Filomat 38:6 (2024), 1929–1946
https://doi.org/10.2298/FIL2406929H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Constructing relay fusion frames in Hilbert spaces

**Guoqing Hong[a], Pengtong Li[b]**

[a]*School of Science, Henan Institute of Technology, Xinxiang, 453003, China.*
[b]*Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing, 210016, China.*

**Abstract.** In this work, we start by introducing a general methodology to generate new relay fusion frames from given ones, namely the Spatial Complement Method, and analyze the relationships between the parameters of the original and the new relay fusion frame. We then present another simple approach to obtain relay fusion frames by considering fusion frames for its components. An explicit characterization concerning the existence of Parseval relay fusion frame consisting of two initial subspaces is given. Moreover, we obtain a necessary and sufficient condition under which the spatial complements of alternate dual relay fusion frames remain to be alternate dual relay fusion frames. Some results about Bessel relay fusion sequences are included. Finally, several examples are also given.

## 1. Introduction and preliminaries

Throughout the present paper, $\mathbb{I}$, $\mathbb{J}$ and $\mathbb{K}$ will denote generic countable (or finite) index sets. Let $\mathcal{H}$ and $\mathcal{K}$ (resp. $\mathcal{K}_i, i \in \mathbb{I}$) be separable complex Hilbert spaces and let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ (resp. $\mathcal{B}(\mathcal{H}, \mathcal{K}_i), i \in \mathbb{I}$) be the space of all the bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ (resp. $\mathcal{K}_i, i \in \mathbb{I}$). If $\mathcal{H} = \mathcal{K}$ we write $\mathcal{B}(\mathcal{H})$. For an operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, ran $T$ denotes the range of $T$, ker $T$ the nullspace of $T$, $T^* \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ the adjoint of $T$. We use $I_{\mathcal{H}}$ (resp. $I_{\mathcal{K}}$) to denote the identity operator on $\mathcal{H}$ (resp. $\mathcal{K}$). If $W \subseteq \mathcal{H}$ and $V \subseteq \mathcal{K}$ are two closed subspaces, we let $\pi_W \in \mathcal{B}(\mathcal{H})$ and $\tau_V \in \mathcal{B}(\mathcal{K})$ denote the orthogonal projections onto the subspaces $W$ and $V$, respectively. In particular, we use the notation $\{W_i\}_{i\in\mathbb{I}} \sqsubset \mathcal{H}$ to represent a family of closed subspaces $\{W_i\}_{i\in\mathbb{I}}$ of a Hilbert space $\mathcal{H}$, for the sake of brevity.

Frames are generalizations of orthonormal bases in Hilbert spaces. A frame as well as an orthonormal basis allows each element in the underlying Hilbert space to be written as an unconditionally convergent infinite linear combination of the frame elements; however, in contrast to the situation for a basis, the coefficient might not be unique. Nice properties of frames make them very useful in characterization of function spaces and other fields of applications such as sigma-delta quantization [2], filter bank theory [3], signal and image processing [5] and wireless communications [13]. The formal definition is as follows.

**Definition 1.1.** *A system $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}}$ of elements in $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist constants $\alpha, \beta > 0$ such that*

$$\alpha\|f\|^2 \leqslant \sum_{i\in\mathbb{I}} |\langle f, f_i\rangle|^2 \leqslant \beta\|f\|^2, \ \forall f \in \mathcal{H}. \tag{1}$$

2020 *Mathematics Subject Classification*. Primary 42C15; Secondary 46C99, 41A58.
*Keywords*. Frame; G-frame; Fusion frame; Relay fusion frame.
Received: 22 August 2020; Accepted: 17 October 2023
Communicated by Dragan S. Djordjević
Research supported by National Natural Science Foundation of China (Nos. 12301149 and 11671201), Henan Provincial Department of Science and Technology Research Project (Nos. 232102210187 and 222102210291) and Key Scientific Research Projects of Colleges and Universities in Henan Province (23B210005).
*Email addresses:* guoqinghong@nuaa.edu.cn (Guoqing Hong), pengtongli@nuaa.edu.cn (Pengtong Li)

Fusion frames were introduced in [6] (under the name *frames of subspaces*) and further developed in [7], and have quickly become a major tool in the implementation of distributed systems. One of the main applications of fusion frames is to sensor networks [8]. It can be regarded as a frame-like collection of subspaces in a Hilbert space, which clearly generalizes classical vector frames. The precise definition is as follows.

**Definition 1.2.** *Let $\{W_i\}_{i\in\mathbb{I}} \sqsubset \mathcal{H}$ and let $\{w_i\}_{i\in\mathbb{I}} \in \ell^\infty(\mathbb{I})$ such that $w_i > 0$ for every $i \in \mathbb{I}$. The pair $\{(W_i, w_i)\}_{i\in\mathbb{I}}$ is a fusion frame for $\mathcal{H}$ if there exist numbers $0 < \alpha \leqslant \beta < \infty$ which satisfy that*

$$\alpha\|f\|^2 \leqslant \sum_{i\in\mathbb{I}} w_i^2\|\pi_{W_i}(f)\|^2 \leqslant \beta\|f\|^2,\ \forall f \in \mathcal{H}.$$

*In this case we say that $\{(W_i, w_i)\}_{i\in\mathbb{I}}$ is an $(\alpha, \beta)$-fusion frame.*

In [17], Sun introduced a generalization of frames, called the *g-frames*, and showed that g-frames include the frames and fusion frames mentioned above and proved that g-frames share many useful properties with frames. However, the generality of this notion is not suitable for modeling distributed processing.

