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Additive maps preserving inner inverses on $\mathcal{B}(\mathcal{X})$

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Abstract. Let $\mathcal{B}(\mathcal{X})$ be the algebra of all bounded linear operators on a complex Banach space $\mathcal{X}$. In this paper, we determine the structures of all additive surjective maps on $\mathcal{B}(\mathcal{X})$ preserving inner inverses in both directions.

1. Introduction and Notations

Over the last few decades there has been a considerable interest in the so-called linear preserver problems. One of the most famous problems in this direction is Kaplansky's conjecture [7] asking whether every unital linear surjective map between two semi-simple Banach algebras which preserves invertibility is a Jordan homomorphism. For more details on linear preserver problems, we refer the reader to [3, 5, 8] and the references therein. As we know, generalized inverse is a very important concept in operator theory(cf.[1, 4, 6]). Therefore, a lot of studies have been done on the subject of linear or additive preserver problems with respect to different kinds of generalized inverses. In [9], the authors initiated the study of linear maps preserving generalized invertibility. It has been shown that such maps preserve the ideal of compact operators in both directions and their induced maps on the Calkin algebra are Jordan automorphisms. Then a remarkable improvement was achieved in [10]. By reducing the condition of linearity, Boudi [2] characterized additive maps preserving strongly generalized inverses. We note that inner inverse is an elementary notion in generalized inverse theory. That is, let $\mathcal{A}$ be an algebra and $a, b \in \mathcal{A}$. If $aba = a$, then b is an inner inverse of a. Of course, if b is a generalized inverse of a described in [2] or in [9, 10], then b is an inner inverse of a. Motivated by those discussions, we characterize additive surjective maps preserving inner inverses in both directions.

Let $\mathcal{X}$ be a complex Banach space, $\mathcal{B}(\mathcal{X})$ the algebra of all bounded linear operators on $\mathcal{X}$ and $\mathcal{F}(\mathcal{X})$ the ideal of all finite rank operators. For an operator $T \in \mathcal{B}(\mathcal{X})$, write $ker(T)$ for its kernel, $ran(T)$ for its range and T^* for its adjoint on the topological dual space $\mathcal{X}^*$. For every nonzero $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$, the symbol $x \otimes f$ stands for the rank-one bounded linear operators defined by $(x \otimes f)z = f(z)x$ for all $z \in \mathcal{X}$. Note that every operator of rank one can be written in this form. The operator $x \otimes f$ is an idempotent if and only if $f(x) = 1$ and $x \otimes f$ is a nilpotent if and only if $f(x) = 0$. The set of all idempotents in $\mathcal{B}(\mathcal{X})$ will be denoted by $\mathcal{P}(\mathcal{X})$. Recall that $P, Q \in \mathcal{P}(\mathcal{X})$ are orthogonal if $PQ = QP = 0$ and $P \leq Q$ if $PQ = QP = P$. As usual, we denote respectively by $\mathbb{C}$ and $\mathbb{Q}$ the complex number field and the rational number field. Without any confusion, I denotes the identity operator on any Banach space.

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Let $A, B \in \mathcal{B}(\mathcal{X})$. If $ABA = A$, then we say that B is an inner inverse of A. We say that a map $\varphi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ preserves inner inverses in both directions if

$$\varphi(A)\varphi(B)\varphi(A) = \varphi(A) \Leftrightarrow ABA = A$$

for all $A, B \in \mathcal{B}(\mathcal{X})$. In this paper, we will characterize an additive surjective map φ on $\mathcal{B}(\mathcal{X})$ preserving inner inverses in both directions.

2. Main results

Let φ be an additive map on $\mathcal{B}(\mathcal{X})$. We completely determine all forms of maps preserving inner inverses in both directions on $\mathcal{B}(\mathcal{X})$.

Theorem 2.1 *Let $\mathcal{X}$ be an infinite dimensional complex Banach space and $\varphi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ an additive surjective map. Then φ preserves inner inverses in both directions if and only if there exist a scalar $\alpha \in \{1, -1\}$ and either a bijective bounded linear, or conjugate linear operator $A : \mathcal{X} \to \mathcal{X}$ such that*

$$\varphi(T) = \alpha ATA^{-1} \quad \text{for all } T \in \mathcal{B}(\mathcal{X});$$

or a bijective bounded linear, or conjugate linear operator $B : \mathcal{X}^* \to \mathcal{X}$ such that

$$\varphi(T) = \alpha BT^*B^{-1} \quad \text{for all } T \in \mathcal{B}(\mathcal{X}).$$

In the second case, $\mathcal{X}$ must be a reflexive Banach space.

