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Projection-type methods for nonlinear integral equations with non-smooth kernels

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Abstract. In this paper, we explore two methods for estimating the solution of *Urysohn* integral equations with a *Green*'s function type kernel: the *Kantorovich* approach and a projection-type method. Either the orthogonal projection or an interpolatory projection onto the space of piecewise polynomials of degre $\leq r$ is used as the approximating operator. Compared to the projection-type solutions, it is shown that if the right hand side of the operator equation is only continuous, then the iterated *Kantorovich* solution converge more rapidly. However, the projection-type method has lower computational costs. Several numerical examples are provided to validate the theoretical estimates.

1. Introduction

We consider the following *Urysohn* integral equation defined on $\mathbb{X} = L^\infty[0,1]$ by

$$x(s) - \int_0^1 \kappa(s,t,x(t))dt = f(t), \quad s \in [0,1] \tag{1.1}$$

where $f \in \mathbb{X}$, the kernel $\kappa(s,t,u)$ is a real valued non-smooth function and $x \in \mathbb{X}$ is the unknown function. Equation (1.1) include the following special case of *Hammerstein* equation

$$x(s) - \int_0^1 \kappa(s,t)\psi(t,x(t))dt = f(t), \tag{1.2}$$

where $\psi \in \mathcal{C}([0,1] \times \mathbb{R})$. For the solution of (1.1), there are a number of numerical approaches available. *Atkinson* and *Potra* [7] investigated projection and iterated projection methods and *Atkinson* and *Potra* [8] studied the discrete version of *Galerkin* and iterated *Galerkin* methods. *Kulkarni* and *Nidhin* [15] suggested an alternative approach, the modified projection method, for solving (1.1) with a continuous kernel, and *Grammont et al.* [14] investigated a more general type of kernels. Convergence of the iterated modified projection approach is demonstrated to be faster than that of the iterated projection solution. Specifically for orthogonal projection, the discrete variant of the modified projection method is examined in *Kulkarni* and *Rakshit* [12].

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Kumar and *Sloan* presented a new collocation approach in [17] for solving the *Hammerstein* problem (1.2), and *Kumar*[16] investigated the superconvergence features of this method. *Allouch et al.* [3] proposed a superconvergent variant of the *Kumar* and *Sloan* approach that converges as quickly as the modified projection method.
There are also many publications discussing alternative approaches for solving *Hammerstein* equations with smooth kernels using spline quasi-interpolation. (see for instance [9, 10]).
In this study, we employ the classical projection approach to provide several strategies for resolving equations (1.1) and (1.2).
Using piecewise polynomial basis functions, we first establish the *Kantorovich* technique for the numerical solution of (1.1), which is based on "Kantorovich regularization" (*Kantorovich,* 1948). The use of this technique for linear *Fredholm* integral equations is explored in *Schock* [19] and *Sloan* [20], but it does not appear to have been studied yet for nonlinear integral equations with non-smooth kernels. We point out that for solving *Uryshon* equations with smooth kernels, the proposed method and its discrete version are studied in *Allouch et. al* [4] (see also *Grammont et. al* [13]).
For (1.2), we next suggest a redefinition of the *Kumar* and *Sloan* approach, which is called the collocation-type method in the literature, by making use of the orthogonal projection. This last approximation does not seem to have been considered previously for *Green*'s kernels. However, it was analyzed in *Allouch et. al* [5] for solving *Hammerstein* equations with weakly singular kernels. When the orthogonal projection is employed, this technique will be referred to as a *Galerkin*-type method, whereas when the type of projection is not specified, it will be referred to as a projection-type method.
We provide an error analysis for the projection-type approach, and we prove that if the right hand side f of (1.2) is less smooth, the iterated *Kantorovich* solution is generally more accurate than the projection-type methods. However, we will notice that the projection-type method has better performance, in term of the computational cost.

Although spline quasi-interpolation has previously been employed in the treatment of linear *Fredholm* integral equations using *Green*'s kernels (see [2]), the projections operators employed here exhibit greater convergence orders.

Here is a quick overview of the paper. In Section 2, we establish notation, describe the numerical approaches, and recall some useful results. In Section 3, for both the orthogonal projection and the interpolatory projection, the orders of convergence of the given approaches are established. In Section 4, our results are illustrated by numerical tests.

2. Methods and notations

2.1. *Urysohn integral operators of class* $C_2(\alpha,\gamma)$

Let $\Pi = [0,1]\times[0,1]\times\mathbb{R}$. Divide Π into two subests Π_1 and Π_2, where

$$\Pi_1 = \{(s,t,u) : 0 \le s \le t \le 1,\ u \in \mathbb{R}\}$$

and

$$\Pi_2 = \{(s,t,u) : 0 \le t \le s \le 1,\ u \in \mathbb{R}\}.$$

Let α and γ be integers such that $\alpha \ge \gamma$, $\alpha \ge 0$ and $\gamma \ge -1$. The kernel κ defined in (1.1) is assumed to be of the following form

$$\kappa(s,t,u) = \begin{cases} \kappa_1(s,t,u), & (s,t,u) \in \Pi_1, \quad s \ne t \\ \kappa_2(s,t,u), & (s,t,u) \in \Pi_2, \end{cases}$$

where $\kappa_i \in \mathcal{C}^{\alpha}(\Pi_i)$, $i = 1,2$. We assume that if $\gamma \ge 0$, we have $\kappa \in \mathcal{C}^{\gamma}(\Pi)$ and if $\gamma = -1$, then κ may have a discontinuity of the first kind along the line $s = t$. Assume that the partial derivative $\ell(s,t,u) = \frac{\partial\kappa}{\partial u}(s,t,u)$

exists for all $(s,t,u) \in \Pi$ and

$$\ell(s,t,u) = \begin{cases} \ell_1(s,t,u), & (s,t,u) \in \Pi_1, \quad s \neq t \\ \ell_2(s,t,u), & (s,t,u) \in \Pi_2, \end{cases}$$

where $\ell_i \in C^{\alpha}(\Pi_i)$, $i = 1,2$. Following *Atkinson* and *Potra* [7], we say that κ is of class $C_2(\alpha,\gamma)$. Consider the *Urysohn* integral operator denoted by $\mathcal{K}$

$$(\mathcal{K}x)(s) = \int_0^1 \kappa(s,t,x(t))dt, \quad s \in [0,1]. \tag{2.1}$$

The operator $\mathcal{K}$ is compact and is completely continuous from $L^{\infty}[0,1]$ into $C^{\gamma_1}[0,1]$, where

$$\gamma_1 = \min\{\alpha, \gamma+1\}.$$

Moreover, $\mathcal{K}$ is *Fréchet* differentiable and its *Fréchet* derivative at $x \in \mathbb{X}$ is given by

$$(\mathcal{K}'(x)g)(t) = \int_0^1 \frac{\partial \kappa}{\partial u}(s,t,x(t))g(t)dt.$$

In operator form, the integral equation (1.1) can be represented as

$$x - \mathcal{K}(x) = f. \tag{2.2}$$

Let x_0 be the unique solution of (2.2). If $f \in C^{\alpha}[0,1]$, then from Corollary 3.2 of *Atkinson* and *Potra* [7], $x_0 \in C^{\alpha}[0,1]$. As the range of $\mathcal{K}$ is contained in $C^{\gamma_1}[0,1]$, then if $f \in C[0,1]$, we have also $x_0 \in C[0,1]$. For $\delta_0 > 0$, let

$$\mathcal{B}(x,\delta_0) = \{v \in \mathbb{X} : \|x - y\|_{\infty} < \delta_0\}.$$

The operator $\mathcal{K}'$ is *Lipschitz* continuous in a neighborhood $\mathcal{B}(x_0,\delta_0)$ of x_0, that is, there exists a constant Λ such that

$$\|\mathcal{K}'(x_0) - \mathcal{K}'(x)\| \leq \Lambda \|x_0 - x\|_{\infty}, \quad x \in \mathcal{B}(x,\delta_0). \tag{2.3}$$

Note that $\mathcal{K}'(x_0) : L^{\infty}[0,1] \to C[0,1]$ is a compact linear operator (See *Krasnoselskii* [22]). Assume that 1 is not an eigenvalue of $\mathcal{K}'(x_0)$. Then (See *Riesz-Nagy* [21])

$$M = (I - \mathcal{K}'(x_0))^{-1}\mathcal{K}'(x_0)$$

is the compact linear integral operator given by

$$(Mg)(s) = \int_0^1 m(s,t)g(t)dt,$$

and the kernel m has the same smoothness as kernel $\ell_*(s,t) = \ell(s,t,x_0(t))$ of $\mathcal{K}'(x_0)$. (See *Atkinson* and *Potra* [7, Lemma 5.1]). In fact, since $x_0 \in C^{\alpha}[0,1]$, it follows that

$$m \in C^{\alpha}\{0 \leq s \leq t \leq 1\} \quad \text{and} \quad m \in C^{\alpha}\{0 \leq t \leq s \leq 1\}.$$

If $\gamma \geq 0$, then $m \in C^{\gamma}([0,1] \times [0,1])$, whereas for $\gamma = -1$, the kernel m may have a discontinuity of the first kind along the line $s = t$. Following *Chatelin* and *Lebbar* [11], the class of the kernel m is denoted by $C(\alpha,\gamma)$.

