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Betti numbers of edge ideals of some graphs with application to graphs assigned to groups

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Abstract. The article presents the Betti numbers of multiple complete split-like graphs, clique stars and their generalization. As an applications, we also give the Betti numbers of the graphs defined on groups, like power graphs of groups and commuting graphs of non-abelian groups. Also, we give their extremal Betti numbers and their projective dimension.

1. The first section

2. Introduction

For a polynomial ring $R = \mathbb{K}[x_1, x_2, \ldots, x_N]$ over a field $\mathbb{K}$ with standard degree grading. To every finite simple graph G with vertex set $V(G) = \{x_1, x_2, \ldots, x_N\}$ and edge set $E(G)$, we can associate its *edge ideal* $I(G)$ (see, Villarreal [23]) defined as $I(G) = (x_i x_j | x_i, x_j \in E(G)) \subseteq R$. The quotient $R/I(G)$ is known as *edge ring* of G. By Hilbert-Syzygy theorem, the graded R-module, $R/I(G)$ exhibits a unique minimal $\mathbb{N}$-graded free resolution

$$0 \to \bigoplus_{j=p+1}^{s_p} R(-j)^{\beta_{p,j}} \to \cdots \longrightarrow \bigoplus_{j=i+1}^{s_i} R(-j)^{\beta_{i,j}} \longrightarrow \cdots \to \bigoplus_{j=2}^{s_1} R(-j)^{\beta_{1,j}} \to R \to R/I(G) \to 0,$$

of length $p \leq n$. The number p is the length of the minimal graded free resolution of $R/I(G)$, and is called the projective dimension of $R/I(G)$, written as $\operatorname{pd}(R/I(G))$ (or shortly $\operatorname{pd}(G)$). $R(-j)$ is a graded free R-module of rank one generated in degree j and the number $\beta_{i,j}$ of generators of ith syzygy module in degree j is called the ith graded *Betti number* of $R/I(G)$ in degree j, denoted by $\beta_{i,j}(R/I(G))$ (or simply $\beta_{i,j}(G)$). There are particular cases and equivalent ways to find the Betti numbers of $I(G)$, but since $I(G)$ is a square-free monomial ideal, so our principal tool to study $\beta_{i,j}(I(G))$ shall be Hochster's formula (see, [13, 20]). The free resolution of $I(G)$ encodes several homological invariants of $I(G)$ which are intimately related to the graph invariants of G. Two such important invariants are (Castelnuovo-Mumford) regularity, which is defined as

$$\operatorname{reg}^{\mathbb{K}}(I(G)) = \max\{j - i | \beta_{i,j}^{\mathbb{K}}(I(G)) \neq 0\},$$

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and the projective dimension, given as

$$\mathrm{pd}^{\mathbb{K}}(I(G)) = \max\{i|\beta_{i,j}^{\mathbb{K}}(I(G)) \neq 0 \text{ for some } j\}.$$

Many interesting papers can be found in this direction [10, 11, 15, 19, 23]. Mohammadi and Moradi [18] investigated resolutions of unmixed bipartite graphs. Singh and Rohit [21] found the Betti numbers of edge ideals of some split graphs. The Betti numbers, regularity and the projective dimension of $I(G)$ of G, in general, depends on both the graph and the characteristic of underlying field. However, in our study, these invariants are independent of the characteristic of field. Thus, for the sake of brevity, we write $\beta_{i,j}^{\mathbb{K}}(R/I(G)) = \beta_{i,j}(G)$, $\mathrm{reg}^{\mathbb{K}}(R/I(G)) = \mathrm{reg}(G)$ and $\mathrm{pd}^{\mathbb{K}}(R/I(G)) = \mathrm{pd}(G)$. A Betti number $\beta_{i,j}$ is called an *extremal Betti number* if $\beta_{r,s} = 0$ for all $r \geq i, s \geq j+1$ and $s - r \geq j - i$. Extremal Betti numbers of graded algebras are widely studied, for some recent progress see [4, 14, 18] and the references cited therein.

The rest of the paper is organized as: In Section 3, we discuss the Betti numbers of multiple complete split-like graphs, clique stars and the generalized clique stars and give exact formulae for their initial Betti numbers. We also obtain their extremel Betti numbers and the projective dimension. Section 4 and 5 discuses the application of Section 3 to the power graphs of finite groups and the commuting graphs of non-abelian groups. We end up the article with conclusion for future work.

3. Betti numbers of edge ideals of some graphs

Let G be a finite simple (without loops and multiple edges) graph with vertex set $V(G) = \{x_1, x_2, \ldots, x_N\}$ and edge set $E(G)$. A subgraph G' of G is called an *induced subgraph* if two vertices of G' are adjacent if and only if they are adjacent in G. The degree of a vertex $v \in V(G)$ is denoted by d_v. The union of two graphs G_1 and G_2, denoted by $G_1 \cup G_2$, is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. We denote by kG, the union of $k \geq 2$ (integer) copies of G. The join of two graphs G_1 and G_2, denoted by $G_1 * G_2$, is obtained from $G_1 \cup G_2$ along with the edge set $\{xy : \text{ for } x \in V(G_1) \text{ and } y \in V(G_2)\}$. The complement $\overline{G}$ of G is a graph with the same vertex set as of G and the edge set $E(\overline{G}) = E(K_N) \setminus E(G)$. A complete graph K_N on N vertices is a graph in which every pair of distinct vertices are adjacent. A subset $S \subseteq V(G)$ is called an independent (stable) set if its induced subgraph is totally disconnected (isomorphic to complement of clique). However, a subset C of $V(G)$ is called a clique if the induced subgraph on C is a complete graph. A graph G is known as chordal graph if it does not contain an induced cycle of length greater than or equal to 4. A graph G is said to be co-chordal if $\overline{G}$ is chordal.

A simplicial complex on the vertex set $V(\Delta) = \{x_1, x_2, \ldots, x_N\}$ is a collection of subsets of $V(\Delta)$ such that each singleton $\{x_i\} \in \Delta$ for each i, and $B \in \Delta$ for each $B \subseteq F$ with $F \in \Delta$, that is, roughly saying that Δ is closed under inclusion. An element of Δ is known as face of Δ and the maximal faces of Δ under inclusion are called facets. A face $F \in \Delta$ is called an i-dimensional face (or i-face) if $|F| - 1 = i$. The dimension of Δ, denoted by dim, is defined to be d if $\max\{|F| \mid F \in \Delta\} = d + 1$. We represent the number of connected components of Δ by $\mathrm{comp}(\Delta)$. Δ' said to be subcomplex of Δ if $\Delta' \subseteq \Delta$. The induced subcomplex Δ_S on a subset S of $V(\Delta)$ is a simplicial complex $\Delta_S = \{F \in \Delta \mid F \subseteq W\}$. A subcomplex of Δ is said to be full provided every face of Δ having its elements in $V(\Delta)$ also belongs to it. If Δ and Δ' are two simplicial complexes such that $V(\Delta) \cap V(\Delta') = \emptyset$, then their join is the simplicial complex $\Delta * \Delta' = \{\sigma \cup \tau \mid \sigma \in \Delta, \tau \in \Delta'\}$.

Let G be a finite simple graph with vertex set $V(G) = \{x_1, x_2, \ldots, x_N\}$. Then the simplicial complex

$$\Delta(G) = \{S \mid S \text{ is an independent subset of } V(G)\}$$

on $V(G)$ is known as the independent complex of G. Given a simplicial complex Δ with vertex set $\{x_1, x_2, \ldots, x_N\}$, the squarefree monomial ideal I_Δ in the polynomial ring $R = \mathbb{K}[x_1, x_2, \ldots, x_N]$ generated by all squarefree monomials $x_{i_1} x_{i_2} \ldots x_{i_p}$ such that $\{x_{i_1}, x_{i_2}, \ldots, x_{i_p}\}$ is not a face of Δ is known as Stanley-Reisner ideal, that is,

$$I_\Delta = \{x_{i_1} x_{i_2} \ldots x_{i_p} \mid \{x_{i_1}, x_{i_2}, \ldots, x_{i_p}\} \notin \Delta\} \subset R.$$

The quotient ring $\mathbb{K}[\Delta] = R/I_\Delta$ is known as the Stanley-Reisner ring of Δ. Conversely, for each squarefree monomial ideal $I \subset R = \mathbb{K}[x_1, x_2, \ldots, x_N]$ there is a simplicial complex Δ on vertex set $\{x_1, x_2, \ldots, x_N\}$ such that $I = I_\Delta$. Therefore, for an edge ideal $I(G)$ in the polynomial ring $R = \mathbb{K}[x_1, x_2, \ldots, x_N]$, the simplicial complex $\Delta(G)$ associated to graph G on vertex set $\{x_1, x_2, \ldots, x_N\}$ given by

$$\Delta(G) = \{\{x_{i_1} x_{i_2} \ldots x_{i_p}\} \subseteq V \mid \{x_{i_1} x_{i_2} \ldots x_{i_p}\} \text{ is an stable set}\},$$

is such that $I(G) = I_{\Delta(G)}$.

Next, we state an interesting result known as Hochster's formula [13] (also see, [20]), which is an important tool for the computation of graded Betti numbers of Stanley-Reisner ring $\mathbb{K}[\Delta]$. This formula describes the graded Betti numbers of I_Δ in terms of the dimensions of the reduced homology of Δ.

Theorem 3.1 ([13]). *The graded Betti number $\beta_{i,j}$ of the Stanley-Reisner ring $\mathbb{K}[\Delta] = R/I_\Delta$ in degree j is given by*

$$\beta_{i,j}(\mathbb{K}[\Delta]) = \sum_{\substack{S \subseteq V \\ |S|=j}} dim_{\mathbb{K}} \widetilde{H}_{j-i-1}(\Delta_S; \mathbb{K}), \tag{1}$$

for each $i, j \geq 0$.

A connected graph G is called a split graph if its vertex set can be put as a disjoint union of a clique and a stable set. In addition, if each vertex of a clique is connected to every vertex of a stable set, then we say G is the complete split graph. Further if there are n number of cliques $K_b, b \geq 2$ on disjoint vertex sets such that each vertex of such cliques are joined to every vertex of a stable set say of cardinality a, we obtain a multiple complete split-like graph, denoted by $MCS^a_{b,n}$. Thus, the multiple complete split-like graph G can be written as $G \cong MCS^a_{b,n} = \overline{K}_a * nK_b$. If we replace a stable set $\overline{K}_a$ by a clique K_a, then we obtain a clique star $CS^a_{b,n} = K_a * nK_b$. If we put $n = 1$ in $MCS^a_{b,n}$, we obtain $MCS^a_{b,1} \cong CS^a_b$, where CS^a_b is a complete split graph with clique size b and a stable set of size a. Next, we discuss the Betti numbers of a multiple complete split-like graph, a clique star and its generalizations.

