mathbb{H}}} \geq min(\beta_1, 1)\|w\|_{\mathbb{H}}^{p-1}.$$

Therefore, since $p > 1$, we have

$$\frac{\langle \mathcal{A}w, w\rangle}{\|w\|_{\mathbb{H}}} \longrightarrow +\infty \text{ as } \|w\|_{\mathbb{H}} \longrightarrow +\infty,$$

that is, $\mathcal{A}$ is coercive.

4.2.3. Continuity of the operator $\mathcal{A}$

We need to show that the operator $\mathcal{A}$ is continuous, i.e. We will show that $\mathcal{A}w_n \longrightarrow \mathcal{A}w$ in $\mathbb{H}^*$, that is to say we need to show the following convergences

$$B_k w_n \longrightarrow B_k w \quad \text{in} \quad L^{p'}(\mathcal{D}, \vartheta_1), \tag{4}$$

$$Nw_n \longrightarrow Nw \quad \text{in} \quad L^{t'}(\mathcal{D}, \phi), \tag{5}$$

and

$$Jw_n \longrightarrow Jw \quad \text{in} \quad L^{p'}(\mathcal{D}, \vartheta_2). \tag{6}$$

Let $w_n \longrightarrow w$ in $\mathbb{H}$ as $n \longrightarrow \infty$. Then $\nabla w_n \longrightarrow \nabla w$ in $(L^p(\mathcal{D}, \vartheta_1))^i$. Hence, thanks to Theorem 2.6, there exist a sub sequence (w_{n_m}) and $\psi \in L^p(\mathcal{D}, \vartheta_1)$ such that

$$\begin{aligned} &\nabla w_{n_m}(z) \longrightarrow \nabla w(z), \quad \text{a.e. in } \mathcal{D} \\ &|\nabla w_{n_m}(z)| \leq \psi(z), \quad \text{a.e. in } \mathcal{D}. \end{aligned} \tag{7}$$

The following notes are required to demonstrate this convergence.
Not 1:
For $k = 1, ..., n$, we define the operator

$$\begin{aligned} &B_k : H \longrightarrow L^{p'}(\mathcal{D}, \vartheta_1) \\ &(B_k w)(z) = \mathcal{K}_k(z, \nabla w(z)). \end{aligned}$$

We need to show that $B_k w_n \longrightarrow B_k w$ in $L^{p'}(\mathcal{D}, \vartheta_1)$.
In Banach spaces, we will use the convergence principle and the Lebesgue theorem .

- Let $w \in \mathbb{H}$. Using **(II)** and Theorem 2.7(with $\nu = 1$), we obtain

$$\begin{aligned} \|B_k w\|^{p'}_{L^{p'}(\mathcal{D},\vartheta_1)} &= \int_{\mathcal{D}} |B_k w(z)|^{p'} \vartheta_1 dz = \int_{\mathcal{D}} |\mathcal{K}_k(z, \nabla w)|^{p'} \vartheta_1 dz \\ &\leq \int_{\mathcal{D}} \left(\tau + h|\nabla w|^{p-1}\right)^{p'} \vartheta_1 dz \\ &\leq M_p \int_{\mathcal{D}} \left(\tau^{p'} + h^{p'}|\nabla w|^p\right) \vartheta_1 dz \\ &\leq M_p \left[\|\tau\|^{p'}_{L^{p'}(\mathcal{D},\vartheta_1)} + \|h\|^{p'}_{L^\infty(\mathcal{D})}\|\nabla w\|^p_{L^p(\mathcal{D},\vartheta_1)}\right] \\ &\leq M_p \left[\|\tau\|^{p'}_{L^{p'}(\mathcal{D},\vartheta_1)} + \|h\|^{p'}_{L^\infty(\mathcal{D})}\|w\|^p_{\mathbb{H}}\right], \end{aligned}$$

  where the constant M_p depends only on p.