2.2. Approximating projection operators

For any integer n, let

$$\Delta^{(n)} : 0 = t_0 < t_1 < \ldots < t_n = 1,$$

be a quasi-uniform partition of $[0, 1]$, that is

$$\sup_n q^{(n)} < \infty, \quad \text{where} \quad q^{(n)} = \max_{1\le i,j\le n} \frac{h_i^{(n)}}{h_j^{(n)}} \quad \text{and} \quad h_i^{(n)} = t_i - t_{i-1}.$$

Let $\Delta_i^{(n)} = [t_{i-1}, t_i]$ and $h^{(n)} = \max_{1\le i\le n} h_i^{(n)}$. In order to keep notations as simple as possible, from here on, we will no longer use the index (n) when referring to the partition or its elements. For $\nu \ge 0$, set

$$C_\Delta^\nu = \left\{ y \in L^\infty[0, 1] : y_i = y|_{\Delta_i} \in C^\nu(\Delta_i),\ i = 1, \ldots, n \right\}.$$

For $x \in C^j[0, 1]$, we define

$$\|x\|_{j,\infty} = \sum_{i=0}^{j} \|x^{(i)}\|_\infty,$$

where $x^{(i)}$ denotes the i^{th} derivative of x. For $y \in C_\Delta^0 = C_\Delta$, the following notations will be used

$$\|y\|_{2,\Delta_i} = \|y_i\|_2, \quad \|y\|_{\infty,\Delta_i} = \|y_i\|_\infty, \quad \|y\|_\infty = \max_{1\le i\le n} \|y_i\|_\infty.$$

Hence, we obtain the following bound

$$\|y\|_{2,\Delta_i} \le h_i^{1/2} \|y\|_{\infty,\Delta_i} \le h_i^{1/2} \|y\|_\infty, \quad i = 1, \ldots, n. \tag{2.4}$$

Let $\mathbb{P}_r$ denote the set of all polynomials of degree $\le r$, where r is a given integer and let $\mathbb{X}_n$ be the set of functions belonging to $\mathbb{P}_r$ on each subinterval Δ_i.

Let $\eta_0, \eta_1, \ldots,$ be the sequence of orthonormal polynomials in $L^2[0, 1]$ i.e. η_p is a polynomial of degree p, and

$$\langle \eta_p, \eta_q \rangle = \delta_{pq} \quad \text{for all} \quad p, q \ge 0.$$

For $i = 1, \ldots, n$ define η_{ip} on $[t_{i-1}, t_i]$ by

$$\eta_{ip}(t_{i-1} + \tau h_i) = h_i^{-1/2} \eta_p(\tau), \quad 0 \le \tau \le 1,$$

and then extend by zero to $[0, 1]$. The set

$$\{\eta_{ip},\ 1 \le i \le n,\ 0 \le p \le r\} \tag{2.5}$$

form an orthonormal basis for $\mathbb{X}_n$ and the restriction to $L^\infty[0, 1]$ of the orthogonal projection π_n^G from $L^2[0, 1]$ to $\mathbb{X}_n$ is given by

$$\pi_n^G g = \sum_{i=1}^{n} \sum_{p=0}^{r} \langle g, \eta_{ip} \rangle \eta_{ip} \tag{2.6}$$

and satisfies

$$\langle \pi_n^G g, \eta_{ip} \rangle = \langle g, \eta_{ip} \rangle, \quad 1 \le i \le n,\ \ 0 \le p \le r. \tag{2.7}$$

For $g \in C_\Delta$, let $\pi_n^C g$ denote the unique piecewise polynomial of degree r that satisfies

$$(\pi_n^C g)(\tau_{ip}) = g(\tau_{ip}), \quad 1 \le i \le n,\ \ 0 \le p \le r, \tag{2.8}$$

where the collocation points are

$$\tau_{ip} = (i - 1 + \tau_p)h_i, \quad 1 \le i \le n, \ \ 0 \le p \le r \tag{2.9}$$

and $\{\tau_0, \tau_1, \dots, \tau_r\}$ are the $r+1$ *Gauss* points in $[0,1]$. This map can be extended to $L^\infty[0,1]$ and then $\pi_n^C : L^\infty[0,1] \to \mathbb{X}_n$ is a projection. In the *Lagrange* form π_n^C is

$$\pi_n^C g = \sum_{i=1}^{n} \sum_{p=0}^{r} g(\tau_{ip}) l_{ip},$$

where $\{l_{ip}, \ 1 \le i \le n, \ 0 \le p \le r\}$ is the *Lagrange* basis of $\mathbb{X}_n$ satisfying

$$l_{ip}(\tau_{jq}) = \delta_{ij}\delta_{pq}, \quad 1 \le i, j \le n, \ \ 0 \le p, q \le r.$$

From here on, for notational convenience, we will write π_n^G or π_n^C as π_n.
The projection π_n converge to identity operator pointwise on $\mathcal{C}[0,1]$ and, for $g \in C_\Delta^\alpha$, (see *Chatelin* and *Lebbar* [11])

$$\|(I - \pi_n)g\|_\infty \le C_1 \|g^{(\beta)}\|_\infty h^\beta, \tag{2.10}$$

where

$$\beta = \min\{\alpha, r+1\}$$

and C_1 is a constant independent of n. Moreover, the projection π_n is uniformly bounded with respect to n, i.e.

$$p = \sup_n \|\pi_n|_{C_\Delta}\| < \infty. \tag{2.11}$$

Let

$$\beta_1 = \min\{\beta, \gamma + 1\} \quad \text{and} \quad \beta_2 = \min\{\beta, \gamma + 2\}.$$

For $\mu = 1, \dots, \beta_2$, if $g \in C_\Delta^\mu$, then, additionally, we have again, from *Chatelin* and *Lebbar* [11],

$$\|(I - \pi_n)g\|_\infty \le C_1 \|g^{(\mu)}\|_\infty h^\mu. \tag{2.12}$$

The following result is quoted from [15, Lemma 2.2].
For $g \in C_\Delta$, let $\pi_{n,i} g = (\pi_n g)|_{\Delta_i}$. If $g \in C_\Delta^\alpha$, then

$$\|(I - \pi_{n,i})g_i\|_{\infty,\Delta_i} \le C_1 \|g_i^{(\beta)}\|_{\infty,\Delta_i} h_i^\beta, \quad 1 \le i \le n. \tag{2.13}$$

Henceforth, we assume that C is a generic constant independent of n. According to *Grammont et al.* [14], if $g \in C_\Delta$, then

$$\|(\mathcal{K}'(x_0)g)^{(\mu)}\|_\infty \le C\|g\|_\infty, \quad 0 \le \mu \le \gamma_1 + 1. \tag{2.14}$$