Theorem 3.2. *Let $G \cong MCS^a_{b,n}$ be a complete split like graph of order $N \geq 3$ and let $l^j_t, t = 1, 2, \ldots, n+1$ and $j = 1, 2, \ldots, n$ be positive integers. Then the initial Betti numbers of G are*

$$\begin{aligned}
\beta_{i,i+1}(G) = {} & n \cdot i\binom{b}{i+1} + n \sum_{\substack{l^1_1+l^1_2=i+1 \\ l^1_1,l^1_2 \geq 1}} l^1_2 \binom{a}{l^1_1}\binom{b}{l^1_2} + \binom{n}{2} \sum_{\substack{l^2_1+l^2_2+l^2_3=i+1 \\ l^2_1,l^2_2,l^2_3 \geq 1}} \binom{a}{l^2_1}\binom{b}{l^2_2}\binom{b}{l^2_3} \\
& + \binom{n}{3} \sum_{\substack{l^3_1+\cdots+l^3_4=i+1 \\ l^3_1,l^3_2,l^3_3,l^3_4 \geq 1}} \binom{a}{l^3_1}\binom{b}{l^3_2}\binom{b}{l^3_3}\binom{b}{l^3_4} + \binom{n}{4} \sum_{\substack{l^4_1+\cdots+l^4_5=i+1 \\ l^4_1,l^4_2,l^4_3,l^4_4,l^4_5 \geq 1}} \binom{a}{l^4_1}\binom{b}{l^4_2}\binom{b}{l^4_3}\binom{b}{l^4_4}\binom{b}{l^4_5} + \\
& \vdots \\
& + \binom{n}{n-1} \sum_{\substack{l^{n-1}_1+\cdots+l^{n-1}_n=i+1 \\ l^{n-1}_j \geq 1, j=1,2,\ldots,n}} \binom{a}{l^{n-1}_1}\binom{b}{l^{n-1}_2}\binom{b}{l^{n-1}_3}\cdots\binom{b}{l^{n-1}_{n-2}}\binom{b}{l^{n-1}_{n-1}}\binom{b}{l^{n-1}_n} \\
& + \sum_{\substack{l^n_1+\cdots+l^n_{n+1}=i+1 \\ l^n_j \geq 1, j=1,2,\ldots,n,n+1}} \binom{a}{l^n_1}\binom{b}{l^n_2}\binom{b}{l^n_3}\cdots\binom{b}{l^n_{n-2}}\binom{b}{l^n_{n-1}}\binom{b}{l^n_n}\binom{b}{l^n_{n+1}}.
\end{aligned}$$

Proof. Let $G \cong \overline{K}_a * (nK_b)$ be the multiple complete split-like graph of order $a + nb$, where $a, b, n \geq 1$ are positive integers. Let $\Delta = \Delta(G)$ be the simplicial complex of G. Let V_1 denote the vertices of $\overline{K}_a$ and let

$U_j = V(K_b)$, for $j = 2, 3, \ldots, n+1$. Thus, by using Theorem 1, we have

$$\beta_{i,i+1}(G) = \sum_{\substack{S \subseteq V \\ |S|=i+1}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}),$$

where $V = V(G)$ and $\Delta = \Delta(G)$.

We note that $V_1 = \{x_1, x_2, \ldots, x_a\}$ is an independent subset of G of cardinality a and each of $U_j = \{y_{i1}, y_{i2}, \ldots, y_{ib}\}$ is a clique of same size. So, the above expression can be put as

$$\beta_{i,i+1}(G) = \sum_{\substack{S \subseteq V_1 \\ |S|=i+1}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}) + \sum_{\substack{S \subseteq U_j \\ |S|=i+1}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}) + \sum_{\substack{S \in \mathcal{S} \\ |S|=i+1}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}), \tag{2}$$

where $\mathcal{S} = \{S \subset V(G) \mid |S| - 1 = i,\ S \cap V_1 \neq \emptyset$ and $S \cap \left(U_{k_1} \cup U_{k_2} \cup \cdots \cup U_{k_t}\right) \neq \emptyset\}$, for $1 \leq t \leq n$ and $k_1 < k_2 < \cdots < k_t$.

For $S \subseteq V_1$, it is clear that Δ_S is $a-1$-simplex $\langle x_1, x_2, \ldots, x_a \rangle$ subcomplex of Δ and it has zero reduced homology. Thus comp(S) is one and $\dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}) = 0$. Thus, it follows that $\sum_{\substack{S \subseteq U_j \\ |S|=i+1}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}) = 0$. Again, for $S \subset U_1$, Δ_S is a disjoint union of $|S|$ simplexes of dimension zero $\langle y_{jk} \rangle$, $1 \leq k \leq |S|$ and Δ_S has a non-zero reduced homology. Thus, such a subset contributes $|S| - 1 = i$ to $\beta_{i,i+1}(G)$ and besides that the number of subsets of U_1 which contain exactly $i+1$ elements are $\binom{|U_1|}{i+1} = \binom{b}{i+1}$, since U_1 is a clique of size b. Therefore, $i\binom{b}{i+1}$ is the total contribution for $S \subseteq U_1 : |S| = i+1$ for $\beta_{i,i+1}(G)$. Similarly, repeating the same process with the remaining subsets U_j, $j = 2, 3, \ldots, n$, we see that $i\binom{b}{i+1}$ is repeated $n-1$ times and from Equation 2, we have

$$\beta_{i,i+1}(G) = ni\binom{b}{i+1} + \sum_{\substack{S \in \mathcal{S} \\ |S|=i+1}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}). \tag{3}$$

Next, we calculate the quantity $\sum_{\substack{S \in \mathcal{S} \\ |S|=i+1}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K})$ where where $\mathcal{S} = \{S \subset V(G) \mid |S| - 1 = i,\ S \cap V_1 \neq \emptyset$ and $S \cap \left(U_{k_1} \cup U_{k_2} \cup \cdots \cup U_{k_t}\right) \neq \emptyset\}$, for $1 \leq t \leq n$ and $k_1 < k_2 < \cdots < k_t$. Therefore for any $S \in \mathcal{S}$, $\dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}) = \text{comp}(\Delta_S) \neq 0$ and has a non-zero contribution for $\beta_{i,i+1}(G)$. The following choices for such $S \in \mathcal{S}$ are:

Case (1). $t = 1$.

$$S \subseteq V_1 \cup U_{k_1} \text{ such that } S \cap V_1 \neq \emptyset \text{ and } S \cap U_{k_1} \neq \emptyset, \text{where } 1 \leq k_1 \leq n.$$

First for $k_1 = 1$, and we see that Δ_S is a disjoint union of $a-1$-simplex and 0-simplexes. Let l_1^1 and l_2^2 be the positive integers such that $|S \cap V_1| = l_1^1$ and $|S \cap U_1| = l_2^1$. Then in this case Δ_S has $l_2^1 + 1$ connected components and such a subset will contribute l_2^1 to $\beta_{i,i+1}(G)$, since $\dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}) + 1 = \text{comp}\Delta_S$. Thus the net contributions of these type of subsets to $\beta_{i,i+1}(G)$ is

$$\sum_{\substack{l_1^1 + l_2^1 = i+1 \\ l_1^1, l_2^1 \geq 0}} l_2^1 \binom{|V_1|}{l_1^1}\binom{|U_1|}{l_2^1} = \sum_{\substack{l_1^1 + l_2^1 = i+1 \\ l_1^1, l_2^1 \geq 0}} l_2^1 \binom{a}{l_1^1}\binom{b}{l_2^1}.$$

Repeating the same process for $j = 2, 3, \ldots, n-1, n$, we have n above type of contributions to $\beta_{i,i+1}(G)$.

Case (2). $t = 2$.

$$S \subseteq V_1 \cup U_{k_1} \cap U_{k_2} \text{ such that } S \cap V_1 \neq \emptyset, S \cap U_{k_1} \neq \emptyset \text{ and } S \cap U_{k_2} \neq \emptyset, \text{for } 1 \leq k_1 < k_2 \leq n.$$

For $k_1 = 1$ and $k_2 = 2$, let $l_t^2, t = 1,2,3$ be the positive integers such that $|S \cap V_1| = l_1^2, |S \cap U_1| = l_2^2$ and $|S \cap U_2| = l_3^3$. In this case Δ_S contains two disjoint simplexes namely $a-1$-simplex and the induced simplex of $\Delta_{U_1} * \Delta_{U_2}$ and such a subset S will contribute 1 to $\beta_{i,i+1}(G)$. So, the total contributions of S to $\beta_{i,i+1}$ is

$$\sum_{\substack{l_1^2+l_2^2+l_3^2=i+1\\ l_1^2,l_2^2,l_3^2\geq 0}} \binom{|V_1|}{l_1^2}\binom{|U_1|}{l_2^2}\binom{|U_2|}{l_3^2} = \sum_{\substack{l_1^2+l_2^2+l_3^2=i+1\\ l_1^2,l_2^2,l_3^2\geq 0}} \binom{a}{l_1^2}\binom{b}{l_2^2}\binom{b}{l_3^2}.$$

We are done yet, since we considered only one case k_2, the other cases are yet to be considered. There are still $n-2$ possibilities of k_2 (it can be $U_3, U_4, \ldots U_n$). It follows that with $k_1 = 1$ there are $n-1$ choices for k_2. Similarly, for $k_1 = 2$, k_2 can be chosen in $n-2$ ways, for $k_1 = 3$, k_2 can be chosen $n-3$ ways, so on $\ldots$, for $k_1 = n-2$, k_2 can be chosen in 2 ways, lastly for $k_1 = n-1$, we are left with $k_2 = n$. Summing all such possibilities, U_{k_1} and U_{k_2} can be chosen in $(n-1)+(n-2)+\cdots+3+2+1 = \frac{n(n-1)}{2} = \binom{n-1}{2}$ ways. Therefore the net contribution of S to the $\beta_{i,i+1}$ is

$$\binom{n}{2} \sum_{\substack{l_1^2+l_2^2+l_3^2=i+1\\ l_1^2,l_2^2,l_3^2\geq 0}} \binom{a}{l_1^2}\binom{b}{l_2^2}\binom{b}{l_3^2}.$$

From the above calculations, we see that for any subset $S \in \mathcal{S} = \{S \subset V(G) \mid |S|-1 = i,\ S \cap V_1 \neq \emptyset$ and $S \cap \left(U_{k_1} \cup U_{k_2} \cup \cdots \cup U_{k_t}\right) \neq \emptyset\}$, with $t \geq 3$, Δ_S consists of two connected components, since it is disjoint union of $a-1$-simplex and the induced simplex of $\Delta_{U_{k_1}} * \Delta_{U_{k_2}} * \cdots * \Delta_{U_{k_t}}$ for $t \geq 3$. So with $t \geq 3$ and for any $S \in \mathcal{S}$, $\dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}) = \text{comp}(\Delta_S) - 1 = 2 - 1 = 1$. Next, we consider the other cases along with the total number of such subsets.

Case (3). With the similar procedure as above, for $t = 3$, the net contribution of any subset S intersecting non-trivially V_1 and the three mutually disjoint subsets U_α, U_β and U_η, $1 \leq \alpha < \beta < \eta \leq n$, the total contribution of such a subset to $\beta_{i,i+1}$ is

$$\binom{n}{3} \sum_{\substack{l_1^2+l_2^2+l_3^2=i+1\\ l_1^3,l_2^3,l_3^3,l_4^3\geq 0}} \binom{a}{l_1^3}\binom{b}{l_2^3}\binom{b}{l_3^3}\binom{b}{l_4^3},$$

where $l_t^3, t = 1,2,3,4$ are positive integers satisfying $|S \cap V_1| = l_1^3, |S \cap U_\alpha| = l_2^3, |S \cap U_\beta| = l_3^3$ and $|S \cap U_\eta| = l_4^3$.