- Let $w_n \longrightarrow w$ in $\mathbb{H}$ as $n \longrightarrow \infty$. By **(II)** and (7), we obtain

$$\begin{aligned}
\|B_k w_{n_m} - B_k w\|^{p'}_{L^{p'}(\mathcal{D},\vartheta_1)} &= \int_{\mathcal{D}} |B_k w_{n_m}(z) - B_k w(z)|^{p'} \vartheta_1 dz \\
&\leq \int_{\mathcal{D}} \left(|\mathcal{K}_k(z,\nabla w_{n_m})| + |\mathcal{K}_k(z,\nabla w)|\right)^{p'} \vartheta_1 dz \\
&\leq M_p \int_{\mathcal{D}} \left(|\mathcal{K}_k(z,\nabla w_{n_m})|^{p'} + |\mathcal{K}_k(z,\nabla w)|^{p'}\right) \vartheta_1 dz \\
&\leq M_p \int_{\mathcal{D}} \left[\left(\tau + h|\nabla w_{n_m}|^{p-1}\right)^{p'} + \left(\tau + h|\nabla w|^{p-1}\right)^{p'}\right] \vartheta_1 dz \\
&\leq M_p \int_{\mathcal{D}} \left[\left(\tau + h\psi^{p-1}\right)^{p'} + \left(\tau + h\psi^{p-1}\right)^{p'}\right] \vartheta_1 dz \\
&\leq 2M_p M_p' \int_{\mathcal{D}} \left(\tau^{p'} + h^{p'}\psi^p\right) \vartheta_1 dz \\
&\leq 2M_p M_p' \left[\|\tau\|^{p'}_{L^{p'}(\mathcal{D},\vartheta_1)} + \|h\|^{p'}_{L^\infty(\mathcal{D})}\|\psi\|^p_{L^p(\mathcal{D},\vartheta_1)}\right].
\end{aligned}$$

Hence, thanks to (**I**), we get, as $n \longrightarrow \infty$

$$B_k w_{n_m}(z) = \mathcal{K}_k(z,\nabla w_{n_m}(z)) \longrightarrow \mathcal{K}_k(z,\nabla w(z)) = B_k w(z), \quad \text{a.e. } z \in \mathcal{D}.$$

Therefore, by Lebesgue's theorem, we obtain

$$\|B_k w_{n_m} - B_k w\|_{L^{p'}(\mathcal{D},\vartheta_1)} \longrightarrow 0,$$

that is,

$$B_k w_{n_m} \longrightarrow B_k w \quad \text{in} \quad L^{p'}(\mathcal{D},\vartheta_1).$$

Finally, in view to convergence principle in Banach spaces, we have

$$B_k w_n \longrightarrow B_k w \quad \text{in} \quad L^{p'}(\mathcal{D},\vartheta_1). \tag{8}$$

Not 2:

We define the operator

$$\begin{aligned}
&N : \mathbb{H} \longrightarrow L^{t'}(\mathcal{D},\phi) \\
&(Nw)(z) = b(z,\Delta w(z)).
\end{aligned}$$

In this not, we will show that $Nw_n \longrightarrow Nw$ in $L^{t'}(\mathcal{D},\phi)$.

- Let $w \in H$. Using **(II)** we obtain

$$\begin{aligned}
\|Nw\|^{t'}_{L^{t'}(\mathcal{D},\phi)} &= \int_{\mathcal{D}} |a(z,\Delta w)|^{t'} \phi dz \\
&\leq M_0^{t'} \int_{\mathcal{D}} \left(1 + |\Delta w|^{t-1}\right)^{t'} \phi dz \\
&\leq M_0^{t'} M_t \int_{\mathcal{D}} \left(1 + |\Delta w|^t\right) \phi dz \\
&\leq M_0^{t'} M_t \left[(\phi(D))^{1/t'} + \|\Delta w\|^t_{L^t(\mathcal{D},\phi)}\right] \\
&\leq M_0^{t'} M_t \left[(\phi(D))^{1/t'} + \|w\|^t_{\mathbb{H}}\right],
\end{aligned}$$

where the constant M_t depends only on t.