2.3. Kantorovich method for Urysohn equations

For our convenience we let

$$y = \mathcal{K}(x). \tag{2.15}$$

Thus, writing the solution of (2.2) as $x = y + f$, we have

$$y = \mathcal{K}(y + f). \tag{2.16}$$

The *Kantorovich* method, is obtained by applying the projection method to equation (2.16). Thus, the approximate solution is

$$x_n^K = y_n + f, \tag{2.17}$$

where y_n satisfies

$$y_n - \pi_n \mathcal{K}(y_n + f) = 0. \tag{2.18}$$

The theoretical advantage of the proposed method is that the inhomogeneous term is now 0 rather than $\pi_n f$ in projection methods which may be smoother than f.
Observe that the aforementioned equations can be reduced to a single equation for x_n

$$x_n^K - \pi_n \mathcal{K}(x_n^K) = f. \tag{2.19}$$

We notice that this form is directly introduced in [13] to define the *Kantorovich* method. Throughout this paper, this method will be referred to as the *Kantorovich-Galerkin* method when an orthogonal projection is used, and the *Kantorovich*-collocation method when an interpolatory projection is employed. Finally, the iterated *Kantorovich* approximation is defined by

$$\begin{aligned} \widetilde{x}_n^K &= \mathcal{K}(x_n^K) + f, \\ &= \widetilde{y}_n + f, \end{aligned} \tag{2.20}$$

where $\widetilde{y}_n = \mathcal{K}(y_n + f)$. From (2.18) and (2.20) we observe that $y_n = \pi_n \widetilde{y}_n$, and hence

$$\widetilde{y}_n - \mathcal{K}(\pi_n \widetilde{y}_n + f) = 0. \tag{2.21}$$

For the implementation of the method, we define

$$F_n(v) = v - \pi_n \mathcal{K}(v + f).$$

Then, equation (2.18) becomes

$$F_n(y_n) = 0.$$

This last equation is solved iteratively by using the *Newton-Kantorovich* method. For an initial approximation $y_n^{(0)}$, define

$$y_n^{(k+1)} = y_n^{(k)} - [F_n'(y_n^{(k)})]^{-1} F_n(y_n^{(k)}),$$

where $F_n'(y_n^{(k)})$ is the *Fréchet* derivative of F_n given by

$$F_n'(y_n^{(k)}) = I - \pi_n \mathcal{K}'(y_n^{(k)} + f).$$

By a simple calculus, we get

$$y_n^{(k+1)} - \pi_n \mathcal{K}'(y_n^{(k)} + f) y_n^{(k+1)} = \pi_n \mathcal{K}(y_n^{(k)} + f) - \pi_n \mathcal{K}'(y_n^{(k)} + f) y_n^{(k)}. \tag{2.22}$$

Since $y_n^{(k)} \in \mathbb{X}_n$, we can write for the orthogonal projection

$$y_n^{(k)} = \sum_{j=1}^{N} \langle y_n^{(k)}, \varphi_j \rangle \varphi_j = \sum_{j=1}^{N} v_n^{(k)}(j) \varphi_j,$$

where $N = n(r + 1)$ and $\{\varphi_1, \ldots, \varphi_N\}$ is the orthonormal ordered basis of $\mathbb{X}_n$ given by (2.5). Then, (2.22) is equivalent to the following linear system of size N

$$(I - A_n^{(k)}) v_n^{(k+1)} = r_n^{(k)},$$

where for $i, j = 1, \dots, N$,

$$\begin{aligned} A_n^{(k)}(i,j) &= \langle \mathcal{K}'(y_n^{(k)} + f)\varphi_j, \varphi_i\rangle, \\ r_n^{(k)}(i) &= \langle \mathcal{K}(y_n^{(k)} + f), \varphi_i\rangle - (A_n^{(k)} v_n^{(k)})(i). \end{aligned} \tag{2.23}$$

Let $\{L_1, \dots, L_N\}$ be the *Lagrange* basis of $\mathbb{X}_n$ satisfying $L_i(s_j) = \delta_{ij}$, where $\{s_1, \dots, s_N\}$ are the ordered interpolation points given by (2.9). For the interpolatory projection, we can write

$$y_n^{(k)} = \sum_{j=1}^{N} y_n^{(k)}(s_j) L_j = \sum_{j=1}^{N} v_n^{(k)}(j) L_j.$$

Then, we obtain the system of linear equations

$$(I - B_n^{(k)}) v_n^{(k+1)} = q_n^{(k)},$$

where for $i, j = 1, \dots, N$,

$$\begin{aligned} B_n^{(k)}(i,j) &= (\mathcal{K}'(y_n^{(k)} + f) L_i)(s_j), \\ q_n^{(k)} &= \mathcal{K}(y_n^{(k)} + f)(s_i) - (B_n^{(k)} v_n^{(k)})(i). \end{aligned} \tag{2.24}$$

2.4. Projection-type method for Hammerstein equations

Let $\Psi : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ be the *Nemytskii* bounded and continuous operator defined by

$$\Psi(x)(t) = \psi(t, x(t)), \quad x \in \mathcal{C}[0,1], \quad t \in [0,1]$$

and let T be the linear integral operator with a kernel κ of class $C(\alpha, \gamma)$ that is,

$$(Tx)(t) = \int_0^1 \kappa(s,t) x(t) dt, \quad t \in [0,1], \quad x \in \mathbb{X}. \tag{2.25}$$

With this notation, the *Hammerstein* equation (1.2) takes the following form

$$x - T\Psi(x) = f. \tag{2.26}$$

It is more convenient to set

$$z(t) = \psi(t, x(t)) = \psi(t, Tz(t) + f(t)), \quad t \in [0,1].$$

Thus, we obtain the equivalent equation for the function z

$$z = \Psi(Tz + f). \tag{2.27}$$

The projection method for (2.27) is seeking an approximate solution $z_n \in \mathbb{X}_n$ which satisfies the operator equation

$$z_n = \pi_n \Psi(Tz_n + f). \tag{2.28}$$

The desired projection-type solution x_n^S is then defined to be

$$x_n^S = Tz_n + f$$

which means that

$$x_n^S = T\pi_n \Psi(x_n^S) + f. \tag{2.29}$$

Let

$$F_n(v) = v - \pi_n \Psi(Tv + f).$$

Then, equation (2.28) becomes

$$F_n(z_n) = 0. \tag{2.30}$$

The *Fréchet* derivative of F_n is given by

$$F'_n(v) = I - \pi_n \Psi'(Tv + f)T.$$

The *Newton-Kantorovich* method for solving (2.30) iteratively give for an initial approximation $z_n^{(0)}$

$$z_n^{(k+1)} - \pi_n \Psi'(Tz_n^{(k)} + f)Tz_n^{(k+1)} = \pi_n \Psi(Tz_n^{(k)} + f) - \pi_n \Psi'(Tz_n^{(k)} + f)Tz_n^{(k)}. \tag{2.31}$$

In the case of the orthogonal projection, $z_n^{(k)} = \sum_{j=1}^{N} v_n^{(k)}(j)\varphi_j$, and (2.31) is equivalent to the following linear system of size N

$$(I - A_n^{(k)})v_n^{(k+1)} = r_n^{(k)},$$

where

$$\begin{aligned} A_n^{(k)}(i,j) &= \langle \Psi'(Tz_n^{(k)} + f)T\varphi_j, \varphi_i \rangle, \quad i,j = 1,\dots,N, \\ r_n^{(k)}(i) &= \langle \Psi(Tz_n^{(k)} + f), \varphi_i \rangle - (A_n^{(k)} v_n^{(k)})(i), \end{aligned} \tag{2.32}$$

while for the interpolatory projection $z_n^{(k)} = \sum_{j=1}^{N} v_n^{(k)}(j)L_j$, the system of linear equations is

$$(I - B_n^{(k)})v_n^{(k+1)} = q_n^{(k)},$$

where

$$\begin{aligned} B_n^{(k)}(i,j) &= [\Psi'(Tz_n^{(k)} + f)TL_i](t_j), \quad i,j = 1,\dots,N, \\ q_n^{(k)} &= \Psi(Tz_n^{(k)} + f)(s_i) - (B_n^{(k)} v_n^{(k)})(i). \end{aligned} \tag{2.33}$$

The following interesting observation was made in many papers (see for instance [6, 17]). The integrals in the linear systems (2.23) and (2.24) must be computed at each step of the iteration. However, since in (2.32) and (2.33), the coefficients $v_n^{(k)}(j)$ involving in the expression of $z_n^{(k)}$ can be extracted out of the operator T, the integrals will depends only on the basis, not on the unknowns $v_n^{(k)}(j)$ and this make the computations of the integrals necessary only once throughout the iteration process. Therefore, in the *Kumar* and *Sloan* method, the number of integrals to be calculated is significantly lower than in the *Kantorovich* method.

3. Convergence rates

3.1. Kantorovich method

Let $x_0 \in \mathbb{X}$ be the unique solution (1.1). For $i = 1, 2$, define

$$\begin{aligned} A_i &= \max\left\{ \left| \frac{\partial^\mu \kappa_i}{\partial s^\mu}(s,t,u) \right| : (s,t,u) \in \Phi_i,\ \mu = 0,\dots,\alpha \right\}, \\ A &= \max\{A_1, A_2\}, \end{aligned}$$

where

$$\Phi_i = \{(s,t,u) : (s,t,u) \in \Pi_i, |u| \le \|x_0\|_\infty\}.$$

It is straightforward that

$$\|(\mathcal{K}(x_0))^{(\mu)}\|_\infty \le A, \quad \mu = 0,\dots,\alpha. \tag{3.1}$$

The following result is crucially used (see [7, Theorem 4.1]).