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Case (n-1). For $t = n-1$, we must choose $n-1$ subsets among n subsets $U_j, j = 1,2,\ldots,n$, which can be chosen in $\binom{n}{n-1}$ ways. Let S be a subset which intersects non-trivially S_1 and the remaining $n-1$ subsets among $U_t, t = 1,2,\ldots,n$ and let l_i^{n-1} be the positive integers such that $|S \cap V_1| = l_1^{n-1}$ and each U_t have l_i^{n-1} elements common with S. The total contributions of such subsets to $\beta_{i,i+1}(G)$ is

$$\binom{n}{n-1} \sum_{\substack{l_1^{n-1}+\cdots+l_n^{n-1}=i+1\\ l_j^{n-1}\geq 1, j=1,2,\ldots,n}} \binom{a}{l_1^{n-1}}\binom{b}{l_2^{n-1}}\binom{b}{l_3^{n-1}}\cdots\binom{b}{l_{n-2}^{n-1}}\binom{b}{l_{n-1}^{n-1}}\binom{b}{l_n^{n-1}}.$$

Case (n). For the last case with $t = n$. Let $S \subseteq V_1 \cup U_1 \cup \cdots \cup U_n$ and let $l_t^n, t = 1,2,\ldots,n$ be the positive integers such that $|S \cap V_1| = l_1^n, |S \cap U_2| = l_2^n, \ldots, |S \cap U_{n-1}| = l_n^n$ and $|S \cap U_n| = l_{n+1}^n$. As Δ_S has two connected components, the total contributions of such subsets to $\beta_{i,i+1}(G)$ is

$$\sum_{\substack{l_1^n+\cdots+l_{n+1}^n=i+1\\ l_j^n\geq 1, j=1,2,\ldots,n,n+1}} \binom{a}{l_1^n}\binom{b}{l_2^n}\binom{b}{l_3^n}\cdots\binom{b}{l_{n-2}^n}\binom{b}{l_{n-1}^n}\binom{b}{l_n^n}\binom{b}{l_{n+1}^n}.$$

Using all these values in Equations (2) and (3), we obtain the result. □

For $n = 1$, the following result gives the Betti numbers of the complete split graph $CS_b^a = \overline{K}_a * K_b$, already found in [21].

Corollary 3.3. *Let CS_b^a be a complete split graph of order $N = a + b$. Then the Betti numbers of CS_b^a are*

$$\beta_{i,i+1}(G) = i\binom{b}{i+1} + \sum_{\substack{l_1^1+l_2^1=i+1\\ l_1^1,l_2^1\geq 1}} l_2^1\binom{a}{l_1^1}\binom{b}{l_2^1}.$$

For $a = 0$, we get the Betti numbers of the complete graph $CS_b^0 \cong K_b$ and for $b = 1$, we get the Betti numbers of star graph $K_{a,1}$ as given below

$$\beta_i(CS_b^0) = i\binom{b}{i+1}, \quad \text{and} \quad \beta_i(CS_0^a) = \binom{a}{i}.$$

The following is an immediate consequence of Theorem 3.2.

Corollary 3.4. *Let G be the multiple complete split-like graph. Then for every $i \geq a + nb$, we have*

$$\beta_{i,i+1}(G) = 0.$$

We will illustrate Theorem 3.2 with the help of the following example.

Example 3.5. *For $a = 3, b = 3$ and $n = 5$ and using Theorem 3.2, the initial Betti numbers of the multiple complete split-like graph $G \cong MCS_{3,5}^3$ are given below:*

$$\begin{aligned}\beta_{i,i+1}(G) &= 5\cdot i\binom{3}{i+1} + 5\sum_{\substack{l_1^1+l_2^1=i+1\\ l_1^1,l_2^1\geq 1}} l_2^1\binom{3}{l_1^1}\binom{3}{l_2^1} + \binom{5}{2}\sum_{\substack{l_1^2+l_2^2+l_3^2=i+1\\ l_1^2,l_2^2,l_3^2\geq 1}}\binom{a}{l_1^2}\binom{b}{l_2^2}\binom{b}{l_3^2}\\ &+\binom{5}{3}\sum_{\substack{l_1^3+\cdots+l_4^3=i+1\\ l_1^3,l_2^3,l_3^3,l_4^3\geq 1}}\binom{3}{l_1^3}\binom{3}{l_2^3}\binom{3}{l_3^3}\binom{3}{l_4^3} + \binom{5}{4}\sum_{\substack{l_1^4+\cdots+l_5^4=i+1\\ l_1^4,l_2^4,l_3^4,l_4^4,l_5^4\geq 1}}\binom{3}{l_1^4}\binom{3}{l_2^4}\binom{3}{l_3^4}\binom{3}{l_4^4}\binom{3}{l_5^4}\\ &+\sum_{\substack{l_1^5+l_2^5+l_3^5+l_4^5+l_5^5+l_6^5=i+1\\ l_1^5,l_2^5,l_3^5,l_4^5,l_5^5,l_6^5\geq 1}}\binom{3}{l_1^5}\binom{3}{l_2^5}\binom{3}{l_3^5}\binom{3}{l_4^5}\binom{3}{l_5^5}\binom{3}{l_6^5}.\end{aligned}$$

Now, substituting particular values of i in the above expression, we have

$$\beta_{1,2}(G) = 5\cdot 1\binom{3}{2} + 5\binom{3}{1}\binom{3}{1} = 15 + 45 = 60$$

$$\begin{aligned}\beta_{2,3}(G) &= 5\cdot 2\binom{3}{3} + 5\sum_{l_1^1+l_2^1=3} l_2^1\binom{3}{l_1^1}\binom{3}{l_2^1} + \binom{5}{2}\sum_{l_1^2+l_2^2+l_3^2=3}\binom{3}{l_1^2}\binom{3}{l_2^2}\binom{3}{l_3^2}\\ &= 10 + 5\left[1\binom{3}{1}\binom{3}{2} + 2\binom{3}{2}\binom{3}{1}\right] + 10\binom{3}{1}\binom{3}{1}\binom{3}{1} = 10 + 135 + 270 = 415\end{aligned}$$

$$\beta_{3,4}(G) = 0 + 5 \sum_{l_1^1+l_2^1=4} l_2^1 \binom{3}{l_1^1}\binom{3}{l_2^1} + \binom{5}{2} \sum_{l_1^2+l_2^2+l_3^2=4} \binom{3}{l_1^3}\binom{3}{l_2^3}\binom{3}{l_3^3}$$
$$+ \binom{5}{3} \sum_{l_1^3+l_2^3+l_3^3+l_4^3=4} \binom{3}{l_1^4}\binom{3}{l_2^4}\binom{3}{l_3^4}\binom{3}{l_4^4} = 5\Big[\binom{3}{1}\binom{3}{3} + 2\binom{3}{2}\binom{3}{2} + 3\binom{3}{3}\binom{3}{1}\Big]$$
$$+ \binom{5}{2}\Big[\binom{3}{1}\binom{3}{1}\binom{3}{2} + \binom{3}{1}\binom{3}{2}\binom{3}{1} + \binom{3}{2}\binom{3}{1}\binom{3}{1}\Big] + \binom{5}{3}\binom{3}{1}\binom{3}{1}\binom{3}{1}\binom{3}{1}$$
$$= 150 + 810 + 810 = 1770$$

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$$\beta_{15,16}(G) = \sum_{l_1^5+l_2^5+l_3^5+l_4^5+l_5^5+l_6^5=16} \binom{3}{l_1^5}\binom{3}{l_2^5}\binom{3}{l_3^5}\binom{3}{l_4^5}\binom{3}{l_5^5}\binom{3}{l_6^5} = \binom{3}{1}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}$$
$$+ \binom{3}{3}\binom{3}{1}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3} + \binom{3}{3}\binom{3}{3}\binom{3}{1}\binom{3}{3}\binom{3}{3}\binom{3}{3} + \binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{1}\binom{3}{3}\binom{3}{3}$$
$$+ \binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{1}\binom{3}{3} + \binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{1} + \binom{3}{2}\binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}$$
$$+ \binom{3}{2}\binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{3} + \binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{3} + \binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{3}$$
$$+ \binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{2} + \binom{3}{3}\binom{3}{2}\binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{3} + \binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{3}$$
$$+ \binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{3} + \binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{2} + \binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{2}\binom{3}{3}\binom{3}{3}$$
$$+ \binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{2}\binom{3}{3} + \binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{2} + \binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{2}\binom{3}{3}$$
$$+ \binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{2} + \binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{2} = 18 + 45 + 36 + 27 + 18 + 9 = 153$$

$$\beta_{16,17}(G) = \sum_{l_1^5+l_2^5+l_3^5+l_4^5+l_5^5+l_6^5=17} \binom{3}{l_1^5}\binom{3}{l_2^5}\binom{3}{l_3^5}\binom{3}{l_4^5}\binom{3}{l_5^5}\binom{3}{l_6^5} = \binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}$$
$$+ \binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3} + \binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{3} + \binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{3}$$
$$+ \binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{3} + \binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{2} = 3 + 3 + 3 + 3 + 3 + 3 = 18$$

$$\beta_{17,18}(G) = \sum_{l_1^5+l_2^5+l_3^5+l_4^5+l_5^5+l_6^5=18} \binom{3}{l_1^5}\binom{3}{l_2^5}\binom{3}{l_3^5}\binom{3}{l_4^5}\binom{3}{l_5^5}\binom{3}{l_6^5} = \binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3} = 1$$

The following tables gives exactly the same Betti numbers (4-th row) of $MCS_{3,5}^3$ using the computer calculations with the help of Macaulay 2 (see [12]).

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
total:	1	60	505	2430	8035	19467	35753	50827	56688	50045	35171	19725	8799	3077	815	153	18	1
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	60	415	1770	5610	13569	25389	37323	43615	40755	30459	18109	8463	3045	815	153	18	1
2:	.	.	90	390	670	570	240	40	.	.	.	.	.	.	.	.	.	.
3:	.	.	.	270	1350	2790	3050	1860	600	80	.	.	.	.	.	.	.	.
4:	.	.	.	.	405	2295	5535	7365	5840	2760	720	80	.	.	.	.	.	.
5:	.	.	.	.	.	243	1539	4239	6633	6450	3992	1536	336	32	.	.	.	.

Figure 1: Betti table of the minimal free resolution of $R/I(MCS_{3,5}^3)$.