- Let $w_m \to w$ in $\mathbb{H}$ as $m \to 0$. We need to show that $Nw_m \to Nw$ in $L^{t'}\left(\mathcal{D},\phi\right)$. If $w_m \to w$ in H then $\Delta w_m \to \Delta w$ in $L^t\left(\mathcal{D},\phi\right)$. Using Theorem 2 , there exist a subsequence $\{w_{m_k}\}$ and a function $\Phi \in L^t\left(\mathcal{D},\phi\right)$ such that

$$\Delta w_{m_k}(z) \to \Delta w(z) \text{ a.e. in } \mathcal{D}$$
$$\left|\Delta w_{m_k}(z)\right| \leq \Phi(z) \text{ a.e. in } \mathcal{D}.$$

By **(II)** , we get

$$\begin{aligned}
\|Nw_{n_m} - Nw\|^{t'}_{L^{t'}(\mathcal{D},\phi)} &= \int_{\mathcal{D}} \left|Nw_{n_m}(z) - Nw(z)\right|^{t'} \phi dz \\
&\leq \int_{\mathcal{D}} \left(|a(z,\Delta w_{n_m})| + |a(z,\Delta w)|\right)^{t'} \phi dz \\
&\leq M_t \int_{\mathcal{D}} \left(|a(z,\Delta w_{n_m})|^{t'} + |a(z,\Delta w)|^{t'}\right)\phi dz \\
&\leq M_0^{t'} M_t \int_{\mathcal{D}} \left[\left(1 + |\Delta w_{n_m}|^{t-1}\right)^{t'} + \left(1 + |\Delta w|^{t-1}\right)^{t'}\right]\phi dz \\
&\leq M_0^{t'} M_t \int_{\mathcal{D}} \left[\left(1 + |\Phi|^{t-1}\right)^{t'} + \left(1 + \Phi^{t-1}\right)^{t'}\right]\phi dz \\
&\leq 2M_0^{t'} M_t M_t' \left[(\phi(D))^{1/t'} + \|\Phi\|^{t}_{L^t(\mathcal{D},\phi)}\right],
\end{aligned}$$

next, using condition **(I)**, we deduce, as $n \longrightarrow \infty$

$$Nw_{n_m}(z) = a(z,\Delta w_{n_m}(z)) \longrightarrow a(z,\Delta w(z)) = Nw(z), \quad \text{a.e. } z \in \mathcal{D}.$$

Therefore, by the Lebesgue's theorem, we obtain

$$\|Nw_{n_m} - Nw\|_{L^{t'}(\mathcal{D},\phi)} \longrightarrow 0,$$

that is,

$$Nw_{n_m} \longrightarrow Nw \quad \text{in} \quad L^{t'}(\mathcal{D},\phi).$$

We conclude, from the convergence principle in Banach spaces, that

$$Nw_n \longrightarrow Nw \quad \text{in} \quad L^{t'}(\mathcal{D},\phi). \tag{9}$$

Not 3:
We define the operator

$$\begin{aligned} &J : H \longrightarrow L^{p'}(\mathcal{D},\vartheta_2) \\ &(Jw)(z) = |w(z)|^{p-2}w(z). \end{aligned}$$

In this not, we will demonstrate that $Jw_n \longrightarrow Jw$ in $L^{p'}(\mathcal{D},\vartheta_2)$.

- Let $w \in \mathbb{H}$. Using remark 3.2, we have

$$\begin{aligned}
\|Jw\|^{p'}_{L^{p'}(\mathcal{D},\vartheta_2)} &= \int_{\mathcal{D}} |Jw|^{p'} \vartheta_2 dz \\
&= \int_{\mathcal{D}} |w|^{(p-1)p'} \vartheta_2 dz \\
&= \int_{\mathcal{D}} |w|^{p} \vartheta_2 dz \\
&\leq M_{p,p'} \|w\|^{p}_{\mathbb{H}}.
\end{aligned}$$

- Let $w_n \longrightarrow w$ in $\mathbb{H}$ as $n \longrightarrow \infty$. Then $w_n \longrightarrow w$ in $L^p(\mathcal{D},\vartheta_2)$. Hence, thanks to Theorem 2.6, there exist a subsequence (w_{n_m}) and $\varphi \in L^p(\mathcal{D},\vartheta_2)$ such that