**Definition 1.3.** *A sequence $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i) : i \in \mathbb{I}\}$ is called a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i : i \in \mathbb{I}\}$ if there exist two positive constants $\alpha$ and $\beta$ such that*

$$\alpha\|f\|^2 \leqslant \sum_{i\in\mathbb{I}} \|\Lambda_i(f)\|^2 \leqslant \beta\|f\|^2,\ \forall f \in \mathcal{H}.$$

*In this case we say that $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i) : i \in \mathbb{I}\}$ is an $(\alpha, \beta)$-g-frame.*

Non-orthogonal fusion frames as another generalization of fusion frames introduced in [10] in order to achieve sparsity of the fusion frame operator. The basic observation in [10] is that replacing orthogonal projections in the original definition of fusion frames by non-orthogonal projections onto the same subspaces can result in a fusion frame operator which is much sparser. Recall that a non-orthogonal projection onto a closed subspace $V$ of a Hilbert space $\mathcal{H}$ is a linear mapping $P_V$ from $\mathcal{H}$ onto $V$ which satisfies $P_V^2 = P_V$.

**Definition 1.4.** *Let $\{V_i\}_{i\in\mathbb{I}} \sqsubset \mathcal{H}$ and let $\{v_i\}_{i\in\mathbb{I}}$ be a family of positive weighting scalers. We say $\{(P_{V_i}, v_i)\}_{i\in\mathbb{I}}$ is a non-orthogonal fusion frame for $\mathcal{H}$ if there exist constants $0 < \alpha \leqslant \beta < \infty$ which satisfy that*

$$\alpha\|f\|^2 \leqslant \sum_{i\in\mathbb{I}} v_i^2\|P_{V_i}(f)\|^2 \leqslant \beta\|f\|^2,\ \forall f \in \mathcal{H},$$

In [14], the authors introduced the idea of r-fusion frames and showed that this includes more other cases of generalizations of frames concept and proved that many basic properties can be derived within this more general context. As r-fusion frame is an extension of fusion frame, it is more suitable for applications requiring three-stage (local-relay-global) signal/data analysis, which is mainly used in areas requiring distributed relay processing [15]. We now make the formal definition of the objects that we shall be studying.

**Definition 1.5.** *Let $\{W_i\}_{i\in\mathbb{I}} \sqsubset \mathcal{H}$ and let $\{V_{ij}\}_{j\in\mathbb{J}_i} \sqsubset \mathcal{K}_i$ for each $i \in \mathbb{I}$. Let $\{v_{ij}\}_{i\in\mathbb{I}, j\in\mathbb{J}_i}$ be a family of positive weights and $T_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i)$ for all $i \in \mathbb{I}$. Then the quadruple $\{(W_i, V_{ij}, T_i, v_{ij})\}_{i\in\mathbb{I}, j\in\mathbb{J}_i}$ (denoted by $\mathcal{R}$ for short) is said to be a relay fusion frame, or simply r-fusion frame, if there exist constants $0 < \alpha \leqslant \beta < \infty$ such that*

$$\alpha\|f\|^2 \leqslant \sum_{i\in\mathbb{I}, j\in\mathbb{J}_i} v_{ij}^2\|\tau_{V_{ij}} T_i \pi_{W_i}(f)\|^2 \leqslant \beta\|f\|^2,\ \forall f \in \mathcal{H}. \tag{2}$$

*In this case we say that $\mathcal{R}$ is an $(\alpha, \beta)$-r-fusion frame.*

The numbers $\alpha$ and $\beta$ are called *lower* and *upper r-fusion frame bounds*. We call $\mathcal{R}$ a *Parseval* r-fusion frame if $\alpha = \beta = 1$. An *$\alpha$-tight* r-fusion frame provided that $\alpha = \beta$. If $\mathcal{R}$ satisfies the second inequality in (2), then it is said to be a *Bessel r-fusion sequence* in $\mathcal{H}$ with Bessel r-fusion bound $\beta$. The operators $T_i$, spaces $\mathcal{K}_i$, $i \in \mathbb{I}$ are called *relay operators* and *relay spaces*, respectively. Moreover, if $\{f : \tau_{V_{ij}} T_i \pi_{W_i}(f) = 0, i \in \mathbb{I}, j \in \mathbb{J}_i\} = \{0\}$, then we say that $\mathcal{R}$ is *r-complete*.

One easily verifies that, in view of the given definitions, frames, fusion frames, non-orthogonal fusion frames and g-frames are all special cases of this notion. As well as vector frames, there are some bounded operators associated to an r-fusion frame. First, we set the Hilbert space $\mathcal{R}_{\ell^2} := \bigoplus_{i\in\mathbb{I},j\in\mathbb{J}_i} V_{ij}$ (endowed with the $\ell^2$ norm) and we define the *analysis operator* $T_{\mathcal{R}} : \mathcal{H} \mapsto \mathcal{R}_{\ell^2}$ and the *synthesis operator* $T^*_{\mathcal{R}} : \mathcal{R}_{\ell^2} \mapsto \mathcal{H}$ of $\mathcal{R}$ by

$$T_{\mathcal{R}}(f) = \left\{v_{ij}\tau_{V_{ij}} T_i \pi_{W_i}(f)\right\}_{i\in\mathbb{I},j\in\mathbb{J}_i}, \ \forall f \in \mathcal{H},$$

$$T^*_{\mathcal{R}}(f) = \sum_{i\in\mathbb{I},j\in\mathbb{J}_i} v_{ij}\pi_{W_i} T_i^* f_{ij}, \ f = \{f_{ij}\}_{i\in\mathbb{I},j\in\mathbb{J}_i} \in \mathcal{R}_{\ell^2}.$$