The minimal $\mathbb{N}$-graded free resolution of $R/I(MCS^3_{3,5})$ computed with the help of Macaulay 2 [12] is

$$0 \to R[-18]^{1} \to R[-17]^{18} \to R[-16]^{153} \to R[-15]^{815} \to R[-14]^{3077} \to R[-13]^{8799}$$
$$\to R[-12]^{19725} \to R[-11]^{35171} \to R[-10]^{50045} \to R[-9]^{56688} \to R[-8]^{50827} \to R[-7]^{35753}$$
$$\to R[-6]^{19467} \to R[-5]^{8035} \to R[-4]^{2430} \to R[-3]^{505} \to R[-2]^{60} \to R \to R/I(MCS^3_{3,5}) \to 0.$$

Theorem 3.6. *Let $G \cong S^a_{b,n}$ be a clique star graph of order $N \geq 3$ and let $l^j_t, t = 1, 2, \ldots, n+1$ and $j = 1, 2, \ldots, n$ be positive integers. Then the initial Betti numbers of G are*

$$\beta_{i,i+1}(G) = i\binom{a}{i+1} + n \cdot i\binom{b}{i+1} + n \sum_{\substack{l^1_1+l^1_2=i+1\\ l^1_1,l^1_2\geq 1}} (l^1_1 + l^1_2 - 1)\binom{a}{l^1_1}\binom{b}{l^1_2}$$
$$+ \binom{n}{2} \sum_{\substack{l^2_1+l^2_2+l^2_3=i+1\\ l^2_1,l^2_2,l^2_3\geq 1}} l^2_1 \binom{a}{l^2_1}\binom{b}{l^2_2}\binom{b}{l^2_3} + \binom{n}{3} \sum_{\substack{l^3_1+\cdots+l^3_4=i+1\\ l^3_1,l^3_2,l^3_3,l^3_4\geq 1}} l^3_1 \binom{a}{l^3_1}\binom{b}{l^3_2}\binom{b}{l^3_3}\binom{b}{l^3_4}$$
$$+ \binom{n}{4} \sum_{\substack{l^4_1+\cdots+l^4_5=i+1\\ l^4_1,l^4_2,l^4_3,l^4_4,l^4_5\geq 1}} l^4_1 \binom{a}{l^4_1}\binom{b}{l^4_2}\binom{b}{l^4_3}\binom{b}{l^4_4}\binom{b}{l^4_5} +$$
$$\vdots$$
$$+ \binom{n}{n-1} \sum_{\substack{l^{n-1}_1+\cdots+l^{n-1}_n=i+1\\ l^{n-1}_j\geq 1, j=1,2,\ldots,n}} l^{n-1}_1 \binom{a}{l^{n-1}_1}\binom{b}{l^{n-1}_2}\binom{b}{l^{n-1}_3}\cdots\binom{b}{l^{n-1}_{n-2}}\binom{b}{l^{n-1}_{n-1}}\binom{b}{l^{n-1}_n}$$
$$+ \sum_{\substack{l^n_1+\cdots+l^n_{n+1}=i+1\\ l^n_j\geq 1, j=1,2,\ldots,n,n+1}} l^n_1 \binom{a}{l^n_1}\binom{b}{l^n_2}\binom{b}{l^n_3}\cdots\binom{b}{l^n_{n-2}}\binom{b}{l^n_{n-1}}\binom{b}{l^n_n}\binom{b}{l^n_{n+1}}.$$

Proof. Let $G = S^a_{b,n} \cong K_a * (nK_b)$ be the clique star of order $a + nb$, where $a, b, n \geq 1$ are positive integers. Let $\Delta = \Delta(G)$ be the simplicial complex of G. Let V_1 denote the set of vertices of degree $a - 1 + nb$ and let $U_j = V(K_b)$, for $j = 2, 3, \ldots, n+1$. Thus, by Hochster's formula 1 with $j = i + 1$, we have

$$\beta_{i,i+1}(G) = \sum_{\substack{S\subseteq V\\ |S|=i+1}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}), \tag{4}$$

where $V = V(G)$ and $\Delta = \Delta(G)$.

Since V_1 and each of U_j's are cliques, so Expression (4) can be written as

$$\beta_{i,i+1}(G) = \sum_{\substack{S\subseteq V_1\\ |S|=i+1}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}) + \sum_{\substack{S\subseteq U_j\\ |S|=i+1}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}) + \sum_{\substack{S\in \mathcal{S}\\ |S|=i+1}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}), \tag{5}$$

where $\mathcal{S} = \{S \subset V(G) \mid |S| - 1 = i,\ S \cap V_1 \neq \emptyset$ and $S \cap \left(U_{k_1} \cup U_{k_2} \cup \cdots \cup U_{k_t}\right) \neq \emptyset\}$, for $1 \leq t \leq n$ and $k_1 < k_2 < \cdots < k_t$.

For $S \subseteq V_1$ (respectively U_j), then it follows that Δ_S is a disjoint union of $|S|$ simplexes of dimension 0 and any such subset S of $V_1(U_j)$ will have a non zero contribution $|S| - 1$ to $\beta_{i,i+1}(G)$. Along with this information and recalling that there are n copies of U_j, Equation 5 can be reformulated as:

$$\beta_{i,i+1}(G) = i\binom{|V_1|}{i+1} + n \cdot i\binom{|U_j|}{i+1} + \Theta, \tag{6}$$

where $\Theta = \sum_{\substack{S\in\mathcal{S}\\|S|=i+1}} \dim_{\mathbb{K}}\widetilde{H}_0(\Delta_S;\mathbb{K})$. Next, consider V_1 and one of U_j's (say U_1) and let S be a subset intersecting non-trivially both V_1 and U_1. Let l_1^1 and l_2^1 be the positive integers such that $|S\cap V_1| = l_1^1$ and $|S\cap U_1| = l_2^1$. Also Δ_S is a disjoint union of $|S|$ simplexes of dimension 0, so it will contribute $l_1^1+l_2^1-1$ to $\beta_{i,i+1}(G)$. Therefore, taking into account n choices of U_j, $j = 1,2,\ldots,n$, the total contribution of such subsets to $\beta_{i,i+1}(G)$ is given as

$$n\sum_{\substack{l_1^1+l_2^1=i+1\\ l_1^1,l_2^1\geq 1}} (l_1^1+l_2^1-1)\binom{|V_1|}{i+1}\binom{|U_1|}{i+1}.$$

Further for any subset $S \in \mathcal{S} = \{S \subset V(G) \mid |S|-1 = i,\ S\cap V_1 \neq \emptyset \text{ and } S\cap\left(U_{k_1}\cup U_{k_2}\cup\cdots\cup U_{k_t}\right)\neq\emptyset\}$, with $t \geq 2$, since $t = 1$ is done above. Now, for $t \geq 2$, Δ_S consists of a zero dimensional simplexes and the induced simplex of $\Delta_{U_{k_1}} * \Delta_{U_{k_2}} * \cdots * \Delta_{U_{k_t}}$ where $t \geq 2$. Let $l_1^z, z = 2,3,\ldots,n$ be the positive integer such that $|S\cap V_1| = l_1^z$ and S intersects non-trivially each of the $U_{k_1}\cup U_{k_2}\cup\cdots\cup U_{k_t}$, for $t \geq 2$. Then such a subset S will always contribute l_1^z to $\beta_{i,i+1}(G)$, since $\dim_{\mathbb{K}}\widetilde{H}_0(\Delta_S;\mathbb{K}) = \mathrm{comp}(\Delta_S) - 1 = l_1^z$, for $z = 2,3,\ldots,n$. Now, following the cases (2) to (n) of Theorem 3.2, and using them in (6), the required formulae for $\beta_{i,i+1}$ can be established as in the statement. □

The following is an immediate consequence of Theorem 3.2.

Corollary 3.7. *Let $G \cong S_{b,n}^a$ be the clique star graph. Then for every $i \geq a+nb$, we have*

$$\beta_{i,i+1}(G) = 0.$$

We will illustrate Theorem 3.6 by the following example.

Example 3.8. *For $a = 3, b = 3$ and $n = 5$ and using Theorem 3.6, the initial Betti numbers of the clique star graph $G \cong S_{3,5}^3$ are given below:*

$$\begin{aligned}\beta_{i,i+1}(G) &= i\binom{3}{i+1} + 5\cdot i\binom{3}{i+1} + 5\sum_{\substack{l_1^1+l_2^1=i+1\\ l_1^1,l_2^1\geq 1}}(l_1^1+l_2^1-1)\binom{3}{l_1^1}\binom{3}{l_2^1}\\
&+\binom{5}{2}\sum_{\substack{l_1^2+l_2^2+l_3^2=i+1\\ l_1^2,l_2^2,l_3^2\geq 1}} l_1^2\binom{3}{l_1^2}\binom{3}{l_2^2}\binom{3}{l_3^2} + \binom{5}{3}\sum_{\substack{l_1^3+\cdots+l_4^3=i+1\\ l_1^3,l_2^3,l_3^3,l_4^3\geq 1}} l_1^3\binom{3}{l_1^3}\binom{3}{l_2^3}\binom{3}{l_3^3}\binom{3}{l_4^3}\\
&+\binom{5}{4}\sum_{\substack{l_1^4+l_2^4+l_3^4+l_4^4+l_5^4=i+1\\ l_1^4,l_2^4,l_3^4,l_4^4,l_5^4\geq 1}} l_1^4\binom{3}{l_1^4}\binom{3}{l_2^4}\binom{3}{l_3^4}\binom{3}{l_4^4}\binom{3}{l_5^4}\\
&+\sum_{\substack{l_1^5+l_2^5+l_3^5+l_4^5+l_5^5+l_6^5=i+1\\ l_j^5\geq 1, j=1,2,\ldots,5,6}} l_1^n\binom{3}{l_1^n}\binom{3}{l_2^n}\binom{3}{l_3^n}\binom{3}{l_{n-2}^n}\binom{3}{l_{n-1}^n}\binom{3}{l_n^n}.\end{aligned}$$

Now, substituting particular values of i in the above expression, we have

$$\beta_{1,2}(G) = \binom{3}{2} + 5\binom{3}{2} + 5\binom{3}{1}\binom{3}{1} = 3+15+45 = 60$$

$$\beta_{2,3}(G) = 2\binom{3}{3} + 5\cdot 2\binom{3}{3} + 5\sum_{l_1^1+l_2^1=3}(l_1^1+l_2^1-1)\binom{3}{l_1^1}\binom{3}{l_2^1} + \binom{5}{2}\sum_{l_1^2+l_2^2+l_3^2=3} l_1^2\binom{3}{l_1^2}\binom{3}{l_2^2}\binom{3}{l_3^2}$$

$$= 2 + 10 + 5\Big[2\binom{3}{1}\binom{3}{2} + 2\binom{3}{2}\binom{3}{1}\Big] + 10\binom{3}{1}\binom{3}{1}\binom{3}{1} = 2 + 10 + 180 + 270 = 462$$

$$\beta_{3,4}(G) = 0 + 5\sum_{l_1^1+l_2^1=4}(l_1^1 + l_2^1 - 1)\binom{3}{l_1^1}\binom{3}{l_2^1} + \binom{5}{2}\sum_{l_1^2+l_2^2+l_3^2=4} l_1^2\binom{3}{l_1^3}\binom{3}{l_2^3}\binom{3}{l_3^3}$$
$$+ \binom{5}{3}\sum_{l_1^3+l_2^3+l_3^3+l_4^3=4} l_1^3\binom{3}{l_1^4}\binom{3}{l_2^4}\binom{3}{l_3^4}\binom{3}{l_4^4} = 5\Big[3\binom{3}{1}\binom{3}{3} + 3\binom{3}{2}\binom{3}{2} + 3\binom{3}{3}\binom{3}{1}\Big]$$
$$+ 10\Big[\binom{3}{1}\binom{3}{1}\binom{3}{2} + \binom{3}{1}\binom{3}{2}\binom{3}{1} + 2\binom{3}{2}\binom{3}{1}\binom{3}{1}\Big] + 10\binom{3}{1}\binom{3}{1}\binom{3}{1}\binom{3}{1}$$
$$= 225 + 1080 + 810 = 2115$$