If the kernel κ is of class $C_2(\alpha,\gamma)$, the *Urysohn* operator $\mathcal{K}$ is a continuous operator on C_Δ^ν into $C_\Delta^{\min\{\alpha,\gamma+\nu+2\}}$, $\nu \ge 0$.

Theorem 3.1. *Let the kernel κ be of class $C_2(\alpha,\gamma)$ and assume that 1 is not an eigenvalue of $\mathcal{K}'(x_0)$. Then there exists a real number $\delta_0 > 0$ such that the approximate equation (2.19) has a unique solution x_n^K in $\mathcal{B}(x_0,\delta_0)$ for a sufficiently large n. Moreover, for $f \in \mathcal{C}^\alpha[0,1]$*

$$\|x_n^K - x_0\|_\infty = \mathcal{O}(h^\beta), \tag{3.2}$$

whereas for $f \in \mathcal{C}[0,1]$

$$\|x_n^K - x_0\|_\infty = \mathcal{O}(h^{\beta_2}). \tag{3.3}$$

Proof. Since the *Kantorovich* method is equivalent to a projection method for the quantity $y_0 = \mathcal{K}(x_0)$, and since $x_n^K - x_0 = y_n - y_0$ the error bound follow immediately from the analysis of the projection method. Indeed, we derive from [7, Theorem 2.2]

$$\|x_n^K - x_0\|_\infty \le C\|(I - \pi_n)y_0\|_\infty. \tag{3.4}$$

The operator $\mathcal{K}$ is a continuous map from C_Δ^α to C_Δ^α. Thus, if $f \in \mathcal{C}^\alpha[0,1]$, $\mathcal{K}(x_0) \in C_\Delta^\alpha$ and it follows from (2.13) and (3.1) that

$$\begin{aligned}\|(I - \pi_n)\mathcal{K}(x_0)\|_\infty &\le C_1\|(\mathcal{K}(x_0))^{(\beta)}\|_\infty h^\beta,\\ &\le C_1 A h^\beta.\end{aligned}$$

Hence, the estimate (3.2) is a consequence of (3.4).
Next, we recall that for $f \in \mathcal{C}[0,1]$, we have $x_0 \in \mathcal{C}[0,1]$. Furthermore, the operator $\mathcal{K}$ is a continuous map from C_Δ to $C_\Delta^{\gamma_2}$, where

$$\gamma_2 = \min\{\alpha, \gamma + 2\},$$

Consequently, if we take (2.12), we can say that

$$\|(I - \pi_n)\mathcal{K}(x_0)\|_\infty \le C_1\|(\mathcal{K}(x_0))^{(\beta_2)}\|_\infty h^{\beta_2}. \tag{3.5}$$

We now deduce (3.3) from (3.1) and (3.4). This completes the proof. □

The following estimates are provided by *Chatelin* and *Lebbar* [11].
Let T be a linear integral operator with kernel $\kappa \in C(\alpha,\gamma)$. Then, for any $x \in C_\Delta^\alpha$

$$\|T(I - \pi_n^G)x\|_\infty \le c_2\|x^{(\beta)}\|_\infty h^{\beta+\beta_2}. \tag{3.6}$$

In addition, if $\alpha \ge r + 1$,

$$\|T(I - \pi_n^C)x\|_\infty \le c_2\|x\|_{\beta_3,\infty} h^{\beta_3}, \tag{3.7}$$

where

$$\beta_3 = \min\{\alpha, 2r + 2, r + \gamma + 3\}.$$

Theorem 3.2. *Let the kernel κ be of class $C_2(\alpha,\gamma)$ and let $\widetilde{x}_n^K$ be the iterated* Kantorovich *solution defined by (2.20). If $f \in \mathcal{C}^\alpha[0,1]$, then for the orthogonal projection*

$$\|\widetilde{x}_n^K - x_0\|_\infty = \mathcal{O}(h^{\beta+\beta_2}), \tag{3.8}$$

while for the interpolatory projection

$$\|\widetilde{x}_n^K - x_0\|_\infty = \mathcal{O}(h^{\beta_3}). \tag{3.9}$$

Proof. First we observe that $\widetilde{x}_n^K - x_0 = \widetilde{y}_n - y_0$. It then follows, by essentially the same argument as for the iterated projection method, but now x_0 is replaced by $\mathcal{K}(x_0)$ that (see equation (5.12) in [7])

$$\begin{aligned}\widetilde{x}_n^K - x_0 &= (I + M\pi_n)\Big(\mathcal{K}(x_n^K) - \mathcal{K}'(x_0)(x_n^K - x_0) - \mathcal{K}(x_0)\Big)\\ &\quad - M(I - \pi_n)\mathcal{K}'(x_0)(x_n^K - x_0) - M(I - \pi_n)\mathcal{K}(x_0).\end{aligned} \tag{3.10}$$

By applying the mean-value theorem for operators to $\mathcal{K}$ and using the *Lipschitz* continuity of $\mathcal{K}'$, we get

$$\begin{aligned}&\|\mathcal{K}(x_n^K) - \mathcal{K}'(x_0)(x_n^K - x_0) - \mathcal{K}(x_0)\|\\ &\qquad = \|[\mathcal{K}'(x_n^K + \theta(x_0 - x_n^K)) - \mathcal{K}'(x_0)](x_n^K - x_0)\|,\\ &\qquad \leq \gamma(1-\theta)\|x_n^K - x_0\|_\infty^2,\end{aligned} \tag{3.11}$$

where $0 < \theta < 1$. As $\mathcal{K}(x_0) \in C_\Delta^\alpha$ and $m \in C(\alpha, \gamma)$, then using (3.6) and (3.7), we respectively obtain

$$\|M(I - \pi_n^G)\mathcal{K}(x_0)\|_\infty = \mathcal{O}(h^{\beta+\beta_2}) \tag{3.12}$$

and

$$\|M(I - \pi_n^C)\mathcal{K}(x_0)\|_\infty = \mathcal{O}(h^{\beta_3}). \tag{3.13}$$

In addition, as stated in the proof of Lemma 2.1 in [14],

$$\|(I - \pi_n)\mathcal{K}'(x_0)\|_\infty = \mathcal{O}(h^{\beta_2}). \tag{3.14}$$

Combining (3.8) with the estimates (3.10)-(3.14) and making use of $\beta_3 \leq \beta + \beta_2$, the remarks

$$\min\{2\beta, \beta + \beta_2\} = \beta + \beta_2$$

and

$$\min\{2\beta, \beta + \beta_2, \beta_3\} = \min\{\beta + \beta_2, \beta_3\} = \beta_3$$

ends the proof. □

Theorem 3.3. *Let the kernel κ be of class $C_2(\alpha, \gamma)$ and let $\widetilde{x}_n^K$ be the iterated* Kantorovich-Galerkin *solution defined by (2.20). If $f \in \mathcal{C}[0,1]$, then there holds*

$$\|\widetilde{x}_n^K - x_0\|_\infty = \mathcal{O}(h^{2\beta_2}). \tag{3.15}$$

Proof. For a fixed $s \in [0,1]$, let $m_s(t) = m(s,t),\ t \in [0,1]$. Using the orthogonality of π_n^G,

$$\begin{aligned}[M(I - \pi_n^G)\mathcal{K}(x_0)](s) &= \Big\langle m_s, (I - \pi_n^G)y_0\Big\rangle\\ &= \Big\langle (I - \pi_n^G)m_s, (I - \pi_n^G)y_0\Big\rangle\\ &= \sum_{j=1}^n \Big\langle (I - \pi_n^G)m_s, (I - \pi_n^G)y_0\Big\rangle_j,\end{aligned}$$

where

$$\Big\langle (I - \pi_n^G)m_s, (I - \pi_n^G)y_0\Big\rangle_j = \int_{t_{j-1}}^{t_j} [(I - \pi_n^G)m_s](t)[(I - \pi_n^G)y_0](t)dt.$$

It results now, from the *Cauchy-Schwarz* inequality that

$$\Big|[M(I - \pi_n^G)\mathcal{K}(x_0)](s)\Big| \leq \sum_{j=1}^n \|(I - \pi_n^G)m_s\|_{2,\Delta_j}\|(I - \pi_n^G)y_0\|_{2,\Delta_j}. \tag{3.16}$$

For $j = 1, \dots, n$, the bounds (2.4) and (3.5) allows us to write

$$\begin{aligned} \|(I - \pi_n^G)y_0\|_{2,\Delta_j} &\leq C_1 h_j^{\beta_2} \|y_0^{(\beta_2)}\|_{2,\Delta_j}, \\ &\leq C_1 h_j^{\beta_2+1/2} \|y_0^{(\beta_2)}\|_{\infty}. \end{aligned} \quad (3.17)$$