By composing $T_{\mathcal{R}}$ and $T^*_{\mathcal{R}}$, we obtain the *r-fusion frame operator* $S_{\mathcal{R}}$ for $\mathcal{R}$ defined by

$$S_{\mathcal{R}}(f) = T^*_{\mathcal{R}} T_{\mathcal{R}}(f) = \sum_{i\in\mathbb{I},j\in\mathbb{J}_i} v_{ij}^2 \pi_{W_i} T_i^* \tau_{V_{ij}} T_i \pi_{W_i}(f), \ \forall f \in \mathcal{H}.$$

The r-fusion frame operators exhibit important properties similar to those of the frame operators. For example, if $\mathcal{R}$ is an r-fusion frame with frame bounds $\alpha$ and $\beta$, then the frame operator for $\mathcal{R}$ is a bounded, positive, self-adjoint, invertible operator on $\mathcal{H}$ with $\alpha I_{\mathcal{H}} \leqslant S_{\mathcal{R}} \leqslant \beta I_{\mathcal{H}}$.

To summarize, r-fusion frame is a new signal representation method that uses collections of relay subspaces instead of vectors to represent signals. Such a representation provides significant flexility compared to classical frame representations. The rich structure of the r-fusion frames framework allows us to capture more complicated signal modes, which in practical terms adds new aspects to the frame theory. Nevertheless, as mentioned in [15], the theory of the r-fusion frame is far from fully developed, and we hope that this work can stimulate the research on this interesting subject.

The present paper is organized as follows. In Sect.2, we give some methods to obtain r-fusion frames and present new types of r-fusion frames in Hilbert spaces. Owing to the fact that the relay operators are involved, several known results can be derived from our results by proper choices of operators and parameters. We first introduce a general way, namely the Spatial Complement Method, to generate new r-fusion frames from existing ones and analyze the connections between the parameters of the original and the new r-fusion frame. Motivated by the idea of spatial complements, we define relay spatial complements and dual spatial complements, and derive some related results. Some of the remarks are also customized about the relationships among them. We then obtain new r-fusion frames by considering fusion frames for its components. Moreover, an explicit characterization of Parseval r-fusion frame consisting of two initial subspaces is given. We also show that under some conditions r-fusion frames are stable, which will play an important role in studying r-fusion frames for Hilbert spaces. The main purpose of Sect.3 is to investigate Bessel r-fusion sequences and alternate dual r-fusion frames. The condition that the spatial complements of alternate dual r-fusion frames to be again alternate dual r-fusion frames is determined, and it is applied to fusion frames. Some results about Bessel r-fusion sequences are obtained. Finally, several concrete examples are also given.

## 2. Construction of new r-fusion frames

In this section, we first present a general way, namely the Spatial Complement Method, for constructing a new r-fusion frame from a given r-fusion frame and establish the relationship between the parameters of the two r-fusion frames. This idea was first developed in [9] for constructing tight fusion frames with given parameters, and later refined in [4]. It seems to be a natural method to generate a new r-fusion frame from a given r-fusion frame. We begin by defining the notion of spatial complement r-fusion frame for a given r-fusion frame. Since the case of multi-relay space is similar (just with a more involved notation), we only consider the situation of single relay space.

**Definition 2.1.** *Let* $\{(W_i, V_i, T, v_i)\}_{i\in\mathbb{I}}$ *be an r-fusion frame for* $\mathcal{H}$*. Then we call the family* $\{(W_i^\perp, V_i, T, v_i)\}_{i\in\mathbb{I}}$ *the spatial complement to* $\{(W_i, V_i, T, v_i)\}_{i\in\mathbb{I}}$*, if* $\{(W_i^\perp, V_i, T, v_i)\}_{i\in\mathbb{I}}$ *is also an r-fusion frame, where* $W_i^\perp$ *is the orthogonal complement of* $W_i$*.*

It is readily to see that a "dual" relation also holds in this case, that is, $\{(W_i, V_i, T, v_i)\}_{i\in\mathbb{I}}$ is also a spatial complement for $\{(W_i^\perp, V_i, T, v_i)\}_{i\in\mathbb{I}}$. In what follows, given an r-fusion frame, we will implicitly assume that the r-fusion frame bounds are optimal and attainable, i.e. there exist elements of initial Hilbert space $\mathcal{H}$ such that equal signs can occur in r-fusion frame inequality (2). Armed with the Definition 2.1, we can now state and prove our first result.

**Theorem 2.2.** *Let relay operator* $T : \mathcal{H} \mapsto \mathcal{K}$ *be an isometry such that* $\{(W_i, \mathcal{K}, T, v_i)\}_{i\in\mathbb{I}}$ *is an* $(\alpha, \beta)$*-r-fusion frame for* $\mathcal{H}$ *with* $\sum_{i\in\mathbb{I}} v_i^2 < \infty$*. Then the following conditions are equivalent.*

(i) $\bigcap_{i\in\mathbb{I}} W_i = \{0\}$.

(ii) $\beta < \sum_{i\in\mathbb{I}} v_i^2$.

(iii) *The family* $\{(W_i^\perp, \mathcal{K}, T, v_i)\}_{i\in\mathbb{I}}$ *is a* $\left(\sum_{i\in\mathbb{I}} v_i^2 - \beta, \sum_{i\in\mathbb{I}} v_i^2 - \alpha\right)$*-r-fusion frame.*

*Proof.* (i) ⇒ (ii): By supposition, for any $f \in \mathcal{H}$, we have

$$\sum_{i\in\mathbb{I}} v_i^2 \|\tau_{\mathcal{K}} T \pi_{W_i}(f)\|^2 \leqslant \sum_{i\in\mathbb{I}} v_i^2 \|f\|^2.$$

Duo to $\beta$ is optimal, this implies that

$$\beta \leqslant \sum_{i\in\mathbb{I}} v_i^2. \tag{3}$$

We claim that equality sign holds precisely in inequality (3) if and only if $\bigcap_{i\in\mathbb{I}} W_i \neq \{0\}$.