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$$\beta_{15,16}(G) = \sum_{l_1^5+l_2^5+l_3^5+l_4^5+l_5^5+l_6^5=16} l_1^5\binom{3}{l_1^5}\binom{3}{l_2^5}\binom{3}{l_3^5}\binom{3}{l_4^5}\binom{3}{l_5^5}\binom{3}{l_6^5} = 1\binom{3}{1}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}$$
$$+ 3\binom{3}{3}\binom{3}{1}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3} + 3\binom{3}{3}\binom{3}{3}\binom{3}{1}\binom{3}{3}\binom{3}{3}\binom{3}{3} + 3\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{1}\binom{3}{3}\binom{3}{3}$$
$$+ 3\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{1}\binom{3}{3} + 3\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{1} + 2\binom{3}{2}\binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}$$
$$+ 2\binom{3}{2}\binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{3} + 2\binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{3} + 2\binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{3}$$
$$+ 2\binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{2} + 3\binom{3}{3}\binom{3}{2}\binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{3} + 3\binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{3}$$
$$+ 3\binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{3} + 3\binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{2} + 3\binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{2}\binom{3}{3}\binom{3}{3}$$
$$+ 3\binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{2}\binom{3}{3} + 3\binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{2} + 3\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{2}\binom{3}{3}$$
$$+ 3\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{2} + 3\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{2}$$
$$= 3 + 5\cdot 9 + 2\cdot 5\cdot 9 + 3\cdot 10\cdot 9 = 408$$

$$\beta_{16,17}(G) = \sum_{l_1^5+l_2^5+l_3^5+l_4^5+l_5^5+l_6^5=17} l_1^5\binom{3}{l_1^5}\binom{3}{l_2^5}\binom{3}{l_3^5}\binom{3}{l_4^5}\binom{3}{l_5^5}\binom{3}{l_6^5} = 2\binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}$$
$$+ 3\binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3} + 3\binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{3}\binom{3}{3} + 3\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{3}\binom{3}{3}$$
$$+ 3\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{2}\binom{3}{3} + 3\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{2} = 6 + 9 + 9 + 9 + 9 + 9 = 51$$

$$\beta_{17,18}(G) = \sum_{l_1^5+l_2^5+l_3^5+l_4^5+l_5^5+l_6^5=18} l_1^5\binom{3}{l_1^5}\binom{3}{l_2^5}\binom{3}{l_3^5}\binom{3}{l_4^5}\binom{3}{l_5^5}\binom{3}{l_6^5} = 3\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3}\binom{3}{3} = 3$$

The following tables (Figure 2) gives exactly the same Betti numbers (4-th row) of $MCS_{3,5}^3$ using the computer calculations with the help of Macaulay 2 (see [12]). The minimal $\mathbb{N}$-graded free resolution of $R/I(S_{3,5}^3)$ computed with the help of Macaulay 2 [12] is

$$0 \to R[-18]^3 \to R[-17]^{51} \to R[-16]^{408} \to R[-15]^{2040} \to R[-14]^{7172} \to R[-13]^{18900}$$
$$\to R[-12]^{38744} \to R[-11]^{63056} \to R[-10]^{82220} \to R[-9]^{86003} \to R[-8]^{71848} \to R[-7]^{47492}$$
$$\to R[-6]^{24472} \to R[-5]^{9610} \to R[-4]^{2775} \to R[-3]^{552} \to R[-2]^{63} \to R \to R/I(S_{3,5}^3) \to 0.$$

Theorem 3.6 can be generalized to a more result in the following theorem. Let $\mathbb{S}_{b,c,n}^a$ be the clique star of order $N = a + b + cn$, such that a vertices of K_a are connected to every vertices of K_b and K_c (which are n copies in $\mathbb{S}_{b,c,n}^a$).

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
total:	1	63	552	2775	9610	24472	47492	71848	86003	82220	63056	38744	18900	7172	2040	408	51	3
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	
1:	.	63	462	2115	7185	18574	37128	58344	72930	72930	58344	37128	18564	7140	2040	408	51	3
2:	.	.	90	390	670	570	240	40	.	.	.	.	.	.	.	.	.	.
3:	.	.	.	270	1350	2790	3050	1860	600	80	.	.	.	.	.	.	.	.
4:	.	.	.	.	405	2295	5535	7365	5840	2760	720	80	.	.	.	.	.	.
5:	.	.	.	.	.	243	1539	4239	6633	6450	3992	1536	336	32	.	.	.	.

Figure 2: Betti table of the minimal free resolution of $R/I(S^3_{3,5})$.

Theorem 3.9. *Let $G \cong \mathbb{S}^a_{b,c,n}$ be a graph of order $N = a+b+cn \geq 3$ and let r^g_k, with $k = 1, 2, \ldots, n+2$, $g = 1, 2, \ldots, n+1$ and l^j_t, with $t = 1, 2, \ldots, n+1$, $j = 1, 2, \ldots, n$ be positive integers. Then the initial Betti numbers of G are*

$$\begin{aligned}
\beta_{i,i+1}(G) = {} & i\binom{a}{i+1} + i\binom{b}{i+1} + n\cdot i\binom{c}{i+1} + \sum_{\substack{r^1_1+r^1_2=i+1\\ r^1_1,r^1_2\geq 1}} i\binom{a}{r^1_1}\binom{b}{r^1_2} + n\sum_{\substack{l^1_1+l^1_2=i+1\\ l^1_1,l^1_2\geq 1}} i\binom{a}{l^1_1}\binom{c}{l^1_2}\\
& + n\sum_{\substack{r^2_1+r^2_2+r^2_3=i+1\\ r^2_1,r^2_2,r^2_3\geq 1}} r^2_1\binom{a}{r^2_1}\binom{b}{r^2_2}\binom{c}{r^2_3} + \binom{n}{2}\sum_{\substack{l^2_1+l^2_2+l^2_3=i+1\\ l^2_1,l^2_2,l^2_3\geq 1}} l^2_1\binom{a}{l^2_1}\binom{c}{l^2_2}\binom{c}{l^2_3}\\
& + \binom{n}{2}\sum_{\substack{r^3_1+r^3_2+r^3_3=i+1\\ r^3_1,r^3_2,r^3_3\geq 1}} r^3_1\binom{a}{r^3_1}\binom{b}{r^3_2}\binom{c}{r^3_3}\binom{c}{r^3_4} + \binom{n}{3}\sum_{\substack{l^3_1+\cdots+l^3_4=i+1\\ l^3_1,l^3_2,l^3_3,l^3_4\geq 1}} l^3_1\binom{a}{l^3_1}\binom{c}{l^3_2}\binom{c}{l^3_3}\binom{c}{l^3_4}\\
& +\\
& \vdots\\
& +\\
& + \binom{n}{n-2}\sum_{\substack{r^{n-1}_1+\cdots+r^{n-1}_n=i+1\\ r^{n-1}_j\geq 1, j=1,2,\ldots,n}} r^{n-1}_1\binom{a}{r^{n-1}_1}\binom{b}{r^{n-1}_2}\binom{c}{r^{n-1}_3}\cdots\binom{c}{r^{n-1}_n}\\
& + \binom{n}{n-1}\sum_{\substack{l^{n-1}_1+\cdots+l^{n-1}_n=i+1\\ l^{n-1}_j\geq 1, j=1,2,\ldots,n}} l^{n-1}_1\binom{a}{l^{n-1}_1}\binom{c}{l^{n-1}_2}\cdots\binom{c}{l^{n-1}_n}\\
& + \binom{n}{n-1}\sum_{\substack{r^n_1+\cdots+r^n_{n+1}=i+1\\ r^n_j\geq 1, j=1,2,\ldots,n+1}} r^n_1\binom{a}{r^n_1}\binom{b}{r^n_2}\binom{c}{r^n_3}\cdots\binom{c}{r^n_{n+1}}\\
& + \sum_{\substack{l^n_1+\cdots+l^n_{n+1}=i+1\\ l^n_j\geq 1, j=1,2,\ldots,n,n+1}} l^n_1\binom{a}{l^n_1}\binom{c}{l^n_2}\binom{c}{l^n_3}\cdots\binom{c}{l^n_{n+1}}\\
& + \sum_{\substack{r^{n+1}_1+\cdots+r^{n+1}_{n+2}=i+1\\ r^{n+1}_j\geq 1, j=1,2,\ldots,n+1,n+2}} r^{n+1}_1\binom{a}{r^{n+1}_1}\binom{b}{r^{n+1}_2}\binom{c}{r^{n+1}_3}\cdots\binom{c}{r^{n+1}_{n+2}}.
\end{aligned}$$

Next, we find the extremal Betti numbers and the projective dimension of the graphs considered in Theorems 3.2, 3.6 and 3.9.

Theorem 3.10. *The following hold for the extremal Betti numbers of a graph of order N.*

(i) *The extremal Betti number of* $MSC^a_{b,n}$ *is* $\beta_{N-1,N} = 1$

(ii) *The extremal Betti number of* $S^a_{b,n}$ *is* $\beta_{N-1,N} = a$

(iii) *The extremal Betti number of* $\mathbb{S}^a_{b,c,n}$ *is* $\beta_{N-1,N} = a$

Proof. We prove the more general case (iii), other can be similarly proved.
Let $G \cong \mathbb{S}^a_{b,c,n}$ be a graph of order N. Since $\mathrm{reg}(I(G)) = \mathrm{reg}(G) + 1$, where $\mathrm{reg}(G) = \mathrm{reg}(R/I(G))$. So, it implies that a Betti number $\beta_{i,j}$ of the edge ring of G is the extremal Betti number if $j = i+1$ and $i = \max\{r|\beta_{r,r+1} = 0\}$. Let $\Delta = \Delta(G)$, by Hochster's formula, we have

$$\beta_{i,i+1}(G) = \sum_{\substack{S\subseteq V\\ |S|=i+1}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}).$$

It is well known that $\dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K})$ is one less than the number of connected components of Δ. So, $\beta_{i,i+1}(G)$ is non-zero if the number of connected components of S is greater than one. Let $S = V(G)$. Then $S = \Delta$. Recall that $V' = V(K_a)$ form a clique in G and each vertex of V' is connected to every other vertex of G. Also the induced subcomplex $\Delta_{V'}$ of Δ consists of $|V'|$ zero-dimensional facets. Hence, Δ has exactly a facets of dimension 0. Further $\Delta_{V(G)\setminus V'}$ is an induced subcomplex of Δ. Thus, the $\mathrm{comp}(V(G) \setminus V') = 1$. Therefore, the number of connected components of Δ is exactly $a + 1$ and we have

$$\beta_{N-1,N}(G) = a.$$

□

From the above theorem and the fact that $\mathrm{pd}(G) \geq |V(H)| - 1$, where H is induced sugbraph of G and its complement is disconnected (see [15]). We have the following consequence for the projective dimension of the graphs considered in Theorems 3.2, 3.6 and 3.9 and the proof is immediate form Theorem 3.10.