Also, [11, Lemma 9] tells us that if $s \in (t_{i-1}, t_i)$, then

$$\|(I - \pi_n^G)m_s\|_{2,\Delta_j} = \begin{cases} \mathcal{O}(h_j^{\beta+1/2}), & j \neq i, \\ \mathcal{O}(h_i^{\beta_1+1/2}), & j = i, \end{cases} \quad (3.18)$$

whereas for $s \in \Delta$,

$$\|(I - \pi_n^G)m_s\|_{2,\Delta_j} = \mathcal{O}(h_j^{\beta+1/2}), \quad j = 1, \dots, n. \quad (3.19)$$

These results implies that

$$\|M(I - \pi_n^G)\mathcal{K}(x_0)\|_\infty = \mathcal{O}(h^{\beta_2+\min\{\beta,\beta_1+1\}}). \quad (3.20)$$

Since $\min\{\beta, \beta_1+1\} = \beta_2$, then combining (3.3), (3.10), (3.14) with (3.20) yields (3.21). The proof is finished. □

Theorem 3.4. *Let the kernel κ be of class $C_2(\alpha, \gamma)$ and let $\widetilde{x}_n^K$ be the iterated* Kantorovich*-collocation solution defined by (2.20). If $f \in \mathcal{C}[0,1]$, then if $\gamma \geq 0$ we have*

$$\|\widetilde{x}_n^K - x_0\|_\infty = \mathcal{O}(h^{\min\{2\beta_2, r+1\}}). \quad (3.21)$$

Proof. Arguing as in the proof of Theorem 3.1.2, we can deduce from (3.3),(3.10),(3.11),(3.13) and (3.14) that

$$\|\widetilde{x}_n^K - x_0\|_\infty \leq \|M(I - \pi_n^C)y_0\|_\infty + \mathcal{O}(h^{2\beta_2}). \quad (3.22)$$

Lemma 11 in *Chatelin* and *Lebbar* [11] states that for any $s \in [0,1]$

$$\begin{aligned} M(I - \pi_n^C)y_0(s) &= \sum_{j=1}^{n} \left\langle (I - \pi_n^G)m_s \delta_j^{r+1} y_0, v \right\rangle_j, \\ &= \sum_{j=1}^{n} \langle (I - \pi_n^G)m_s \delta_j^{r+1} y_0, (I - \pi_n^G)v \rangle_j, \end{aligned}$$

where $\delta_j^{r+1} y_0(s) = [\tau_{j0}, \dots, \tau_{jr}, s]y_0$ denote the divided difference of y_0 at $\{\tau_{j0}, \dots, \tau_{jr}, s\}$ and

$$v_j(s) = \prod_{p=0}^{r} (s - \tau_{jp}), \quad 1 \leq j \leq n.$$

Therefore, using the *Cauchy-Schwarz* inequality, we obtain

$$|M(I - \pi_n^C)y_0(s)| \leq \sum_{j=1}^{n} \|(I - \pi_n^G)(m_s \delta_j^{r+1} y_0)\|_{\infty,\Delta_j} \|(I - \pi_n^G)v_j\|_{\infty,\Delta_j}. \quad (3.23)$$

From Lemma 2.2 in [15], it follows that

$$\|(I - \pi_n^G)v_j\|_{\infty,\Delta_j} \leq C(r+1)! h_j^{r+1}. \quad (3.24)$$

Hence

$$|M(I-\pi_n^C)y_0(s)| \leq c\left(\sum_{j=1}^{n} \|(I-\pi_n^G)(m_s\delta_j^{r+1}y_0)\|_{\infty,\Delta_j}\right)h^{r+1}.$$

Using the technique employed in [15, Lemma 3.1], we are able to demonstrate that

$$\sup_{s\in[0,1]}\left|[\tau_{j0},\ldots,\tau_{jr},s]y_0\right| \leq C_3.$$

Thus,

$$\|M(I-\pi_n^C)y_0\|_\infty \leq CC_3(1+p)\|m\|_\infty h^{r+1}.$$

Combining the above inequality with (3.22), the desired estimate follows. □

If $r=0$ and $\alpha \geq 1$, we have $\beta_2 = 1$. Hence, it follows from (3.21) that

$$\|\widetilde{x}_n^K - x_0\|_\infty = \mathcal{O}(h).$$

We now show that, if $\gamma = 0$, then the above order of convergence can be increased to h^2.

Theorem 3.5. *Let $\widetilde{x}_n^K$ be the iterated* Kantorovich-*collocation solution defined by (2.20). If $f \in \mathcal{C}[0,1]$, then for $\kappa \in C_2(\alpha,0)$ with $\alpha \geq 1$, we have*

$$\|\widetilde{x}_n^K - x_0\|_\infty = \mathcal{O}(h^2).$$

Proof. For $r=0$, let $\tau^j = \tau_{j0} = \frac{t_{j-1}+t_j}{2}$ be the collocation points. By (2.4) and (3.24), one has

$$\|(I-\pi_n^G)v_j\|_{2,\Delta_j} \leq ch_j^{3/2}, \tag{3.25}$$

where $v_j(s) = (s-\tau^j)$. By using the mean-value theorem,

$$\delta_j^1 y_0 = \frac{y_0(s)-y_0(\tau^j)}{s-\tau^j} = y_0^{(1)}(\sigma_j), \quad \sigma_j \in (t_{j-1},s).$$

Since $y_0^{(1)} \in C_\Delta^{\gamma_2-1} = C_\Delta^1$, it is obvious that the kernel $my_0^{(1)} \in C(\min\{\alpha,\gamma+1\},\gamma) = C(1,0)$. As $\beta = \beta_1 = \beta_2 = 1$, we conclude from (3.18) and (3.19) that

$$\|(I-\pi_n^G)m_s\delta_j^1 y_0\|_{2,\Delta_j} = \mathcal{O}(h_j^{3/2}), \quad s\in[0,1]. \tag{3.26}$$

The proof of the required estimate is accomplished by substituting (3.25) and (3.26) into (3.23) and combining with (3.22), respectively. □

3.2. *Projection-type method*

For the remainder of the paper, we will assume that the kernel κ of the *Hammerstein* integral operator $T\Psi$ is of class $C(\alpha,\gamma)$. When f is smooth, the following result demonstrates that the projection-type approach converges as quickly as the iterated *Kantorovich* method.

Theorem 3.6. *Let x_n^S be the projection-type solution defined by (2.29). Suppose that x_0 is the unique solution of (1.2) and that 1 is not an eigenvalue of $(T\Psi)'(x_0)$. If $\psi \in \mathcal{C}^\alpha([0,1]\times\mathbb{R})$ and $f \in \mathcal{C}^\alpha[0,1]$, then for the Galerkin-type method*

$$\|x_n^S - x_0\|_\infty = \mathcal{O}(h^{\beta+\beta_2}), \tag{3.27}$$

whereas for the collocation-type method

$$\|x_n^S - x_0\|_\infty = \mathcal{O}(h^{\beta_3}). \tag{3.28}$$

Proof. Firstly, we define

$$z_0(t) = \psi(t, x_0(t)), \quad t \in [0, 1].$$

In `Theorem 2` of *Kumar* `[16]` it was shown that

$$\|x_n^S - x_0\|_\infty \leq c\|T(I - \pi_n^C)z_0\|_\infty \tag{3.29}$$

and this estimate is valid not just for π_n^C, but also for the orthogonal projection. Let

$$\Psi_p = \max_{t\in[0,1]} \left|\frac{\partial^p \Psi}{\partial t^p}(t, x_0(t))\right|, \quad p = 0, \ldots, \alpha.$$

Therefore, we have from (3.6) and (3.7)

$$\begin{aligned} \|T(I - \pi_n^G)z_0\| &\leq C_2\|z_0^{(\beta)}\|_\infty h^{\beta+\beta_2}, \\ &\leq C_2\Psi_\beta h^{\beta+\beta_2} \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} \|T(I - \pi_n^C)z_0\| &\leq C_2\|z_0\|_{\beta_3,\infty} h^{\beta_3}, \\ &\leq C_2\left(\sum_{i=0}^{\beta_3} \Psi_i\right) h^{\beta_3}. \end{aligned} \tag{3.31}$$

Combining (3.29) with the aforementioned estimates yields the desired results. This reach the proof. □

If f is not smooth, the convergence order of the *Galerkin*-type solution is lower than that of the iterated *Kantorovich-Galerkin* solution, as stated in the result below.