"⇒" Assume that $\beta = \sum_{i\in\mathbb{I}} v_i^2$. By using the r-fusion frame property, this yields that there exists some $f \in \mathcal{H}$ so that

$$\beta\|f\|^2 = \sum_{i\in\mathbb{I}} v_i^2 \|\tau_{\mathcal{K}} T \pi_{W_i}(f)\|^2 = \sum_{i\in\mathbb{I}} v_i^2 \|\pi_{W_i}(f)\|^2 = \sum_{i\in\mathbb{I}} v_i^2 \|f\|^2,$$

which implies that

$$f \in \bigcap_{i\in\mathbb{I}} W_i \neq \{0\}.$$

"⇐" Obvious.

(ii) ⇒ (iii): We start from the inequality

$$\alpha I_{\mathcal{H}} \leqslant \sum_{i\in\mathbb{I}} v_i^2 \pi_{W_i} T^* \tau_{\mathcal{K}} T \pi_{W_i} = \sum_{i\in\mathbb{I}} v_i^2 \pi_{W_i} \leqslant \beta I_{\mathcal{H}},$$

from which we deduce that

$$\left(\sum_{i\in\mathbb{I}} v_i^2 - \beta\right) I_{\mathcal{H}} \leqslant \sum_{i\in\mathbb{I}} v_i^2 (I_{\mathcal{H}} - \pi_{W_i}) \leqslant \left(\sum_{i\in\mathbb{I}} v_i^2 - \alpha\right) I_{\mathcal{H}}.$$

Exploiting the fact that $\sum_{i\in\mathbb{I}} v_i^2 > \beta$, we conclude that

$$\{(W_i^{\perp}, \mathcal{K}, T, v_i)\}_{i\in\mathbb{I}} = \{(I_{\mathcal{H}} - \pi_{W_i})\mathcal{H}, \mathcal{K}, T, v_i)\}_{i\in\mathbb{I}}$$

is a $\left(\sum_{i\in\mathbb{I}} v_i^2 - \beta, \sum_{i\in\mathbb{I}} v_i^2 - \alpha\right)$-r-fusion frame.

(iii) ⇒ (i): From [15, Theorem 2.12], we know that a necessary condition for a setting to be an r-fusion frame is that the initial subspaces can span the whole ambient space $\mathcal{H}$. Assume that (i) is false and we shall obtain a contradiction. If there exists a vector $0 \neq f \in \bigcap_{i\in\mathbb{I}} W_i$, then $f \perp W_i^{\perp}$ for all $i \in \mathbb{I}$, and therefore, $\{W_i^{\perp}\}_{i\in\mathbb{I}}$ does not span $\mathcal{H}$. This is a contradiction to (iii). □

We point out here that the hidden assumption that the optimal r-fusion frame bounds can be reached cannot be removed, since there may not be any element in the ambient Hilbert space $\mathcal{H}$ found that make the equal signs in the r-fusion frame inequality (2) true, cf. [11, Proposition 5.4.4].

In case relay operator $T$ is merely a bounded operator, we have the following weak result.

**Theorem 2.3.** *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\{(W_i, \mathcal{K}, T, v_i)\}_{i\in\mathbb{I}}$ is an $(\alpha, \beta)$-r-fusion frame for $\mathcal{H}$. If there exist constants $\lambda, \mu > \beta$ so that*

$$\mu\|f\|^2 \leqslant \sum_{i\in\mathbb{I}} v_i^2 \|T(f)\|^2 \leqslant \lambda\|f\|^2, \ \forall f \in \mathcal{H}, \tag{4}$$

*then the family $\{(W_i^{\perp}, \mathcal{K}, T, v_i)\}_{i\in\mathbb{I}}$ is a $\left((\sqrt{\mu} - \sqrt{\beta})^2, (\sqrt{\lambda} + \sqrt{\beta})^2\right)$-r-fusion frame.*

*Proof.* Assume that $\{(W_i, \mathcal{K}, T, v_i)\}_{i\in\mathbb{I}}$ is an $(\alpha, \beta)$-r-fusion frame for $\mathcal{H}$. Let $f$ be an arbitrary element of $\mathcal{H}$. Observe that

$$\left(\sum_{i\in\mathbb{I}} v_i^2 \|\tau_{\mathcal{K}} T \pi_{W_i^{\perp}}(f)\|^2\right)^{\frac{1}{2}} = \left(\sum_{i\in\mathbb{I}} v_i^2 \|T(I_{\mathcal{H}} - \pi_{W_i})(f)\|^2\right)^{\frac{1}{2}}.$$

By the triangle inequality, we have

$$\left(\sum_{i\in\mathbb{I}} v_i^2 \|T(I_{\mathcal{H}} - \pi_{W_i})(f)\|^2\right)^{\frac{1}{2}} \geqslant \left(\sum_{i\in\mathbb{I}} v_i^2 \|T(f)\|^2\right)^{\frac{1}{2}} - \left(\sum_{i\in\mathbb{I}} v_i^2 \|\tau_{\mathcal{K}} T \pi_{W_i}(f)\|^2\right)^{\frac{1}{2}}$$

We see from (4) that

$$\left(\sum_{i\in\mathbb{I}} v_i^2 \|T(f)\|^2\right)^{\frac{1}{2}} - \left(\sum_{i\in\mathbb{I}} v_i^2 \|\tau_{\mathcal{K}} T \pi_{W_i}(f)\|^2\right)^{\frac{1}{2}} \geqslant (\sqrt{\mu} - \sqrt{\beta})\|f\|.$$