Let $n > 1$ and let $M_n(\mathbb{C})$ be the algebra of all complex $n \times n$ matrices. For any ring isomorphism τ on $\mathbb{C}$ and $T = (t_{ij}) \in M_n(\mathbb{C})$, we define $T_\tau = (\tau(t_{ij}))$ and $T^{tr} = (t_{ji})$.

Theorem 2.2 *Let $\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be an additive surjective map. Then φ preserves inner inverses in both directions if and only if there exist an invertible matrix $A \in M_n(\mathbb{C})$ and a ring automorphism $\tau : \mathbb{C} \to \mathbb{C}$ such that either $\varphi(T) = \alpha AT_\tau A^{-1}$ for all $T = (t_{ij}) \in M_n(\mathbb{C})$ or $\varphi(T) = \alpha AT_\tau^{tr} A^{-1}$ for all $T = (t_{ij}) \in M_n(\mathbb{C})$, where $\alpha = \pm 1$.*

In order to prove Theorems 2.1 and 2.2, we need some lemmas firstly. In the sequel, we assume that $\varphi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ is an additive surjective map preserving inner inverses in both directions.

Lemma 2.3. *φ is injective.*

Proof. We firstly prove that $\varphi(I) \neq 0$. Assume on the contrary that $\varphi(I) = 0$. By the surjectivity of φ, there exists a nonzero operator $S \in \mathcal{B}(\mathcal{X})$ such that $\varphi(S) = I$. Note that $I^3 = I$. Thus $\varphi(S)^3 = \varphi(S)$, which implies that $S^3 = S$. Note that every additive map is $\mathbb{Q} -$ *linear*. It easily follows that $\varphi(S + rI) = I$ for all $r \in \mathbb{Q}$, and then $\varphi(S + rI)^3 = \varphi(S + rI)$. This shows that $(S + rI)^3 = S + rI$. By a simple calculation, we can get $Ir^2 + 3Sr + (3S^2 - I) = 0$ for all $r \in \mathbb{Q}$, a contradiction.

Next we will prove that φ is injective. Suppose on the contrary that there exists $T \neq 0$ such that $\varphi(T) = 0$. Then $\varphi(I + rT) = \varphi(I)$ for all $r \in \mathbb{Q}$. It follows from $\varphi(I)^3 = \varphi(I)$ that $\varphi(I + rT)^3 = \varphi(I + rT)$. This implies that $(I + rT)^3 = I + rT$. Therefore, $T = 0$ by the arbitrariness of r. This is a contradiction. Thus φ is injective. □

Lemma 2.3 ensures that φ is bijective and so φ^{-1} satisfies the same properties as φ. In the following lemma, we denote by $\sigma(A)$ the spectrum of $A \in \mathcal{B}(\mathcal{X})$.

Lemma 2.4. *$\varphi(I) = I$ or $\varphi(I) = -I$.*

Proof. We may assume that $\varphi(A) = I$ by the surjectivity of φ. Since $A^3 = A$, by the spectral mapping theorem, we get that $\sigma(A) \subseteq \{0, -1, 1\}$. We claim that $0 \notin \sigma(A)$. If $0 \in \sigma(A)$ and let f be the characteristic function of $\{0\}$, then f is analytic on a neighborhood of $\sigma(A)$. Put $P = f(A)$. It follows from the Riesz functional calculus that $P \neq 0$, $P^2 = P$ and $PA = AP = 0$. Note that $P(rA+P)P = P$ for all $r \in \mathbb{Q}$. Then $\varphi(P)(rI+\varphi(P))\varphi(P) = \varphi(P)$. Since $\varphi(P)^3 = \varphi(P)$, we have $\varphi(P) = 0$. But φ is injective by Lemma 2.3, a contradiction.