Theorem 3.7. *Suppose that x_0 is the unique solution of (1.2) and that 1 is not an eigenvalue of $(T\Psi)'(x_0)$. Then, if $\psi \in \mathcal{C}([0, 1] \times \mathbb{R})$ and $f \in \mathcal{C}[0, 1]$, the Galerkin-type solution fulfills*

$$\|x_n^S - x_0\|_\infty = \mathcal{O}(h^{\beta_2}). \tag{3.32}$$

Proof. For a fixed $s \in [0, 1]$, let $\kappa_s(t) = \kappa(s, t)$, $t \in [0, 1]$. Arguing as in the proof of `Theorem 3.1.4`, the following upper bound can be estabilished

$$\|T(I - \pi_n^G)z_0\|_\infty \leq \max_{s\in[0,1]} \sum_{j=1}^{n} \|(I - \pi_n^G)\kappa_s\|_{2,\Delta_j}\|(I - \pi_n^G)z_0\|_{2,\Delta_j}. \tag{3.33}$$

In the first place, we may write from (2.4) and (2.12)

$$\|(I - \pi_n^G)z_0\|_{2,\Delta_j} \leq (1 + p)h_j^{1/2}\Psi_0. \tag{3.34}$$

To continue, we have used the same procedure for the kernel m in (3.18) and (3.19), which entails

$$\|(I - \pi_n^G)\kappa_s\|_{2,\Delta_j} = \mathcal{O}(h_j^{1/2+\min\{\beta,\beta_1\}}). \tag{3.35}$$

By combining (3.34) and (3.35) with the inequality (3.33) and the estimate (3.29), we reach the proof of (3.32). □

It should be mentioned that since $\pi_n^C x_n^K = \pi_n^C \widetilde{x}_n^K$, then the two solutions agrees at the collocation points. Therefore x_n^K and $\widetilde{x}_n^K$ converge with the same order at those points. For example under the hypothesis of `Theorem 3.1.2` we have the following superconvergence phenomenon for x_n^K

$$\max_{1\leq i\leq N} |[x_n^K - x_0](t_i)| = \mathcal{O}(h^{\beta_3}).$$

Remark 3.8. *Assume that $f \in \mathcal{C}^{\alpha}[0,1]$ and $\alpha \geq r+1$. If $r \leq \gamma$, then since*

$$\beta = \beta_1 = \beta_2 = r+1$$

and

$$\beta_3 = 2r+2,$$

the following full orders

$$\|x_n^K - x_0\|_\infty = \mathcal{O}(h^{r+1})$$

and

$$\|\widetilde{x}_n^K - x_0\|_\infty = \mathcal{O}(h^{2r+2})$$

corresponding to the case of a smooth kernel are recovered.
It should be mentioned that for the Kantorovich *and the iterated* Kantorovich-Galerkin *methods, the preceding convergence orders also hold when $f \in \mathcal{C}[0,1]$. If $r > \gamma$, then*

$$\beta = r+1,\ \beta_1 = \gamma+1$$

and

$$\beta_2 = \gamma+2,\ \beta_3 = r+\gamma+3.$$

Thus,

$$\|x_n^K - x_0\|_\infty = \mathcal{O}(h^{r+1})$$

and

$$\|\widetilde{x}_n^K - x_0\|_\infty = \mathcal{O}(h^{r+\gamma+3}).$$

If $f \in \mathcal{C}[0,1]$, for the Kantorovich *method and the iterated* Kantorovich-Galerkin *method, we have*

$$\|x_n^K - x_0\|_\infty = \mathcal{O}(h^{\gamma+2})$$

and

$$\|\widetilde{x}_n^K - x_0\|_\infty = \mathcal{O}(h^{2\gamma+4}).$$

For the iterated Kantorovich-*collocation method, if $r > 2\gamma+2$, then*

$$\|\widetilde{x}_n^K - x_0\|_\infty = \mathcal{O}(h^{2\gamma+4}).$$

4. Numerical results

Here, we propose several numerical experiments to demonstrate the effectiveness of the presented methods. Two *Hammerstein* equations having *Green*'s kernels and with exact solutions of varying regularity are considered. We solve the associated linear systems for each test equation and then we compute the infinite norm of the errors with respect to the true value x_0. We also evaluate how well each proposed approach performs in comparison to the other. It should be noted that the integrals in the linear system were computed using a high order *Gauss*-quadrature rule.
We choose $\mathbb{X}_n$ to be the space of piecewise constant functions ($r = 0$) or the space of piecewise linear polynomials ($r = 1$) with respect to the uniform partition of $[0,1]$

$$0 = \frac{1}{n} < \frac{2}{n} < \ldots < \frac{n}{n} = 1.$$

Let π_n^G be the restriction to $L^\infty[0,1]$ of the orthogonal projection from $L^2[0,1]$ to $\mathbb{X}_n$. The operator π_n^C is chosen to be either the interpolatory projection at the n midpoints

$$\tau^i = \frac{2i-1}{2n},\ i = 1,\ldots,n$$

or at the $2n$ *Gauss* points given by

$$\tau_1^i = \frac{2i-1}{2n} - \frac{1}{2n}\frac{1}{\sqrt{3}} \quad \text{and} \quad \tau_2^i = \frac{2i-1}{2n} + \frac{1}{2n}\frac{1}{\sqrt{3}},\ i = 1, \dots, n.$$

Note that the maximum errors $\|x_n^K - x_0\|_\infty$, $\|\widetilde{x}_n^K - x_0\|_\infty$ and $\|x_n^S - x_0\|_\infty$ are approximated respectively by

$$E_K^n = \max_{i=1,2,\dots,10^2} |(x_n^K - x_0)(y_i)|,$$
$$\widetilde{E}_K^n = \max_{i=1,2,\dots,10^2} |(\widetilde{x}_n^K - x_0)(y_i)|$$

and

$$E_S^n = \max_{i=1,2,\dots,10^2} |(x_n^S - x_0)(y_i)|,$$

where y_i are equally spaced points in $[0,1]$. The orders of convergence are calculated using the formulas

$$\delta_K = \frac{\log\left(E_K^n / E_K^{2n}\right)}{\log(2)}, \quad \widetilde{\delta_K} = \frac{\log(\widetilde{E}_K^n / \widetilde{E}_K^{2n})}{\log(2)}, \quad \delta_S = \frac{\log\left(E_S^n / E_S^{2n}\right)}{\log(2)}.$$

Example 1. We consider the following *Hammerstein* equation quoted from [18]

$$x(s) - \int_0^1 \kappa(s,t)\psi(t, x(t))dt = f(s), \quad s \in [0,1]$$

where

$$\kappa(s,t) = \frac{1}{\sigma \sinh \sigma} \begin{cases} \sinh \sigma s \sinh \sigma(1-t), & s \le t \\ \sinh \sigma(1-s) \sinh \sigma t, & t \le s \end{cases}$$

with $\sigma = \sqrt{12}$, and

$$\psi(t, x(t)) = \sigma^2 x(t) - 2(x(t))^3, \quad t \in [0,1].$$

We have $f(s) = \frac{1}{\sinh \sigma}\left\{2 \sinh \sigma(1-s) + \frac{2}{3} \sinh \sigma s\right\}$ and the exact solution is

$$x_0(s) = \frac{2}{2s+1}, \quad s \in [0,1].$$

In this example

$$\alpha = \infty, \quad \gamma = 0, \quad \gamma_1 = 1, \quad \gamma_2 = 2.$$

For $r = 0$, we recall from Remark 3.8 and Theorem 3.6 that the expected orders of convergence in *Kantorovich*, iterated *Kantorovich* and projection-type methods, are respectively,

$$\delta_K = 1,\ \widetilde{\delta_K} = 2 \quad \text{and} \quad \delta_S = 2,$$

whereas for $r = 1$, the orders are as follows

$$\delta_K = 2,\ \widetilde{\delta_K} = 4 \quad \text{and} \quad \delta_S = 4.$$

The numerical outcomes are reported in Tables 1-4.