Hence

$$\left(\sum_{i\in\mathbb{I}} v_i^2 \|\tau_{\mathcal{K}} T \pi_{W_i^{\perp}}(f)\|^2\right)^{\frac{1}{2}} \geqslant (\sqrt{\mu} - \sqrt{\beta})\|f\|.$$

Similarly we can prove that

$$\left(\sum_{i\in\mathbb{I}} v_i^2 \|\tau_{\mathcal{K}} T \pi_{W_i^{\perp}}(f)\|^2\right)^{\frac{1}{2}} \leqslant (\sqrt{\lambda} + \sqrt{\beta})\|f\|.$$

□

More generally, we have

**Theorem 2.4.** *Let $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ such that $\{(W_i, V_i, T, v_i)\}_{i\in\mathbb{I}}$ is an $(\alpha,\beta)$-r-fusion frame for $\mathcal{H}$. If there exist constants $\lambda,\mu > \beta$ so that*

$$\mu\|f\|^2 \leqslant \sum_{i\in\mathbb{I}} v_i^2\|\tau_{V_i}T(f)\|^2 \leqslant \lambda\|f\|^2,\ \forall f \in \mathcal{H}, \tag{5}$$

*then the family $\{(W_i^\perp, V_i, T, v_i)\}_{i\in\mathbb{I}}$ is a $\left((\sqrt{\mu}-\sqrt{\beta})^2,(\sqrt{\lambda}+\sqrt{\beta})^2\right)$-r-fusion frame.*

The following theorem shows that the parameters of the new r-fusion frame can be determined from those of the generating fusion frame prior to the construction.

**Theorem 2.5.** *Let $T : \mathcal{H} \mapsto \mathcal{K}$ be an isometry such that $\{(W_i,\mathcal{K},T,v_i)\}_{i\in\mathbb{I}}$ is an r-fusion frame for $\mathcal{H}$ with the associated spatial complement $\{(W_i^\perp,\mathcal{K},T,v_i)\}_{i\in\mathbb{I}}$. Let $S_{\mathcal{R}}$ denote the frame operator for $\{(W_i,\mathcal{K},T,v_i)\}_{i\in\mathbb{I}}$ with eigenvectors $\{x_j\}_{j\in\mathbb{J}}$ and respective eigenvalues $\{\xi_j\}_{j\in\mathbb{J}}$. Then the r-fusion frame operator for $\{(W_i^\perp,\mathcal{K},T,v_i)\}_{i\in\mathbb{I}}$ possesses the same eigenvectors $\{x_j\}_{j\in\mathbb{J}}$ and respective eigenvalues $\left\{\sum_{i\in\mathbb{I}} v_i^2-\xi_j\right\}_{j\in\mathbb{J}}$.*

*Proof.* By hypothesis, we obtain that

$$\sum_{i\in\mathbb{I}} v_i^2\pi_{W_i}T^*\tau_{\mathcal{K}}T\pi_{W_i}(x_j) = \sum_{i\in\mathbb{I}} v_i^2\pi_{W_i}(x_j) = \xi_j x_j,\ \forall j \in \mathbb{J}.$$

Hence,

$$\sum_{i\in\mathbb{I}} v_i^2(I_{\mathcal{H}}-\pi_{W_i})(x_j) = \left(\sum_{i\in\mathbb{I}} v_i^2-\xi_j\right)x_j,$$

as claimed. □

**Theorem 2.6.** *Let $T : \mathcal{H} \mapsto \mathcal{K}$ be an isometry such that $\{(W_i,\mathcal{K},T,v_i)\}_{i\in\mathbb{I}}$ is an $\alpha$-tight r-fusion frame for $\mathcal{H}$, not all of whose subspaces equal $\mathcal{H}$. Then $\{(W_i^\perp,\mathcal{K},T,v_i)\}_{i\in\mathbb{I}}$ is a $\left(\sum_{i\in\mathbb{I}} v_i^2-\alpha\right)$-tight r-fusion frame.*

*Proof.* Let $f \in \mathcal{H}$. Then we have

$$\begin{aligned}
\sum_{i\in\mathbb{I}} v_i^2\|\tau_{\mathcal{K}}T\pi_{W_i^\perp}(f)\|^2 &= \sum_{i\in\mathbb{I}} v_i^2\|\pi_{W_i^\perp}(f)\|^2\\
&= \sum_{i\in\mathbb{I}} v_i^2\|(I_{\mathcal{H}}-\pi_{W_i})(f)\|^2\\
&= \sum_{i\in\mathbb{I}} v_i^2(\|f\|^2-\|\pi_{W_i}(f)\|^2)\\
&= \left(\sum_{i\in\mathbb{I}} v_i^2-\alpha\right)\|f\|^2.
\end{aligned}$$

It is readily verified that $\sum_{i\in\mathbb{I}} v_i^2-\alpha = 0$ if and only if all initial subspaces $\{W_i\}_{i\in\mathbb{I}}$ equal $\mathcal{H}$. Therefore we have $\alpha < \sum_{i\in\mathbb{I}} v_i^2$, and the application of Theorem 2.2 proves the claim. □

The next two propositions extend (or refine) Theorem 3.9 and Theorem 3.12 of [14]. First we state a useful lemma for our discussion.