$$w_{n_m}(z) \longrightarrow w(z), \quad \text{a.e. in } \mathcal{D}$$

$$|w_{n_m}(z)| \leq \varphi(z), \quad \text{a.e. in } \mathcal{D}.$$

Next, we get

$$\begin{aligned}
\|Jw_{n_m} - Jw\|^{p'}_{L^{p'}(\mathcal{D},\vartheta_2)} &= \int_{\mathcal{D}} \left|Jw_{n_m}(z) - Jw(z)\right|^{p'} \vartheta_2 dz \\
&\leq \int_{\mathcal{D}} \left(|Jw_{n_m}(z)| + |Jw(z)|\right)^{p'} \vartheta_2 dz \\
&\leq M_p \int_{\mathcal{D}} \left(|Jw_{n_m}(z)|^{p'} + |Jw(z)|^{p'}\right)\vartheta_2 dz \\
&\leq M_p \int_{\mathcal{D}} \left(||w_{n_m}|^{p-2}w_{n_m}|^{p'} + ||w|^{p-2}w|^{p'}\right)\vartheta_2 dz \\
&\leq M_p \int_{\mathcal{D}} \left(|w_{n_m}|^{(p-1)p'} + |w|^{(p-1)p'}\right)\vartheta_2 dz \\
&\leq M_p \int_{\mathcal{D}} \left(|w_{n_m}|^{p} + |w|^{p}\right)\vartheta_2 dz \\
&\leq M_p \int_{\mathcal{D}} \left(|\varphi|^{p} + |\varphi|^{p}\right)\vartheta_2 dz \\
&\leq 2M_p \int_{\mathcal{D}} |\varphi|^{p} \vartheta_2 dz \\
&\leq 2M_p \|\varphi\|^{p}_{L^p(\mathcal{D},\vartheta_2)}.
\end{aligned}$$

Therefore, by Lebesgue's theorem, we obtain

$$\|Jw_{n_m} - Jw\|_{L^{p'}(\mathcal{D},\vartheta_2)} \longrightarrow 0,$$

that is,

$$Jw_{n_m} \longrightarrow Jw \quad \text{in} \quad L^{p'}(\mathcal{D},\vartheta_2).$$

We conclude, in view to convergence principle in Banach spaces, that

$$Jw_n \longrightarrow Jw \quad \text{in} \quad L^{p'}(\mathcal{D},\vartheta_2). \tag{10}$$

Finally, let $v \in H$ and using Hölder inequality, we obtain

$$\begin{aligned}
|\mathcal{F}_1(w_n,v) - \mathcal{F}_1(w,v)| &= \left|\int_{\mathcal{D}} \langle \mathcal{K}(z,\nabla w_n) - \mathcal{K}(z,\nabla w), \nabla v\rangle \vartheta_1 dz\right| \\
&\leq \sum_{k=1}^{n} \int_{\mathcal{D}} |\mathcal{K}_k(z,\nabla w_n) - \mathcal{K}_k(z,\nabla w)||D_k v|\vartheta_1 dz \\
&= \sum_{k=1}^{n} \int_{\mathcal{D}} |B_k w_n - B_k w||D_k v|\vartheta_1 dz \\
&\leq \sum_{k=1}^{n} \|B_k w_n - B_k w\|_{L^{p'}(\mathcal{D},\vartheta_1)} \|D_k v\|_{L^p(\mathcal{D},\vartheta_1)} \\
&\leq \left(\sum_{k=1}^{n} \|B_k w_n - B_k w\|_{L^{p'}(\mathcal{D},\vartheta_1)}\right) \|v\|_H,
\end{aligned}$$

and

$$\begin{aligned}
|\mathcal{F}_2(w_n,v) - \mathcal{F}_2(w,v)| &\leq \int_{\mathcal{D}} |a(z,\Delta w_n) - a(z,\Delta w)||\Delta v|\phi dz \\
&= \int_{\mathcal{D}} |Nw_n - Nw||\Delta v|\phi dz \\
&\leq \|Nw_n - Nw\|_{L^{t'}(\mathcal{D},\phi)} \|\Delta v\|_{L^t(\mathcal{D},\phi)} \\
&\leq \|Nw_n - Nw\|_{L^{t'}(\mathcal{D},\phi)} \|v\|_{\mathbb{H}}.
\end{aligned}$$

and by not 4 and Remark 3.2 we get

$$\begin{aligned} |\mathcal{F}_3(w_n,v)-\mathcal{F}_3(w,v)| &\leq \int_{\mathcal{D}} \Big||w_i|^{p-2}w_i - |w|^{p-2}w\Big||v|\vartheta_2 dz \\ &= \int_{\mathcal{D}} |Jw_n - Jw||v|\vartheta_2 dz \\ &\leq M_{p,p'}\|Jw_n - Jw\|_{L^{p'}(\mathcal{D},\vartheta_2)}\|v\|_{\mathbb{H}}. \end{aligned}$$