n	$\|x_n^K - x_0\|_\infty$	δ_K	$\|\widetilde{x}_n^K - x_0\|_\infty$	$\widetilde{\delta}_K$	$\|x_n^S - x_0\|_\infty$	δ_S
2	3.66×10^{-1}	–	4.79×10^{-2}	–	1.77×10^{-2}	–
4	2.61×10^{-1}	0.49	1.02×10^{-2}	2.22	1.02×10^{-2}	1.93
8	1.53×10^{-1}	0.77	2.53×10^{-3}	2.02	2.73×10^{-3}	1.93
16	8.20×10^{-2}	0.90	6.55×10^{-4}	1.95	7.31×10^{-4}	2.01
32	4.23×10^{-2}	0.95	1.61×10^{-4}	2.02	1.80×10^{-4}	2.01
64	2.15×10^{-2}	0.98	3.99×10^{-5}	2.01	4.61×10^{-5}	2.00

Table 1: `Orthogonal projection` $(r = 0)$

n	$\|x_n^K - x_0\|_\infty$	δ_K	$\|\widetilde{x}_n^K - x_0\|_\infty$	$\widetilde{\delta}_K$	$\|x_n^S - x_0\|_\infty$	δ_S
2	4.35×10^{-1}	–	2.28×10^{-2}	–	7.52×10^{-3}	–
4	2.81×10^{-1}	0.63	6.45×10^{-3}	2.13	3.67×10^{-3}	1.04
8	1.58×10^{-1}	0.83	1.61×10^{-3}	2.00	1.29×10^{-3}	1.50
16	8.33×10^{-2}	0.92	3.97×10^{-4}	2.02	3.59×10^{-4}	1.85
32	4.27×10^{-2}	0.96	9.83×10^{-5}	2.01	9.20×10^{-5}	1.96
64	2.16×10^{-2}	0.98	2.47×10^{-5}	1.99	2.31×10^{-5}	1.99

Table 2: `Interpolatory projection` $(r = 0)$

`Figure 1` presents, for the purpose of completeness, error graphs for each of the different approaches when n is equal to 2.

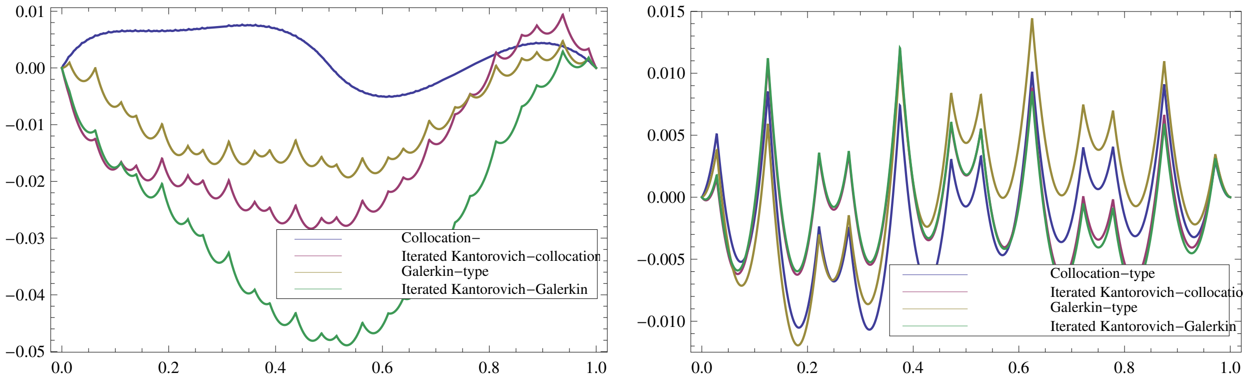

Figure 1: For $r = 0$, we give on the left, the errors of the approximations for `Example 1` produced by *Kantorovich*, iterated *Kantorovich* and projection-type methods for both the orthogonal and the interpolatory projections. On the right, we display the corresponding errors to the case where $r = 1$.

Even though the errors in the infinity norm are essentially identical in the iterated *Kantorovich* and projection-type methods, for both the orthogonal and the interpolatory projections, we notice that the graphical behavior of the errors differs.

n	$\|x_n^K - x_0\|_\infty$	δ_K	$\|\widetilde{x}_n^K - x_0\|_\infty$	$\widetilde{\delta}_K$	$\|x_n^S - x_0\|_\infty$	δ_S
2	1.33×10^{-1}	–	3.91×10^{-3}	–	8.04×10^{-3}	–
4	4.04×10^{-2}	1.73	3.88×10^{-4}	3.33	1.49×10^{-3}	3.09
8	1.07×10^{-2}	1.92	4.41×10^{-5}	3.71	1.03×10^{-4}	3.42
16	2.67×10^{-3}	2.00	3.08×10^{-6}	3.84	6.64×10^{-6}	3.66
32	6.32×10^{-4}	1.08	1.97×10^{-7}	3.96	4.02×10^{-7}	4.06

Table 3: `Orthogonal projection` $(r = 1)$

n	$\|x_n^K - x_0\|_\infty$	δ_K	$\|\widetilde{x}_n^K - x_0\|_\infty$	$\widetilde{\delta}_K$	$\|x_n^S - x_0\|_\infty$	δ_S
2	1.42×10^{-1}	–	3.83×10^{-3}	–	1.66×10^{-2}	–
4	4.13×10^{-2}	1.79	4.01×10^{-4}	3.26	1.48×10^{-3}	3.49
8	1.07×10^{-2}	1.95	2.31×10^{-5}	4.11	9.64×10^{-5}	3.94
16	2.67×10^{-3}	2.00	1.93×10^{-6}	3.58	5.31×10^{-6}	4.18
32	6.61×10^{-4}	2.01	1.22×10^{-7}	3.99	2.90×10^{-7}	4.20

Table 4: `Interpolatory projection` $(r = 1)$

From `Tables 1-4`, it can be seen that the computed orders of convergence match with the theoretical ones. To emphasize the difference between various methods, we compare in `Figure 2` the `CPU time` (in seconds) required to obtain the approximate solutions for different values of n.

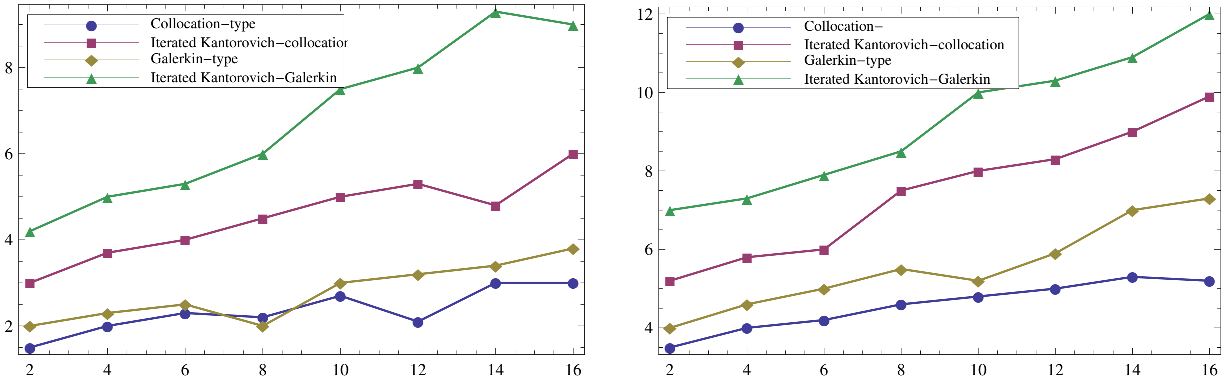

Figure 2: CPU time results for `Example 1`, $r = 0$ (on the left), and $r = 1$ (on the right).

The iterated *Kantorovich* approach is slightly slower than the projection-type method, as can be seen. In addition, given the two approaches, the interpolatory projection requires fewer arithmetic operations than the orthogonal projection.