Corollary 3.11. *If G is any of the graphs* $MSC^a_{b,n}$, $S^a_{b,n}$ *and* $\mathbb{S}^a_{b,c,n}$ *of order N, then* $pd(G) = N - 1$.

4. Betti numbers of some power graphs of non-abelian groups

Kelarev and Quinn [17] introduced the directed power graph of a semigroup S' as a directed graph with vertex set S', where two vertices $x, y \in S'$ are joined by an arc from x to y if and only if $x \neq y$ and $y^i = x$ for some positive integer i. Let $\mathcal{G}$ be a finite group of order N and identity represented by e. Chakrabarty et al. [5] defined the undirected power graph $\mathcal{P}(\mathcal{G})$ of a group $\mathcal{G}$ as an undirected graph with vertex set as $\mathcal{G}$, where two vertices $x, y \in \mathcal{G}$ are adjacent if and only if $x^i = y$ or $y^j = x$, for $2 \leq i, j \leq N$.

The dihedral group of order $N = 2n, n \geq 2$ is denoted by D_{2n} and is represented as follows

$$D_{2n} = \langle a, b \mid a^n = b^2 = e, bab = a^{-1}\rangle.$$

Since $\langle a\rangle$ generates a cyclic subgroup $\{e, a, a^2, \ldots, a^{n-1}\}$ of order n and is isomorphic to $\mathbb{Z}_n$. Also D_{2n} has n elements of order two and they represent K_2's as induced subgraphs in $\mathcal{P}(D_{2n})$. These n elements of order 2 form an independent set of $\mathcal{P}(D_{2n})$ sharing the identity element e. Therefore, the structure of the power graph of the dihedral group D_{2n} can be obtained from the power graph $\mathcal{P}(\mathbb{Z}_n)$ by adding the n pendent vertices at the identity vertex e. If $n = p^z$, where z is a positive integer, then it well know that $\mathcal{P}(\mathbb{Z}_n)$ is the complete graph (see, [5]). Therefore, in this case, the power graph of D_{2n} is

$$\mathcal{P}(D_{2n}) = K_{n-1} * K_1 * \overline{K}_n, \tag{7}$$

since the identity share n pendent vertices (the elements of order two in $\mathcal{P}(D_{2n})$). The graph given in Equation 7 is known as the pineapple graph (a graph obtained from the clique by adding pendent vertices at any vertex of the clique). For $n = 2^2$, the structure of the power graph of D_8 is shown in Figure 4.

Our next result gives the Betti numbers of the pineapple graph and as a consequence, we obtain the Betti numbers of $\mathcal{P}(D_{2n})$.

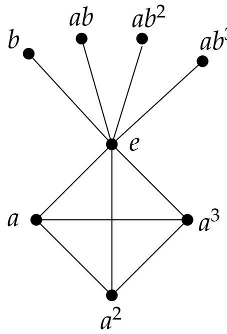

Figure 3: Power graph $\mathcal{P}(D_8)$ of D_8.

Theorem 4.1. *Let $G \cong P_a^b$ be a pineapple graph with clique size a and independent set of size b. Then the initial Betti numbers of G are*

$$\beta_{i,i+1}(G) = i\binom{a}{i+1} + \binom{b}{i} + \sum_{\substack{r_1+r_2=i\\ r_1,r_2\geq 1}} \binom{a-1}{r_1}\binom{b}{r_2},$$

where r_1 and r_2 are positive integers.

Proof. Let G denote the pineapple graph of order $a+b$. Let $\Delta = \Delta(G)$ be the simplicial complex of G. Let $V = V_1 \cup V_2 \cup V_3$ be the vertex set of G, where V_1 consists of the vertices of degree $a-1$, V_2 denote the vertex of $a+b$ and V_2 denote the pendent vertices of G. By Hochster's formula (1), we have

$$\beta_{i,i+1}(G) = \sum_{\substack{S\subseteq V\\ |S|=i+1}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S;\mathbb{K}),$$

where $V = V(G)$ and $\Delta = \Delta(G)$.

As $V_1 \cup V_2$ is a clique and V_3 is an independent subset of G, so from above, we have

$$\beta_{i,i+1}(G) = \sum_{\substack{S\subseteq V_1\cup V_2\\ |S|=i+1}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S;\mathbb{K}) + \sum_{\substack{S\subseteq V_3\\ |S|=i+1}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S;\mathbb{K}) + \sum_{\substack{S\subseteq V_2\cup V_3\\ |S|=i+1}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S;\mathbb{K}) + \Theta, \tag{8}$$

where $\Theta = \sum_{\substack{S\subseteq V\\ |S|=i+1}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S;\mathbb{K})$, and such a subset S satisfies $S \subseteq V \mid |S| = i+1,\ S\cap V_1 \neq \emptyset,\ S \cap V_2 \neq \emptyset$ and $S \cap V_3 \neq \emptyset$.

For $S \subseteq V_1 \cup V_2$, we see that Δ_S consists of simplexes of dimension zero and comp(Δ_S) is same as the size of S. Therefore, for $|S| = i+1$, S will contribute i to $\beta_{i,i+1}(G)$. The total number of choices S intersects $V_1 \cup V_2$ are $\binom{|V_1\cup V_2|}{i+1}$. Also, for $S \subseteq V_3$, Δ_S is a full subcomplex of Δ. So, Δ_S is homotopic to a point and hence $\dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S;\mathbb{K}) = 0$ and contributes zero to $\beta_{i,i+1}(G)$. Again for $S \subseteq V_2 \cup V_3$, then Δ_S is a disjoint union of two simplexes, a point simplex and a full simplex and $\dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S;\mathbb{K}) = 1$, since comp$(\Delta_S) = 2$. Besides choosing one element from V_2 and i elements from V_3, the total number of choices S intersects non-trivially V_2 and V_3 are $\binom{b}{i}$. Lastly, let $S \subset V_1 \cup V_2 \cup V_3$ and let $r_i, i = 1,2,3$ be the positive integers such that $|S \cap V_1| = r_1, |S \cap V_2| = r_2$ and $|S \cap V_3| = r_3$. In this case Δ_S contains two disjoint simplexes namely a point simplex of dimension zero and an induced simplex of $\Delta_{V_1} * \Delta_{V_2}$ and such a subset S will contribute 1 to $\beta_{i,i+1}(G)$. Thus the total contributions of S to $\beta_{i,i+1}$ is

$$\sum_{\substack{r_1+r_2=i\\ r_1,r_2\geq 0}} \binom{a-1}{r_1}\binom{b}{l_2}.$$

With these values in (8), we obtain the required formula. □

The following result is the consequence of above result and gives the Betti numbers of $\mathcal{P}(D_{2p^z}), z \geq 2$.

Corollary 4.2. *The Betti numbers of $\mathcal{P}(D_{2n})$, for $n = p^z, z \geq 2$ are*

$$\beta_{i,i+1}(\mathcal{P}(D_{2n})) = i\binom{n}{i+1} + \binom{n}{i} + \sum_{\substack{r_1+r_2=i \\ r_1,r_2\geq 1}} \binom{n-1}{r_1}\binom{n}{r_2}.$$

The following example illustrates Theorem 4.1 and Corollarly 4.2 for the power graph of D_8.

Example 4.3. *For $a = b = 4$ in Theorem 3.6 (or $n = 4$ in Corollarly 4.2), we have*

$$\beta_{i,i+1}(\mathcal{P}(D_8)) = i\binom{4}{i+1} + \binom{4}{i} + \sum_{\substack{r_1+r_2=i \\ r_1,r_2\geq 1}} \binom{4-1}{r_1}\binom{4}{r_2}.$$

Now, substituting particular values of i in the above expression, we have

$$\beta_{1,2}(\mathcal{P}(D_8)) = \binom{4}{2} + \binom{4}{1} = 6 + 4 = 10$$
$$\beta_{2,3}(\mathcal{P}(D_8)) = 2\binom{4}{3} + \binom{4}{2} + \binom{3}{1}\binom{4}{1} = 8 + 6 + 12 = 26$$
$$\beta_{3,4}(\mathcal{P}(D_8)) = 3\binom{4}{4} + \binom{4}{3} + \binom{3}{1}\binom{4}{2} + \binom{3}{2}\binom{4}{1} = 3 + 4 + 18 + 12 = 37$$
$$\beta_{4,5}(\mathcal{P}(D_8)) = \binom{4}{4} + \binom{3}{1}\binom{4}{3} + \binom{3}{2}\binom{4}{2} + \binom{3}{3}\binom{4}{1} = 1 + 12 + 18 + 4 = 35$$
$$\beta_{5,6}(\mathcal{P}(D_8)) = \binom{3}{1}\binom{4}{4} + \binom{3}{2}\binom{4}{3} + \binom{3}{3}\binom{4}{2} = 3 + 12 + 6 = 21$$
$$\beta_{6,7}(\mathcal{P}(D_8)) = \binom{3}{3}\binom{4}{3} + \binom{3}{2}\binom{4}{4} = 4 + 3 = 7$$
$$\beta_{7,8}(\mathcal{P}(D_8)) = \binom{3}{3}\binom{4}{4} = 1.$$

Table 4 gives exactly the same Betti numbers of $\mathcal{P}(D_8)$ using the computer calculations with the help of Macaulay 2 (see [12]).

```
0  1  2  3  4  5 6 7
total: 1 10 26 37 35 21 7 1
0: 1  .  .  .  .  . . .
1: . 10 26 37 35 21 7 1
```

Figure 4: Betti table of the minimal free resolution of $\mathcal{P}(D_8)$.

The minimal $\mathbb{N}$-graded free resolution of $R/I(D_8)$ computed with the help of Macaulay 2 [12] is

$$0 \to R[-8]^1 \to R[-7]^7 \to R[-6]^{21} \to R[-5]^{35} \to R[-4]^{37} \to R[-3]^{26} \to R[-2]^{10} \to R \to R/I(\mathcal{P}(D_8)) \to 0.$$

Since the regularity of edge ideals is at least 2. The classification of graphs with regularity 2 is referred to as Fröberg's characterization. The following result due to Fröberg characterizes all graphs whose edge ideals have regularity 2.

Lemma 4.4 ([11]). *Let G be a finite simple graph. Then* $\operatorname{reg}(I(G)) = 2$ *if and only if G is a co-chordal graph.*

In the next result we obtain regularity, extremal Betti number and projective dimension of P_a^b.