It now follows that $\sigma(A) \subseteq \{-1, 1\}$. If $\sigma(A) = \{-1\}$ or $\sigma(A) = \{1\}$, we can see that $A = -I$ or $A = I$. If $\sigma(A) = \{1, -1\}$ and put f_1 and f_{-1} are characteristic functions of $\{1\}$ and $\{-1\}$ respectively, then both $f_1(A)$ and $f_{-1}(A)$ are nonzero idempotents such that $\mathcal{X} = f_1(A)\mathcal{X} + f_{-1}(A)\mathcal{X}$ and $A = f_1(A)A + f_{-1}(A)A = A_1 + A_2$. Note that $\sigma(A_1) = \{1\}$ and $\sigma(A_2) = \{-1\}$. It easily follows that $A_1 = I$ on $f_1(A)\mathcal{X}$ as well as $A_2 = -I$ on $f_{-1}(A)\mathcal{X}$ since $A_1^3 = A_1$ as well as $A_2^3 = A_2$. For every nonzero operator $T_0 \in \mathcal{B}(f_{-1}(A)\mathcal{X}, f_1(A)\mathcal{X})$ and $r \in \mathbb{Q}$, put

$$A_T = \begin{pmatrix} I & rT_0 \\ 0 & -I \end{pmatrix} = A + rT,$$

where $T = \begin{pmatrix} 0 & T_0 \\ 0 & 0 \end{pmatrix}$. It is clear that $A_T^3 = A_T$, that is, $(I + r\varphi(T))^3 = I + r\varphi(T)$. It can be obtained by simple calculation that $\varphi(T) = 0$. This contradicts with the injectivity of φ. Therefore, $\varphi(I) = I$ or $\varphi(I) = -I$. □

Without loss of generality, we next assume that $\varphi(I) = I$. It is easy to check that φ preserves idempotents in both directions. Furthermore, We will show that φ preserves rank-one idempotents in both directions.

Lemma 2.5. *φ preserves rank-one idempotents and their orthogonality in both directions.*

Proof. First, we will prove that φ preserves rank-one idempotents in both directions. Let $P = x \otimes f$ be an idempotent. Then $Q = \varphi(P)$ is an idempotent. Suppose on the contrary that Q has rank greater than one. Then there exist two rank-one idempotents Q_1, Q_2 such that $Q_1, Q_2 \leq Q$ and $Q_1Q_2 = Q_2Q_1 = 0$. Obviously, $Q_iQQ_i = Q_i$ for $i = 1, 2$. Thus we have

$$\varphi^{-1}(Q_i)P\varphi^{-1}(Q_i) = \varphi^{-1}(Q_i).$$

This means that $\varphi^{-1}(Q_i)$ is an idempotent of rank one. Then we can assume that $\varphi^{-1}(Q_i) = x_i \otimes f_i$, $i = 1, 2$, where $f_i(x_i) = 1$. Observe that

$$(x_i \otimes f_i)(x \otimes f)(x_i \otimes f_i) = x_i \otimes f_i,$$

we get that $f_i(x)f(x_i) = 1$, that is, $f_i(x) \neq 0$ and $f(x_i) \neq 0$.

On the other hand, it is clear that $Q_1(Q_1 + rQ_2)Q_1 = Q_1$ for every $r \in \mathbb{Q}$. Then

$$\varphi^{-1}(Q_1)(\varphi^{-1}(Q_1) + r\varphi^{-1}(Q_2))\varphi^{-1}(Q_1) = \varphi^{-1}(Q_1).$$

By calculation we have $\varphi^{-1}(Q_1)\varphi^{-1}(Q_2)\varphi^{-1}(Q_1) = 0$, that is, $(x_1 \otimes f_1)(x_2 \otimes f_2)(x_1 \otimes f_1) = 0$. Hence $f_1(x_2) = 0$ or $f_2(x_1) = 0$. We may assume that $f_1(x_2) = 0$ (the case that $f_2(x_1) = 0$ can be considered in a similar way).

Note that $Q_1 + Q_2 \leq Q$. Then we have $(Q_1 + Q_2)Q(Q_1 + Q_2) = Q_1 + Q_2$. Thus

$$(\varphi^{-1}(Q_1) + \varphi^{-1}(Q_2))\varphi^{-1}(Q)(\varphi^{-1}(Q_1) + \varphi^{-1}(Q_2)) = \varphi^{-1}(Q_1) + \varphi^{-1}(Q_2).$$

Since $\varphi^{-1}(Q_i)\varphi^{-1}(Q)\varphi^{-1}(Q_i) = \varphi^{-1}(Q_i)$ for $i = 1, 2$, we have

$$\varphi^{-1}(Q_2)\varphi^{-1}(Q)\varphi^{-1}(Q_1) + \varphi^{-1}(Q_1)\varphi^{-1}(Q)\varphi^{-1}(Q_2) = 0,$$

and then

$$f_2(x)f(x_1)x_2 \otimes f_1 + f_1(x)f(x_2)x_1 \otimes f_2 = 0.$$

Note that $f_2(x)f(x_1) \neq 0$ and $f_1(x)f(x_2) \neq 0$. Then $x_2 \otimes f_1$ and $x_1 \otimes f_2$ are linearly dependent. If x_1 and x_2 are linearly dependent, then $f_1(x_1) = 0$. If f_1 and f_2 are linearly dependent, then $f_2(x_2) = 0$. This contradicts with the fact that $x_1 \otimes f_1$ and $x_2 \otimes f_2$ are idempotents.