**Lemma 2.7.** *([12]) Let $A \in \mathcal{B}(\mathcal{H})$ and $V \subseteq \mathcal{H}$ be a closed subspace. Then*

$$\pi_V A^* = \pi_V A^*\pi_{\overline{AV}}.$$

**Proposition 2.8.** *Let $\{(W_i, V_i, T_i, v_i)\}_{i\in\mathbb{I}}$ be an r-fusion frame for $\mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be an invertible operator. Then $\{(W_i, AV_i, A\tau_{V_i}T_i, v_i)\}_{i\in\mathbb{I}}$ is an r-fusion frame for $\mathcal{H}$.*

*Proof.* Suppose that $\{(W_i, V_i, T_i, v_i)\}_{i\in\mathbb{I}}$ is an r-fusion frame with frame bounds $\alpha$ and $\beta$. We first prove the upper frame bound for $\{(W_i, AV_i, A\tau_{V_i}T_i, v_i)\}_{i\in\mathbb{I}}$. For each $f \in H$, we have

$$\begin{aligned}\sum_{i\in\mathbb{I}} v_i^2\|\tau_{AV_i}A\tau_{V_i}T_i\pi_{W_i}(f)\|^2 &= \sum_{i\in\mathbb{I}} v_i^2\|A\tau_{V_i}T_i\pi_{W_i}(f)\|^2\\ &\leqslant \beta\|A\|^2\|f\|^2.\end{aligned}$$

Now we find a lower frame bound for $\{(W_i, AV_i, A\tau_{V_i}T_i, v_i)\}_{i\in\mathbb{I}}$. Let $f \in H$. Observe that

$$\begin{aligned}\sum_{i\in\mathbb{I}} v_i^2\|\tau_{AV_i}A\tau_{V_i}T_i\pi_{W_i}(f)\|^2 &= \sum_{i\in\mathbb{I}} v_i^2\|A\tau_{V_i}T_i\pi_{W_i}(f)\|^2\\ &\geqslant \sum_{i\in\mathbb{I}} v_i^2\frac{1}{\|A^{-1}\|^2}\|\tau_{V_i}T_i\pi_{W_i}(f)\|^2\\ &\geqslant \frac{\alpha}{\|A^{-1}\|^2}\|f\|^2.\end{aligned}$$

This proves the claim. □

**Proposition 2.9.** *Let $\{(W_i, V_i, T_i, v_i)\}_{i\in\mathbb{I}}$ be an r-fusion frame for $\mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$ be an invertible operator. Then $\{(AW_i, V_i, T_i\pi_{W_i}A, v_i)\}_{i\in\mathbb{I}}$ is an r-fusion frame for $\mathcal{H}$.*

*Proof.* For all $f \in H$, we have

$$\begin{aligned}\sum_{i\in\mathbb{I}} v_i^2\|\tau_{V_i}T_i\pi_{W_i}A\pi_{AW_i}(f)\|^2 &= \sum_{i\in\mathbb{I}} v_i^2\|\tau_{V_i}T_i\pi_{W_i}A(f)\|^2\\ &\leqslant \beta\|A\|^2\|f\|^2.\end{aligned}$$

Now we obtain a lower bound for $\{(AW_i, V_i, T_i\pi_{W_i}A, v_i)\}_{i\in\mathbb{I}}$. Let $f \in H$, we compute

$$\begin{aligned}\sum_{i\in\mathbb{I}} v_i^2\|\tau_{V_i}T_i\pi_{W_i}A\pi_{AW_i}(f)\|^2 &= \sum_{i\in\mathbb{I}} v_i^2\|\tau_{V_i}T_i\pi_{W_i}A(f)\|^2\\ &\geqslant \frac{\alpha}{\|A^{-1}\|^2}\|f\|^2.\end{aligned}$$

The conclusion follows. □

**Example 2.10.** *Let $\{(W_i, V_i, T_i, v_i)\}_{i\in\mathbb{I}}$ be an r-fusion frame for $\mathcal{H}$ with the associated spatial complement $\{(W_i^\perp, V_i, T_i, v_i)\}_{i\in\mathbb{I}}$. Let $\mathcal{F}$ be a frame for relay Hilbert space $\mathcal{K}$. Denote $S_\mathcal{F}$ the frame operators for frame $\mathcal{F}$. Let $\widetilde{V}_i = S_\mathcal{F}^{-1}V_i$ and $\widetilde{T}_i = S_\mathcal{F}^{-1}\tau_{V_i}T_i$. It follows from Theorem 2.8, both $\{(W_i, \widetilde{V}_i, \widetilde{T}_i, v_i)\}_{i\in\mathbb{I}}$ and $\{(W_i^\perp, \widetilde{V}_i, \widetilde{T}_i, v_i)\}_{i\in\mathbb{I}}$ are r-fusion frames for $\mathcal{H}$. Note that in the case of single relay space, local relay dual and global relay dual of relay fusion frames are the same, see Section 3.2 of [14] for further details. Similarly, let $\widehat{W_i^\perp} = S_\mathcal{R}^{-1}W_i^\perp$ and $\widehat{T}_i = T_i\pi_{W_i^\perp}S_\mathcal{R}^{-1}$, where $S_\mathcal{R}$ is the frame operator for $\{(W_i^\perp, V_i, T_i, v_i)\}_{i\in\mathbb{I}}$. Via Theorem 2.9, $\{(\widehat{W_i^\perp}, V_i, \widehat{T}_i, v_i)\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ is also an r-fusion frame for $\mathcal{H}$.*

Inspired by the idea of spatial complements, we now define relay spatial complements.

**Definition 2.11.** *Let $\{(W_i, V_i, T, v_i)\}_{i\in\mathbb{I}}$ be an r-fusion frame for $\mathcal{H}$. Then we call the family $\{(W_i, V_i^\perp, T, v_i)\}_{i\in\mathbb{I}}$ the relay spatial complement to $\{(W_i, V_i, T, v_i)\}_{i\in\mathbb{I}}$, if $\{(W_i, V_i^\perp, T, v_i)\}_{i\in\mathbb{I}}$ is also an r-fusion frame, where $V_i^\perp$ is the orthogonal complement of $V_i$.*

The following theorem is a relay space version of Theorem 2.12 in [15], and will be used in the proof of Theorem 2.13.