Theorem 4.5. *Let* $G \cong P_a^b$ *be the pineapple graph of order* $N = a + b$. *The the follwing hold.*

- **(i)** *The regularity of G is* 2.
- **(ii)** *The extremel Betti number of G is* 1.
- **(iii)** *The projective dimension of G is N.*

Proof. (i) Clearly $\overline{G} \cong K_1 \cup K_b * \overline{K}_{a-1}$ and It is trivial to see that $\overline{G}$ has no induced cycle of length strictly greater than 3. So G is co-chordal and by Lemma 4.4, result follows.
(ii) Also $\operatorname{reg}(I(G)) = \operatorname{reg}(G) + 1$, so, it folows that a Betti number $\beta_{i,j}$ of the edge ring of G is the extremal Betti number if $j = i + 1$ and $i = \max\{r|\beta_{r,r+1} = 0\}$. For $S \subseteq V(G)$ and with $\Delta = \Delta(G)$, it is well known that $\dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K})$ is one less than the number of connected components of Δ. So, $\beta_{i,i+1}(G)$ is non-zero if the number of connected components of S is greater than one. As there is only one dominating vertex (connected to all other vertices) in G, it gives that Δ has one facet of dimension 0 and remaining vertices form an induced subcomplex of Δ. Therefore, the number of connected components of Δ is exactly 2 and we have

$$\beta_{N-1,N}(G) = 1.$$

(iii) follows from (ii). □

The dicyclic groups of order $4n$ are denoted by Q_n and is presented as follows

$$Q_n = \langle a, b \mid a^{2n} = e, b^2 = a^n, ab = ba^{-1} \rangle.$$

If n is a power of 2, then Q_n is called the *generalized quaternion group* of order $4n$. It is clear that $(a^i b)^2 = a^n$ for all $0 \le i \le 2n - 1$, and

$$\langle a^i b \rangle = \langle a^{n+i} b \rangle = \{e, a^i b, a^n, a^{n+i} b\} \text{ for all } 0 \le i \le n - 1. \tag{9}$$

Beside each element of $Q_n - \langle a \rangle$ is of the form $a^i b$ for some $0 \le i \le 2n - 1$. Further, it follows that $\langle a \rangle$ is a cyclic group order $2n$ and its power graph is isomorphic to $\mathcal{P}(\mathbb{Z}_{2n})$. Now, for $n = 2^z, z \ge 2$ it follows that $\mathcal{P}(\mathbb{Z}_{2n}) \cong K_{2n}$, since n is prime power and power graph of prime order is complete (see, [5]) . Again $\mathcal{P}(\mathbb{Z}_{2n}) \cong K_{2n} = K_{2n-2} * K_2$, where $V(K_2) = \{e, a^n\}$. Also, representation given in 9 implies that a^n is adjacent to $a^i b$ for every $0 \le i \le 2n$. Thus, we see that $\{e\}$ and $\{a^n\}$ are adjacent to every other element of Q_{2n} in $\mathcal{P}(Q_n)$. From (9), and $Q_n - \langle a \rangle$, each of the elements $a^i b$ form the cycles C_4's, for some $0 \le i \le 2n - 1$. From this calculation, it follows that the elements $\{e, a^n\}$ of $\mathcal{P}(Q_n)$ are adjacent to every such $a^i b$, for some $0 \le i \le 2n - 1$. Therefore, the power graph of $Q_n, n = 2^z, z \ge 2$ can be written as

$$\mathcal{P}(Q_n) \cong K_{2n-2} * K_2 * nK_2.$$

For $n = 2^2$, the power graph of Q_{16} is shown in Figure 5.

As an application of Theorem 3.9 with $a = 2, b = 2n - 2$ and $c = 2$, we have the following result for the Betti numbers of $\mathcal{P}(Q_n)$.

Theorem 4.6. *Let* $G \cong \mathcal{P}(Q_n)$ *be the power graph of the generalized quaternion group of order* $N = 4n$ *where* $n = 2^z, z \ge 2$ *is a positive integer. Then the initial Betti numbers of G are*

$$\beta_{i,i+1}(G) = i\binom{2}{i+1} + i\binom{2n-2}{i+1} + n \cdot i\binom{2}{i+1} + \sum_{\substack{r_1^1 + r_2^1 = i+1 \\ r_1^1, r_2^1 \ge 1}} i\binom{2}{r_1^1}\binom{2n-2}{r_2^1} + n \sum_{\substack{l_1^1 + l_2^1 = i+1 \\ l_1^1, l_2^1 \ge 1}} i\binom{2}{l_1^1}\binom{2}{l_2^1}$$

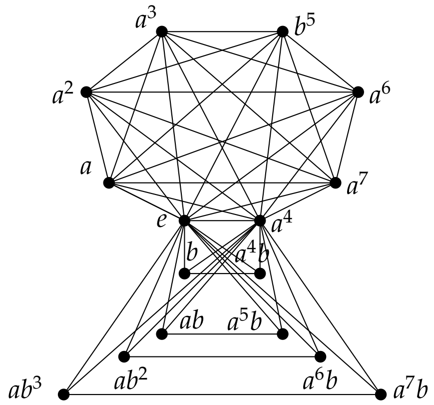

Figure 5: Power graph $\mathcal{P}(Q_{16})$ of Q_{16}.

$$
\begin{aligned}
&+ n \sum_{\substack{r_1^2+r_2^2+r_3^2=i+1\\ r_1^2,r_2^2,r_3^2\geq 1}} r_1^2\binom{2}{r_1^2}\binom{2n-2}{r_2^2}\binom{2}{r_3^2} + \binom{n}{2} \sum_{\substack{l_1^2+l_2^2+l_3^2=i+1\\ l_1^2,l_2^2,l_3^2\geq 1}} l_1^2\binom{2}{l_1^2}\binom{2}{l_2^2}\binom{2}{l_3^2}\\
&+ \binom{n}{2} \sum_{\substack{r_1^3+r_2^3+r_3^3=i+1\\ r_1^3,r_2^3,r_3^3\geq 1}} r_1^3\binom{2}{r_1^3}\binom{2n-2}{r_2^3}\binom{2}{r_3^3}\binom{2}{r_4^3} + \binom{n}{3} \sum_{\substack{l_1^3+\cdots+l_4^3=i+1\\ l_1^3,l_2^3,l_3^3,l_4^3\geq 1}} l_1^3\binom{2}{l_1^3}\binom{2}{l_2^3}\binom{2}{l_3^3}\binom{2}{l_4^3}\\
&+ \cdot \quad \cdot \quad \cdot + \binom{n}{n-1} \sum_{\substack{l_1^{n-1}+\cdots+l_n^{n-1}=i+1\\ l_j^{n-1}\geq 1, j=1,2,\ldots,n}} l_1^{n-1}\binom{2}{l_1^{n-1}}\binom{2}{l_2^{n-1}}\cdots\binom{2}{l_n^{n-1}}\\
&+ \binom{n}{n-1} \sum_{\substack{r_1^n+\cdots+r_{n+1}^n=i+1\\ r_j^n\geq 1, j=1,2,\ldots,n+1}} r_1^n\binom{2}{r_1^n}\binom{2n-2}{r_2^n}\binom{2}{r_3^n}\cdots\binom{2}{r_{n+1}^n}\\
&+ \sum_{\substack{l_1^n+\cdots+l_{n+1}^n=i+1\\ l_j^n\geq 1, j=1,2,\ldots,n,n+1}} l_1^n\binom{2}{l_1^n}\binom{2}{l_2^n}\binom{2}{l_3^n}\cdots\binom{2}{l_{n+1}^n}\\
&+ \sum_{\substack{r_1^{n+1}+\cdots+r_{n+2}^{n+1}=i+1\\ r_j^{n+1}\geq 1, j=1,2,\ldots,n+1,n+2}} r_1^{n+1}\binom{2}{r_1^{n+1}}\binom{2n-2}{r_2^{n+1}}\binom{2}{r_3^{n+1}}\cdots\binom{2}{r_{n+2}^{n+1}}.
\end{aligned}
$$

Now, we find the Betti numbers of cyclic and non-cyclic groups when order is product of two primes. Suppose $\mathcal{G}$ is cyclic group of order $pq, (p < q)$ are primes, then $\mathcal{G}$ has $\phi(n)$ elements elements, which form a clique and each such vertex is of full degree, since they generate all elements. Also, the identity is always adjacent to every other vertex, so it gives us an induced subgraph $K_{\phi(pq+1)}$. Clearly, $\mathcal{G}$ has a unique p-Sylow subgroup and a unique q-Sylow subgroup, and their induced subgraphs are K_p and K_q, respectively. Similarly, if $\mathcal{G}$ is not cyclic, then no elements generates all other elements, so the identity element is the only element adjacent to all other elements of $\mathcal{G}$. In this case, $\mathcal{G}$ has q number of p-Sylow subgroups and a unique q-Sylow subgroup. The above observations are made precise in the following result.

Lemma 4.7 ([7]). *Let $\mathcal{G}$ be a finite group of order pq, where $p < q$, p and q are two distinct primes, and ϕ is the Euler function. Then*

(i) *$\mathcal{G}$ is cyclic if and only if $\mathcal{P}(\mathcal{G}) \cong K_{p-1} * K_{\phi(pq)+1} * K_{q-1}$.*

(ii) *G is non cyclic if and only if $\mathcal{P}(G) \cong qK_{p-1} * K_1 * K_{q-1}$.*

Theorem 4.8. *Let G denote the power graph of a finite group of order pq, where $p < q$, p and q are two distinct primes. Then the following hold.*

(i) *If $\mathcal{G}$ is cyclic, then we have*

$$\beta_{i,i+1}(G) = i\binom{\phi(n)+1}{i+1} + i\binom{p-1}{i+1} + i\binom{q-1}{i} + \sum_{\substack{r_1^1+r_2^1=i+1\\ r_1^1,r_2^1\geq 1}} i\binom{\phi(n)+1}{r_1^1}\binom{p-1}{r_2^1}$$
$$+ \sum_{\substack{l_1^1+l_2^1=i+1\\ l_1^1,l_2^1\geq 1}} i\binom{\phi(n)+1}{l_1^1}\binom{q-1}{l_2^1} + \sum_{\substack{r_1^2+r_2^2+r_3^2=i+1\\ r_1^2,r_2^2,r_3^2\geq 1}} r_1^2\binom{\phi(n)+1}{r_1^2}\binom{p-1}{r_2^2}\binom{q-1}{r_3^2}.$$

(ii) *If $\mathcal{G}$ is non cyclic, then we have*

$$\beta_{i,i+1}(G) = i\binom{q-1}{i+1} + q\cdot i\binom{p-1}{i+1} + i\binom{q-1}{i} + q\cdot i\binom{p-1}{i} + q\sum_{\substack{r_1^1+r_2^1=i\\ r_1^1,r_2^1\geq 1}} \binom{q-1}{r_1^1}\binom{p-1}{r_2^1}$$
$$+ \binom{q}{2}\sum_{\substack{l_1^1+l_2^1=i\\ l_1^1,l_2^1\geq 1}}\binom{p-1}{l_1^1}\binom{p-1}{l_2^1} + \binom{q}{2}\sum_{\substack{r_1^2+r_2^2+r_3^2=i\\ r_1^2,r_2^2,r_3^2\geq 1}}\binom{q-1}{r_1^2}\binom{p-1}{r_2^2}\binom{p-1}{r_3^2}$$
$$+ \binom{q}{3}\sum_{\substack{l_1^2+l_2^2+l_3^2=i\\ l_1^2,l_2^2,l_3^2\geq 1}}\binom{p-1}{l_1^2}\binom{p-1}{l_2^2}\binom{p-1}{l_3^2} + \cdot\ \ \cdot\ \ \cdot$$
$$+ \binom{q}{q-1}\sum_{\substack{l_1^{q-2}+\cdots+l_{q-1}^{q-2}=i\\ l_j^{q-2}\geq 1, j=1,2,\ldots,q-1}}\binom{p-1}{l_1^{q-2}}\binom{p-1}{l_2^{q-2}}\cdots\binom{p-1}{l_{q-1}^{q-2}}$$
$$+ \binom{q}{q-1}\sum_{\substack{r_1^{q-1}+\cdots+r_q^{q-1}=i\\ r_j^{q-1}\geq 1, j=1,2,\ldots,q}}\binom{q-1}{r_1^{q-1}}\binom{p-1}{r_2^{q-1}}\binom{p-1}{r_3^{q-1}}\cdots\binom{p-1}{r_q^{q-1}}$$
$$+ \sum_{\substack{l_1^{q-1}+\cdots+l_q^{q-1}=i\\ l_j^{q-1}\geq 1, j=1,2,\ldots,q}}\binom{p-1}{l_1^{q-1}}\cdots\binom{p-1}{l_q^{q-1}} + \sum_{\substack{r_1^{q}+\cdots+r_{q+1}^{q}=i\\ r_j^{q}\geq 1, j=1,2,\ldots,q+1}}\binom{q-1}{r_1^{q}}\binom{p-1}{r_2^{q}}\cdots\binom{p-1}{r_{q+1}^{q}}.$$

A group $\mathcal{G}$ is said to be an elementary abelian group (sometimes elementary abelian p-group) if every non-trivial element has order p. For an elementary abelian group of prime power order $|\mathcal{G}| = p^z, z \geq 2$, we note that there are $p^z - 1$ elements of order p. Thus, $\mathcal{G}$ has exactly $\frac{p^n-1}{p-1}$ distinct subgroups of order p and has $\frac{p^n-1}{p-1}$ induced subgraphs K_{p-1}. Also, identity is adjacent to all the elements of $\mathcal{G}$. The structure of $\mathcal{G}$ is given in the following result.