Hence, for all $v \in H$, we have

$$\begin{aligned} |\mathcal{F}(w_n,v)-\mathcal{F}(w,v)| &\leq \sum_{j=1}^{3}\Big|\mathcal{F}_j(w_n,v)-\mathcal{F}_j(w,v)\Big| \\ &\leq \Big[\sum_{k=1}^{n}\Big(\|B_k w_n - B_k w\|_{L^{p'}(\mathcal{D},\vartheta_1)}\Big) + M_{p,p'}\|Jw_n - Jw\|_{L^{p'}(\mathcal{D},\vartheta_2)} \\ &\qquad + \|Nw_m - Nw\|_{L^{t'}(\mathcal{D},\phi)}\Big]\|v\|_{\mathbb{H}}. \end{aligned}$$

Then, we get

$$\begin{aligned} \|\mathcal{A}w_n - \mathcal{A}w\|_* &\leq \Big[\sum_{k=1}^{n}\Big(\|B_k w_n - B_k w\|_{L^{p'}(\mathcal{D},\vartheta_1)}\Big) + M_{p,p'}\|Jw_n - Jw\|_{L^{p'}(\mathcal{D},\vartheta_2)} \\ &\qquad + \|Nw_m - Nw\|_{L^{t'}(\mathcal{D},\phi)}\Big]. \end{aligned}$$

Combining (8), (9), and (10), we deduce that

$$\|\mathcal{A}w_n - \mathcal{A}w\|_* \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

that is, $\mathcal{A}w_n \longrightarrow \mathcal{A}w$ in $\mathbb{H}^*$. Hence, $\mathcal{A}$ is continuous and this implies that $\mathcal{A}$ is hemicontinuous.

Therefore, by Theorem 2.9, the operator equation $\mathcal{A}w = \mathcal{G}$ has exactly one solution $w \in \mathbb{H}$ and it is the unique solution for problem (1).

Finally, the proof of Theorem 4.1 is completed.

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Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] C. Allalou, K. Hilal and S. Ait Temghart, Existence of weak solutions for some local and nonlocal p-Laplacian problem. Journal of Elliptic and Parabolic Equations, 8(1), 151-169(2022).
[2] M. A. Ragusa, V. B. Shakhmurov, A Navier–Stokes-Type Problem with High-Order Elliptic Operator and Applications, Mathematics, 8(12), 2256 (2020).
[3] A. Bensoussan, L. Boccardo and F. Murat, On a non linear partial differential equation having natural growth terms and unbounded solution, Annales de l'Institut Henri Poincare (C) Non Linear Analysis, 5(4), Elsevier Masson (1988).
[4] D. Bresch, J. Lemoine, F. Guillen-Gonzalez, A note on a degenerate elliptic equations with applications to lake and seas, Electron. J. Differential Equations, 2004(42), 1-13 (2004).