Example 2. The following second example

$$x(s) - \int_0^1 \kappa(s,t)\psi(t, x(t))dt = f(s), \quad s \in [0,1]$$

is chosen to favour the *Kantorovich* method over the projection-type method, in that κ is the *Green* kernel given by (see `[1]`)

$$\kappa(s,t) = \frac{1}{2(1-\sigma\eta)} \begin{cases} (2st - t^2)(1-\sigma\eta) + s^2 t(\sigma - 1), & t \le \min\{\eta, s\} \\ s^2(1-\sigma\eta) + s^2 t(\sigma-1), & s \le t \le \eta \\ (2st - t^2)(1-\sigma\eta) + s^2(\sigma\eta - t), & \eta \le t \le s \\ s^2(1-t), & \max\{\eta, s\} \le t \end{cases} \tag{4.1}$$

and the inhomogeneous term f is selected so that $x_0(t) = |t - \frac{1}{2}|^{\frac{1}{4}}$. Note that the kernel is discontinuous on the line $t = \eta$.
We choose $\sigma = 2$ and $\eta = \frac{1}{3}$. For $r = 0$, the expected orders of convergence in the *Kantorovich* method and its iterated version, are respectively, 1 and 2, whereas for *Galerkin*-type method the order is 1. The expected orders for $r = 1$, are respectively 2, 4 and 2.
The numerical outcomes are given in `Tables 5-8`.

n	$\|x_n^K - x_0\|_\infty$	δ_K	$\|\widetilde{x}_n^K - x_0\|_\infty$	$\widetilde{\delta}_K$	$\|x_n^S - x_0\|_\infty$	δ_S
2	3.41×10^{-2}	–	8.65×10^{-4}	–	1.11×10^{-2}	–
4	2.27×10^{-2}	0.59	2.71×10^{-4}	1.67	2.95×10^{-3}	1.90
8	1.30×10^{-2}	0.81	7.20×10^{-5}	1.91	7.77×10^{-4}	1.93
16	6.92×10^{-3}	0.91	1.85×10^{-5}	1.96	2.01×10^{-4}	1.95
32	3.57×10^{-3}	0.95	4.70×10^{-6}	1.97	5.16×10^{-5}	1.96
64	1.81×10^{-3}	0.98	1.20×10^{-6}	1.97	1.31×10^{-5}	1.97

Table 5: `Orthogonal projection` $(r = 0)$

n	$\|x_n^K - x_0\|_\infty$	δ_K	$\|\widetilde{x}_n^K - x_0\|_\infty$	$\widetilde{\delta}_K$	$\|x_n^S - x_0\|_\infty$	δ_S
2	3.87×10^{-2}	–	2.29×10^{-4}	–	1.41×10^{-2}	–
4	2.42×10^{-2}	0.68	1.21×10^{-4}	0.93	4.71×10^{-3}	1.58
8	1.34×10^{-2}	0.86	3.60×10^{-5}	1.75	1.57×10^{-4}	1.58
16	7.03×10^{-3}	0.93	9.54×10^{-6}	1.91	5.28×10^{-4}	1.58
32	3.60×10^{-3}	0.97	2.47×10^{-6}	1.95	1.79×10^{-4}	1.57
64	1.82×10^{-3}	0.98	6.41×10^{-7}	1.95	6.08×10^{-5}	1.55

Table 6: `Interpolatory projection` $(r = 0)$

`Figure 3` below shows the graphs of the errors of various methods for $n = 2$.

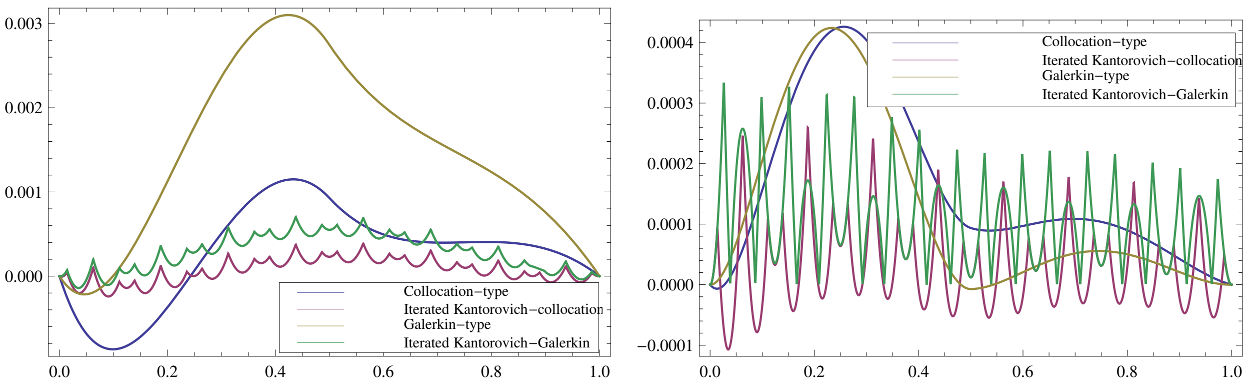

Figure 3: For $r = 0$, we give on the left, the errors of the approximations for `Example 2` produced by *Kantorovich*, iterated *Kantorovich* and projection-type methods for both the orthogonal and the interpolatory projections. On the right, we give the corresponding errors to the case $r = 1$.

n	$\|x_n^K - x_0\|_\infty$	δ_K	$\|\widetilde{x}_n^K - x_0\|_\infty$	$\widetilde{\delta}_K$	$\|x_n^S - x_0\|_\infty$	δ_S
2	1.13×10^{-2}	–	1.68×10^{-4}	–	4.79×10^{-4}	–
4	3.29×10^{-3}	1.78	6.35×10^{-6}	4.72	8.53×10^{-5}	2.49
8	8.74×10^{-4}	1.91	7.23×10^{-7}	3.14	1.50×10^{-5}	2.51
16	2.24×10^{-4}	1.96	7.31×10^{-8}	3.30	2.60×10^{-6}	2.53
32	5.68×10^{-5}	1.98	5.07×10^{-9}	3.85	3.71×10^{-7}	2.81

Table 7: `Orthogonal projection` $(r = 1)$

`Tables 5-8`, illustrate that a high accuracy is obtained by the iterated *Kantorovich* method even when the solution and the right hand side are only continuous.

n	$\|x_n^K - x_0\|_\infty$	δ_K	$\|\widetilde{x}_n^K - x_0\|_\infty$	$\widetilde{\delta}_K$	$\|x_n^S - x_0\|_\infty$	δ_S
2	1.20×10^{-2}	–	1.86×10^{-4}	–	1.38×10^{-3}	–
4	3.36×10^{-3}	1.83	6.75×10^{-6}	4.78	4.57×10^{-4}	1.60
8	8.82×10^{-4}	1.93	7.02×10^{-7}	3.27	1.55×10^{-4}	1.56
16	2.25×10^{-4}	1.97	5.86×10^{-8}	3.58	5.35×10^{-5}	1.53
32	5.70×10^{-5}	1.98	5.78×10^{-9}	3.34	1.87×10^{-5}	1.52

Table 8: `Interpolatory projection` $(r = 1)$

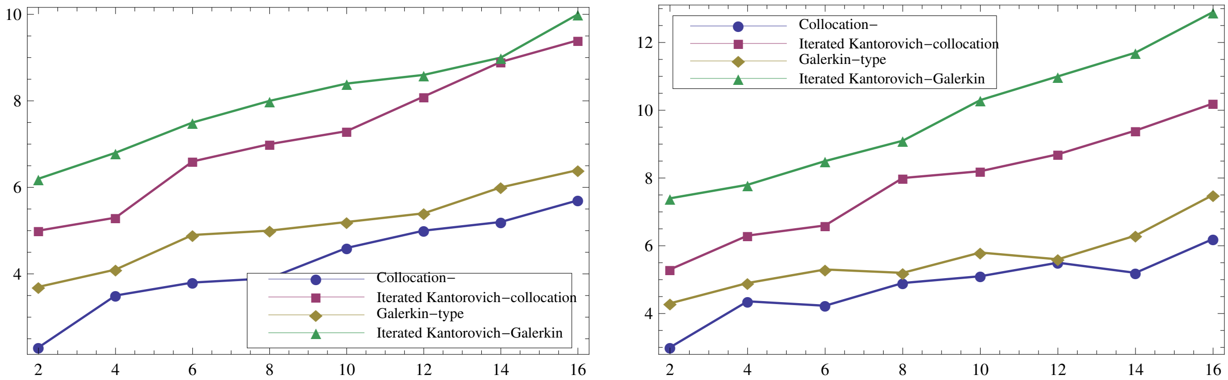

Figure 4: CPU time results for `Example 2`, $r = 0$ (on the left), and $r = 1$ (on the right).

When compared to the projection-type method, the iterated *Kantorovich* method has extremely reasonable computational costs, especially when considering the quality of the generated findings.

Conclusions

To approximatively solve the nonlinear problem (1.1), two efficient numerical approaches based on projection operators have been suggested. Both of the proposed methods for these types of kernels are novel contributions to the literature, since, unlike the standard projection method, the error estimation of the *Kantorovich* method depends precisely on the regularity of Kf instead of f, which is smoother if f has very low smoothness. Further, the *Galerkin*-type method which form a redefinition of the collocation-type method seems to have not been investigated before for *Green*'s kernels. The approximate solution has been obtained at a very low computational cost and by solving a given linear system. The convergence of the methods have been proved, providing superconvergent results even when the solution is only continuous. Moreover, we have shown by some experimental results that our procedure reaches the same accuracy when the solution is sufficiently smooth. We believe that sharper estimates than those stated previously could have been provided, especially in the projection-type method.

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