**Theorem 2.12.** *Let relay operator $T : \mathcal{H} \mapsto \mathcal{K}$ be a surjective such that $\{(\mathcal{H}, V_i, T, v_i)\}_{i\in\mathbb{I}}$ is an r-fusion frame for $\mathcal{H}$. Then $\overline{\text{span}}\{\bigcup_{i\in\mathbb{I}} V_i\} = \mathcal{K}$.*

*Proof.* Since $\{(\mathcal{H}, V_i, T, v_i)\}_{i\in\mathbb{I}}$ is an r-fusion frame for $\mathcal{H}$, by Theorem 2.14 of [15], we know that $T$ is also a injective from $\mathcal{H}$ to $\mathcal{K}$. Now assume that $\overline{\text{span}}\{\bigcup_{i\in\mathbb{I}} V_i\} \neq \mathcal{K}$. Then there exists $0 \neq h \in (\overline{\text{span}}\{\bigcup_{i\in\mathbb{I}} V_i\})^{\perp} \subseteq \mathcal{K}$, such that $\tau_{V_i}(h) = 0$ for every $i \in \mathbb{I}$. Since $T : \mathcal{H} \mapsto \mathcal{K}$ is a surjective, there exists $0 \neq f \in \mathcal{H}$ so that $h = T(f)$. It follows that $\tau_{V_i} T\pi_{\mathcal{H}}(f) = 0$ for each $i \in \mathbb{I}$, which contradicts with the assumption that $\{(\mathcal{H}, V_i, T, v_i)\}_{i\in\mathbb{I}}$ is an r-fusion frame for $\mathcal{H}$. Thus we conclude that $\overline{\text{span}}\{\bigcup_{i\in\mathbb{I}} V_i\} = \mathcal{K}$. □

The following theorem is analogous to Theorem 2.2, which shows that the properties of relay spatial complements are similar to those of spatial complements.

**Theorem 2.13.** *Let relay operator $T : \mathcal{H} \mapsto \mathcal{K}$ be a unitary such that $\{(\mathcal{H}, V_i, T, v_i)\}_{i\in\mathbb{I}}$ is an $(\alpha, \beta)$-r-fusion frame for $\mathcal{H}$ with $\sum_{i\in\mathbb{I}} v_i^2 < \infty$. Then the following conditions are equivalent.*

(i) $\bigcap_{i\in\mathbb{I}} V_i = \{0\}$.

(ii) $\beta < \sum_{i\in\mathbb{I}} v_i^2$.

(iii) *The family $\{(\mathcal{K}, V_i^{\perp}, T, v_i)\}_{i\in\mathbb{I}}$ is a $\left(\sum_{i\in\mathbb{I}} v_i^2 - \beta, \sum_{i\in\mathbb{I}} v_i^2 - \alpha\right)$-r-fusion frame.*

*Proof.* Let us follow the strategy used in the proof of Theorem 2.2.

(i) ⇒ (ii): By assumption, for every $f \in \mathcal{H}$, we have

$$\sum_{i\in\mathbb{I}} v_i^2 \|\tau_{V_i} T(f)\|^2 \leqslant \sum_{i\in\mathbb{I}} v_i^2 \|f\|^2.$$

It follows that

$$\beta \leqslant \sum_{i\in\mathbb{I}} v_i^2.$$

Write $g = T(f)$. It is plain to observe that equality sign holds in the above inequality if and only if

$$g \in \bigcap_{i\in\mathbb{I}} V_i \neq \{0\}.$$

(ii) ⇒ (iii): Since

$$\alpha I_{\mathcal{H}} \leqslant \sum_{i\in\mathbb{I}} v_i^2 T^* \tau_{V_i} T \leqslant \beta I_{\mathcal{H}},$$

we have

$$\left(\sum_{i\in\mathbb{I}} v_i^2 - \beta\right) I_{\mathcal{H}} \leqslant \sum_{i\in\mathbb{I}} v_i^2 T^* (I_{\mathcal{K}} - \tau_{V_i}) T \leqslant \left(\sum_{i\in\mathbb{I}} v_i^2 - \alpha\right) I_{\mathcal{H}}.$$

Applying the fact that $\sum_{i\in\mathbb{I}} v_i^2 > \beta$, this yields that

$$\{(\mathcal{H}, V_i^{\perp}.T, v_i)\}_{i\in\mathbb{I}} = \{(\mathcal{H}, (I_{\mathcal{K}} - \tau_{V_i})\mathcal{K}, T, v_i)\}_{i\in\mathbb{I}}$$

is a $\left(\sum_{i\in\mathbb{I}} v_i^2 - \beta, \sum_{i\in\mathbb{I}} v_i^2 - \alpha\right)$-r-fusion frame.

(iii) ⇒ (i): Assume the opposite, i.e. there exists a vector $0 \neq h \in \bigcap_{i\in\mathbb{I}} V_i$, then $h \perp V_i^{\perp}$ for all $i \in \mathbb{I}$, and therefore, $\{V_i^{\perp}\}_{i\in\mathbb{I}}$ does not span $\mathcal{K}$. Via Theorem 2.12, this is a contradiction to (iii). □

**Remark 2.14.** *We remark here that there also exist relay spatial complement versions of Theorem 2.3, 2.4, 2.5, 2.6, which are similar and the proofs carrying over with small changes. To simplify the description, we omit them. It should be noted that not all properties of the initial subspaces can be directly transferred to the relay subspaces. For instance, the Theorem 2.12 may fail if relay operator T is not a surjective. Recall that there are no additional requirements for relay operators for properties similar to the initial subspaces, cf. [15, Theorem 2.12].*

In view of the notions of spatial complements and relay spatial complements, it is naturally to define dual spatial complements.