It is elementary that φ preserves the orthogonality of rank-one idempotents in both directions. □

For a subset $S \subseteq \mathcal{X}$, the symbol $\bigvee S$ stands for the closed subspace spanned by S, and let $S^{\perp} = \{f \in \mathcal{X}^* : f(x) = 0,\ \forall x \in S\}$. For a subset $M \subseteq \mathcal{X}^*$, let $M_{\perp} = \{x \in \mathcal{X} : f(x) = 0,\ \forall f \in M\}$.

Lemma 2.6. *φ preserves linear spans of idempotents of rank one.*

Proof. Let $x_0 \otimes f_0$ be an idempotent. let $\lambda \in \mathbb{C}$. Put $S = \varphi(\lambda x_0 \otimes f_0)$ and $\varphi(x_0 \otimes f_0) = y_0 \otimes g_0$ with $g_0(y_0) = 1$. It suffices to show that there exists $\mu \in \mathbb{C}$ such that $S = \mu(y_0 \otimes g_0)$. We will complete the proof by two steps.

Step 1. $ran(S) \subseteq \bigvee\{y_0\}$.

Let $g \in \{y_0\}^{\perp}$ and take any nonzero $y \in \mathcal{X}$. We consider the following two cases. If $g(y) \neq 0$, then we may assume without loss of generality that $g(y) = 1$. Put $x \otimes f = \varphi^{-1}(y \otimes g)$. It follows from Lemma 2.5 that $x \otimes f$ is an idempotent. We claim that $Sy \in ker(g)$. Indeed, since $g(y_0) = 0$, we easily get

$$(y \otimes g)(y \otimes g + y_0 \otimes g_0)(y \otimes g) = y \otimes g.$$

Then

$$(x \otimes f)(x \otimes f + x_0 \otimes f_0)(x \otimes f) = x \otimes f,$$

which implies that $(x \otimes f)(x_0 \otimes f_0)(x \otimes f) = 0$. It implies that

$$(x \otimes f)(x \otimes f + \lambda x_0 \otimes f_0)(x \otimes f) = x \otimes f.$$

Thus

$$(y \otimes g)(y \otimes g + S)(y \otimes g) = y \otimes g.$$

This means that $(y \otimes g)S(y \otimes g) = 0$. That is, $g(Sy) = 0$.

If $g(y) = 0$, we can find $y_1 \in \mathcal{X}$ such that $g(y_1) = 1$. Thus $g(y_1 + y) = 1$. By the first case we obtain that $Sy_1 \in ker(g)$ and $S(y_1 + y) \in ker(g)$. Hence $Sy \in ker(g)$.

Therefore, by the choice of g we have $Sy \in \bigvee\{y_0\}$ for every $y \in \mathcal{X}$, that is, $ran(S) \subseteq \bigvee\{y_0\}$.

Step 2. $ran(S^*) \subseteq \bigvee\{g_0\}$.

Let $z \in \{g_0\}_{\perp}$ and take any nonzero $h \in \mathcal{X}^*$. Similar to Step 1, we can easily get $S^*h \in \bigvee\{g_0\}$ for every $h \in \mathcal{X}^*$. Thus $ran(S) \subseteq \bigvee\{g_0\}$.