Lemma 4.9 ([7]). *Let G be an elementary abelian group of order p^n for some prime number p and positive integer n. Then $\mathcal{P}(G) \cong K_1 * lK_{p-1}$, where $l = \frac{p^n-1}{p-1}$.*

Now as the consequence of Theorem 3.6, we have the following result regarding the Betti numbers of the power graph of the elementary abelian group of prime power order.

Theorem 4.10. *Let $\mathcal{G}$ be an elementary abelian group such that $|\mathcal{G}| = p^z$ where p is prime and $z \geq 1$ is an integer. Then the Betti numbers of $\mathcal{P}(\mathcal{G})$ are*

$$\beta_{i,i+1}(G) = l\cdot i\binom{p-1}{i+1} + l\cdot i\binom{p-1}{i} + \binom{l}{2}\sum_{\substack{l_1^1+l_2^1=i\\ l_1^1,l_2^1\geq 1}}\binom{p-1}{l_1^1}\binom{p-1}{l_2^1}$$

$$+\binom{l}{3}\sum_{\substack{l_1^2+l_2^2+l_3^2=i\\ l_1^2,l_2^2,l_3^2\geq 1}} l_1^2\binom{p-1}{l_1^2}\binom{p-1}{l_2^2}\binom{p-1}{l_3^2}+\dots$$

$$+\binom{l}{l-1}\sum_{\substack{l_1^{l-2}+\dots+l_{l-1}^{l-2}=i\\ l_j^{l-2}\geq 1, j=1,2,\dots,l-1}}\binom{p-1}{l_1^{l-2}}\binom{p-1}{l_2^{l-2}}\dots\binom{p-1}{l_{l-1}^{l-2}}$$

$$+\sum_{\substack{l_1^{l-1}+\dots+l_l^{l-1}=i\\ l_j^{l-1}\geq 1, j=1,2,\dots,l}}\binom{p-1}{l_1^{l-1}}\binom{p-1}{l_2^{l-1}}\dots\binom{p-1}{l_n^{l-1}}.$$

5. Betti numbers of commuting graphs of non-abelian groups

Consider a finite group $\mathcal{G}$ of order n with identity e. If $\emptyset \neq X \subseteq \mathcal{G}$ is any set, then the *commuting graph* of $\mathcal{G}$ associated to X, denoted by $C(\mathcal{G}, X)$, defined as the graph with vertex set X and two different vertices x and y are adjacent in $C(\mathcal{G}, X)$ if and only if they commute in X. There is a vast literature available on the commuting graphs of non-abelian groups, the commuting graphs of matrix rings and semirings over finite fields can be seen in [1, 9]. The metric dimension, resolving polynomial, clique number and chromatic number of commuting graphs of the dihedral groups were studied in [3, 6]. Recent results on the commuting graphs of the generalized dihedral groups can be found in [8, 16].

Clearly, the commuting graph $C(\mathbb{Z}_n, \mathbb{Z}_n)$ is the complete graph K_n, as every element of $\mathbb{Z}_n$ commutes with every other element. So, usually the commuting graphs have non-trivial structures for non-abelian groups. Let $Z(\mathcal{G})$ denote the center of group $\mathcal{G}$. It is clear that $Z(D_{2n}) = \left\{e, a^{\frac{n}{2}}\right\}$, for even n and $Z(D_{2n}) = \{e\}$, for odd n. Also, the center of the dicyclic group Q_{4n} is $Z(Q_{4n}) = \{e, a^n\}$. For the commuting graph [3] $G = C(D_{2n}, Z(D_{2n}))$, G is K_1, for odd n and G is K_2, for even n. So, the commuting graphs $C(\mathcal{G}, Z(\mathcal{G}))$ have simple structures as $Z(\mathcal{G})$ usually contains commuting elements. So, it is of interest to consider subsets of $\mathcal{G}$ such that the corresponding commuting graphs have non-trivial structures. For the dihedral group with $X = D_{2n}, n = 2z + 1, z \geq 1$, the identity is the only element adjacent to all other vertices of $C(D_{2n}, Z(D_{2n}))$, while for even n, $\{e, a^{\frac{n}{2}}\}$ are adjacent to every other vertices. This observation is given by Ali, Salman and Huang [3] in the following result.

Lemma 5.1 ([3]). *The commuting graph of the dihedral group D_{2n} is*

$$C(D_{2n}, D_{2n}) = \begin{cases} K_{n-1} * K_1 * \overline{K}_n, & \text{if } n \text{ is odd;} \\ K_{n-2} * K_2 * \frac{n}{2}K_2, & \text{if } n \text{ is even.} \end{cases}$$

The Betti numbers of $C(D_{2n}, D_{2n})$ for odd n are given in Corollary 4.2 and for the even n, the Betti numbers of $C(D_{2n}, D_{2n})$ can be obtained from Theorem 4.6 by replacing n by $\frac{n}{2}$.

The semidihedral group SD_{8n} of order $8n$ is represented by:

$$SD_{8n} = \langle a, b : a^{4n} = e = b^2, ab = ba^{2n-1}\rangle,$$

and in list representation, we have

$$SD_{8n} = \{e, a, a^2, \dots, a^{4n-1}, b, ba, ba^2, \dots, ba^{4n-1}\}.$$

For odd n, it is clear that $Z(SD_{8n}) = \{e, a^n, a^{2n}, a^{3n}\}$ and for even n, $Z(SD_{8n}) = \{e, a^{2n}\}$. Thus, it follows that these center elements are connected to every other vertex in their respective commuting graphs with $X = D_{8n}$. The next lemma gives the complete structure of the commuting graph of the semidihedral group SD_{8n}.

Lemma 5.2 ([22]). *The structure of the commuting graph of SD_{8n} is given as:*

$$\mathcal{C}(SD_{8n}, D_{8n}) = \begin{cases} K_{4n-4} * K_4 * nK_4, & \text{if } n \text{ is odd;} \\ K_{4n-4} * K_2 * 2nK_2, & \text{if } n \text{ is even.} \end{cases}$$

By using Theorem 3.9, the Betti numbers of $\mathcal{C}(SD_{8n}, D_{8n})$ can be obtained as in Theorem 4.6.

The commuting graph $\mathcal{C}(Q_{4n}, Q_{4n})$ [2] of Q_{4n} is $\mathcal{C}(Q_{4n}, Q_{4n}) = K_{2n-2} * K_2 * nK_2$, we note that $\mathcal{C}(Q_{4n}, Q_{4n})$ is isomorphic to $\mathcal{P}(Q_{4n}), n = 2^z, z \geq 2$. Therefore, the Betti numbers of $\mathcal{C}(Q_{4n}, Q_{4n})$ are exactly same as in Theorem 4.6. There are several other non-abelian groups like $U_{m,n}$ of order mn as given below

$$U_{m,n} = \langle a, b | a^{2n} = e, b^m = 3, aba^{-1} = b^{-1} \rangle, m > 2 \text{ and } n > 1,$$

If m is not a multiple of 2, then $\langle a^2 \rangle$ is in the $Z(U_{m,n})$ with cardinality n and each such vertices are connected to every other vertices of $\mathcal{C}(U_{m,n}, U_{m,n})$. For even m, $Z(U_{m,n}) = \langle a^2, a^{\frac{m}{2}} \rangle$ and its order is $2n$. Thus, with this observation, the commuting graph of $U_{m,n}$ (also see [22]) is

$$\mathcal{C}(U_{m,n}, U_{m,n}) = \begin{cases} K_{mn-2n} * K_{2n} * \frac{m}{2} K_{2n}, & \text{if 2 divides } m; \\ K_{mn-n} * K_n * mK_n, & \text{if 2 does not divide } m. \end{cases}$$

The Betti numbers of $\mathcal{C}(U_{m,n}, U_{m,n})$ can be obtained from Theorem 3.9 with $a = 2n, b = mn - 2n, c = 2n$ and $n = \frac{m}{2}$ for $2 \mid m$ and $a = n, b = mn - n, c = n$ and $n = m$ for $2 \nmid m$.

The other well known non-abelian group of order $8n$ is

$$V_{8n} = \langle a, b | a^{2n} = b^4 = e, ba = a^{-1}b^{-1}, b^{-1}a = a^{-1}b \rangle.$$

Similarly, for $\mathcal{G} \cong V_{8n}$, then the center of $\mathcal{G}$ is generated by $\langle b^2 \rangle$ if $2 \nmid n$ and is generated by $\langle a^n, b^2 \rangle$ if $2 \mid n$. The commuting graph of $\mathcal{G}$ (see [22]) is

$$\mathcal{C}(V_{8n}, V_{8n}) = \begin{cases} K_{4n-2} * K_2 * 2nK_n, & \text{if 2 does not } n; \\ K_{4n-4} * K_4 * nK_4, & \text{if 2 divides } n. \end{cases}$$

The corresponding Betti numbers can be obtained from Theorem 3.9 by putting $a = 2, b = 4n - 2, c = n$ and $n = 2n$ provided $2 \nmid n$ and by using $a = 4, b = 4n - 4, c = 4$ for $2 \mid n$.

6. Conclusion

In this article, the formulae for the initial Betti numbers of multiple complete split-like graphs, clique stars and their generalizations are obtained. Also their extremal Betti numbers are given along with their corresponding projective dimensions. The other Betti numbers and the regularity are yet to be discussed, which is non-trivial for such graphs. In the future work, the other Betti numbers, regularity, Hilbert series of such graphs (along with the power graphs of finite groups and commuting graphs of non-abelian groups) can be taken taken into account.

7. Data Availability:

There is no data associated with this article.

8. Conflict of interest

The authors declare that they have no competing interests.

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