[5] M.-M. Boureanu, A new class of nonhomogeneous differential operator and applications to anisotropic systems, Complex Variables and Elliptic Equations, 61(2016), 712–730 (2016).
[6] A. C. Cavalheiro, Existence and Uniqueness of Solution for Navier Problems with Degenerated Operators in Weighted Sobolev Spaces, International Journal of Nonlinear Science 30(1), 3-18 (2020).
[7] M. Chipot, Elliptic Equations: An Introductory Course, Birkh¨auser, Berlin (2009).
[8] F. Colasuonno, P. Pucci and C. Varga, Multiple solutions for an eigenvalue problem involving p-Laplacian type operators, Nonlinear Anal., 75(2012), 4496–4512 (2012).
[9] M. Colombo, Flows of Non-Smooth Vector Fields and Degenerate Elliptic Equations: With Applications to the Vlasov-Poisson and Semigeostrophic Systems, Publications on the Scuola Normale Superiore Pisa, 22, Pisa (2017).
[10] P. Drabek, A. Kufner, and V. Mustonen, Pseudo-monotonicity and degenerated or singular elliptic operators, Bulletin of the Australian Mathematical Society 58(2), 213-221 (1998).
[11] P. Drabek, A. Kufner and F. Nicolosi. Quasilinear elliptic equations with degenerations and singularities, Vol.5, Walter de Gruyter (2011).
[12] E. Fabes, D. S. Jerison and C. E. Kenig, The Wiener test for degenerate elliptic equations, Annales de l'institut Fourier, 32(3), (1982).
[13] S. Fucik, O. John, and A. Kufner, Function Spaces, Noordhoff International Publishing, Leyden. Academia, Publishing House of the Czechoslovak Academy of Sciences, Prague (1977).
[14] E. Guariglia, S. Silvestrov, Fractional-Wavelet Analysis of Positive definite Distributions and Wavelets on $D'(C)$, In: Engineering mathematics II, 337-353 (2016).
[15] E. Guariglia, R. C. Guido, Chebyshev Wavelet Analysis, Journal of Function Spaces, (2022).
[16] J. Garcia-Cuerva and J.L. Rubio de Francia, Weighted norm inequalities and related topics, Elsevier, (2011).
[17] J. Heinonen, T. Kilpelainen and O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Oxford Math, Monographs, Clarendon Press, Oxford (1993).
[18] A. Kufner, Weighted sobolev spaces, Vol. 31, John Wiley and Sons Incorporated (1985).
[19] A. Kufner and B. Opic, How to define reasonably weighted Sobolev spaces, Commentationes Mathematicae Universitatis Carolinae, 25(3), 537-554 (1984).
[20] M. El Ouaarabi, C. Allalou and S. Melliani, Existence result for a Neumann boundary value problem governed by a class of $p(x)$-Laplacian-like equation. Asymptotic Analysis, 132(1-2), 245-259(2023).
[21] M. El Ouaarabi, C. Allalou and S. Melliani, On a class of nonlinear degenerate elliptic equations in weighted Sobolev spaces. Georgian Mathematical Journal, 30(1), 81-94(2023).
[22] M. El Ouaarabi, C. Allalou and S. Melliani, Existence of weak solution for a class of p (x)-Laplacian problems depending on three real parameters with Dirichlet condition. Boletín de la Sociedad Matemática Mexicana, 28(2), 31(2022).
[23] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Transactions of the American Mathematical Society, 207-226 (1972).
[24] B. Muckenhoupt, The equivalence of two conditions for weight functions, Studia Math., 49(1), 101-106 (1974).
[25] S. Polidoro, M. A. Ragusa, Sobolev–Morrey spaces related to an ultraparabolic equation, Manuscripta Mathematica, 96, 371-392 (1998).
[26] S. Ait Temghart, H. El Hammar, C. Allalou and K. Hilala, Existence results for some elliptic systems with perturbed gradient. Filomat, 37(20), 6905-6915(2023).
[27] A. Torchinsky, Real-variable methods in harmonic analysis, Academic Press (1986).
[28] B.O.Turesson. Nonlinear potential theory and weighted Sobolev spaces, Vol. 1736, Springer Science and Business Media (2000).
[29] J. Xiao, Homothetic variant of fractional Sobolev space with application to Navier–Stokes system, Dynamics of Partial Differential Equations, 4(3), 227-245 (2007).
[30] X. Xu, A local partial regularity theorem for weak solutions of degenerate elliptic equations and its application to the thermistor problem, Differential and Integral Equations, 12(1), 83-100 (1999).
[31] E. Zeidler, Nonlinear Functional Analysis and its Applications, vol.I, Springer-Verlag, Berlin (1990).
[32] E. Zeidler, Nonlinear Functional Analysis and its Applications, vol.II/B, Springer-Verlag, New York (1990).
[33] J. Leray and J.-L. Lions, Quelques résultats de Visik sur les problémes elliptiques non linéaires par les méthodes de Minty-Browder, Bulletin de la Société Mathématique de France, 93(1965), 97–107 (1965).
[34] W. Wang and P. Zhao, Nonuniformly nonlinear elliptic equations of p-biharmonic type, J. Math. Anal. Appl., 348(2008), 730–738 (2008).