Therefore, $S = \mu(y_0 \otimes g_0)$ for some scalar $\mu \in \mathbb{C}$. Then φ preserves linear spans of idempotents of rank one. □

Lemma 2.7. *φ maps rank-one nilpotents to nilpotents of rank at most two.*

Proof. Taking any $x \otimes f$ with $f(x) = 0$. Then we can find $g \in \mathcal{X}^*$ such that $g(x) = 1$. Thus $x \otimes g + rx \otimes f = x \otimes (g + rf)$ is an idempotent of rank one for every $r \in \mathbb{Q}$. Set $A = \varphi(x \otimes g)$, $B = \varphi(x \otimes f)$. By Lemma 2.5, both A and $A + rB$ are idempotents of rank one. Since $A^2 = A$ and $(A + rB)^2 = A + rB$, by calculation we get that $B^2 = 0$ and $B = AB + BA$. This implies that B is a nilpotent of rank at most two. □

It is well-known that every operator of rank one is either a scalar multiple of an idempotent or a square-zero operator. By Lemmas 2.5, 2.6 and 2.7, we infer that φ maps $\mathcal{F}(\mathcal{X})$ onto itself.

Proof of Theorem 2.1.

Proof. According to [11, Main Theorem], there exist a scalar $\alpha \in \{1, -1\}$ and either

(i) there exists a bijective bounded linear, or conjugate linear operator $A : \mathcal{X} \to \mathcal{X}$ such that $\varphi(F) = \alpha AFA^{-1}$ for all $F \in \mathcal{F}(\mathcal{X})$; or

(ii) there exists a bijective bounded linear, or conjugate linear operator $B : \mathcal{X}^* \to \mathcal{X}$ such that $\varphi(F) = \alpha BF^*B^{-1}$ for all $F \in \mathcal{B}(\mathcal{X})$. In this case, $\mathcal{X}$ must be a reflexive Banach space.

Assume that φ satisfies (i). For any $T \in \mathcal{B}(\mathcal{X})$, let

$$\psi(T) = \alpha A^{-1}\varphi(T)A.$$

Clearly, ψ satisfies the same properties as φ. Furthermore, $\psi(F) = F$ for all finite rank operators F. It suffices to show that $\psi(T) = T$ for all $T \in \mathcal{B}(\mathcal{X})$. Let $x \in \mathcal{X}$. We will prove the result in the following two cases.

Case 1. $Tx \neq 0$.

For any $f \in \mathcal{X}^*$, we claim that $\langle Tx, f\rangle = 1$ if and only if $\langle \psi(T)x, f\rangle = 1$. Indeed, if $\langle Tx, f\rangle = 1$, then $(x \otimes f)T(x \otimes f) = x \otimes f$ and thus $\psi(x \otimes f)\psi(T)\psi(x \otimes f) = \psi(x \otimes f)$. This means that $(x \otimes f)\psi(T)(x \otimes f) = x \otimes f$, that is, $\langle \psi(T)x, f\rangle = 1$. The converse can be in a similar way. Take an $f \in \mathcal{X}^*$ such that $\langle Tx, f\rangle = 1$. For every $g \in \{Tx\}^{\perp}$, we easily get$\langle Tx, f + g\rangle = 1$. Thus $\langle \psi(T)x, f\rangle = 1$ and $\langle \psi(T)x, f + g\rangle = 1$, which implies that $\langle \psi(T)x, g\rangle = 0$. It now follows that $\psi(T)x \in ker(g)$ for every $g \in \{Tx\}^{\perp}$. Hence $\psi(T)x \in \bigvee\{Tx\}$. This means that $\psi(T)x$ and Tx are linearly dependent. Therefore, $\psi(T)x = \lambda Tx$ for some nonzero scalar $\lambda \in \mathbb{C}$. Note that $(x \otimes f)T(x \otimes f) = x \otimes f$ implies that $(x \otimes f)\psi(T)(x \otimes f) = x \otimes f$ by the assumption on ψ. It entails that $\lambda = 1$, that is, $\psi(T)x = Tx$.

Case 2. $Tx = 0$.

Take any $y \in \mathcal{X}$ with $Ty \neq 0$. Then $\psi(T)x + \psi(T)y = \psi(T)(x + y) = T(x + y) = Ty = \psi(T)y$. Then $\psi(T)x = 0 = Tx$. Thus we have $\psi(T) = T$. Therefore $\varphi(T) = \alpha ATA^{-1}$ for all $T \in \mathcal{B}(\mathcal{X})$. If φ satisfies (ii), then put $\varphi(T) = \alpha B^{-1}\varphi(T)B$, we get in a similar way that $\varphi(T) = T$ for all $T \in \mathcal{B}(\mathcal{X})$. Hence $\varphi(T) = \alpha BT^*B^{-1}$ for all $T \in \mathcal{B}(\mathcal{X})$. □

The proof of Theorem 2.2 follows from Lemma 2.6 and [11, Theorem 4.5].

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