**Definition 2.15.** *Let* $\{(W_i, V_i, T, v_i)\}_{i\in\mathbb{I}}$ *be an r-fusion frame for* $\mathcal{H}$*. Then we call the family* $\{(W_i^\perp, V_i^\perp, T, v_i)\}_{i\in\mathbb{I}}$ *the dual spatial complement to* $\{(W_i, V_i, T, v_i)\}_{i\in\mathbb{I}}$*, if* $\{(W_i^\perp, V_i^\perp, T, v_i)\}_{i\in\mathbb{I}}$ *is also an r-fusion frame, where* $W_i^\perp$ *is the orthogonal complement of* $W_i$ *and* $V_i^\perp$ *is the orthogonal complement of* $V_i$*.*

**Remark 2.16.** *In light of the definition of dual spatial complements, one can consider the dual spatial complement as a spatial complement of the relay space complement of a given r-fusion frame. Likewise, the dual spatial complement can also be regarded as relay spatial complement of spatial complement of a given r-fusion frame. Provided that each of them can form an r-fusion frame for environmental space* $\mathcal{H}$*. For this reason, the connections between them can be studied in a way that mimics the previous work of the present paper. We omit the details.*

Before proceeding, we now provide the following elementary observations between the frames, fusion frames, non-orthogonal fusion frames, g-frames and r-fusion frames, which can help us to better understand the r-fusion frame settings.

**Theorem 2.17.** *With the notation defined as in Section 1.*

(i) *Let* $\{e_{ijk}\}_{k\in\mathbb{K}_{ij}}$ *be an orthonormal basis for the subspaces* $V_{ij}$ *for each* $i \in \mathbb{I},\ j \in \mathbb{J}_i$*. Then* $\{v_{ij}\pi_{W_i}T_i^* e_{ijk}\}_{i\in\mathbb{I},j\in\mathbb{J}_i,k\in\mathbb{K}_{ij}}$ *is a frame for* $\mathcal{H}$ *if and only if* $\{(W_i, V_{ij}, T_i, v_{ij})\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ *is an r-fusion frame for* $\mathcal{H}$*.*

(ii) $\{(W_i, w_i)\}_{i\in\mathbb{I}}$ *is a fusion frame for* $\mathcal{H}$ *if and only if* $\{(\mathcal{H}, W_i, \pi_{W_i}, w_i)\}_{i\in\mathbb{I}}$ *is an r-fusion frame for* $\mathcal{H}$*.*

(iii) $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i) : i \in \mathbb{I}\}$ *is a g-frame for* $\mathcal{H}$ *with respect to* $\{\mathcal{K}_i : i \in \mathbb{I}\}$ *if and only if* $\{(\mathcal{H}, \mathcal{K}_i, \Lambda_i, 1)\}_{i\in\mathbb{I}}$ *is an r-fusion frame for* $\mathcal{H}$*.*

(iv) $\{(P_{V_i}, v_i)\}_{i\in\mathbb{I}}$ *is a non-orthogonal fusion frame for* $\mathcal{H}$ *if and only if* $\{(\mathcal{H}, V_i, P_{V_i}, v_i)\}_{i\in\mathbb{I}}$ *is an r-fusion frame for* $\mathcal{H}$*.*

*Proof.* Statement (i) can be found in [14, Theorem 4.1]. For statements (ii)-(iv), it is easy to observe the relationships from the given definitions. □

In what follows, we introduce another simple technique to obtain relay fusion frames. As defined in [16], we consider a $(\lambda_i, \mu_i)$-g-frame $\{\Lambda_{ij} \in \mathcal{B}(\mathcal{K}_i, V_{ij}) : j \in \mathbb{J}_i\}$ for each $\mathcal{K}_i$ in a g-frame $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i) : i \in \mathbb{I}\}$, such that

$$0 < \lambda = \inf \lambda_i \leqslant \sup \mu_i = \mu < \infty.$$

In this case we say that $\{\Lambda_{ij} \in \mathcal{B}(\mathcal{K}_i, V_{ij}) : j \in \mathbb{J}_i\}$ is $(\lambda, \mu)$-*bounded* for all $i \in \mathbb{I}$.

The following result is a generalization of Theorem 2.2 of [16].

**Theorem 2.18.** *Let* $S_i \in \mathcal{B}(\mathcal{H}), i \in \mathbb{I}$ *and* $T_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i), i \in \mathbb{I}$*. Let* $\{\Lambda_{ij} \in \mathcal{B}(\mathcal{K}_i, V_{ij}) : j \in \mathbb{J}_i\}$ *be a* $(\lambda_i, \mu_i)$*-g-frame for each* $\mathcal{K}_i$ *and suppose that they are* $(\lambda, \mu)$*-bounded. Then the following conditions are equivalent.*

(i) $\{T_iS_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i) : i \in \mathbb{I}\}$ *is a g-frame for* $\mathcal{H}$ *with respect to* $\{\mathcal{K}_i : i \in \mathbb{I}\}$*.*

(ii) $\{\Lambda_{ij}T_iS_i \in \mathcal{B}(\mathcal{H}, V_{ij}) : i \in \mathbb{I}, j \in \mathbb{J}_i\}$ *is a g-frame for* $\mathcal{H}$ *with respect to* $\{V_{ij} : i \in \mathbb{I}, j \in \mathbb{J}_i\}$*.*

*Proof.* To obtain the second statement from the first, let us assume that $\{T_iS_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i) : i \in \mathbb{I}\}$ is an $(\alpha, \beta)$-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i : i \in \mathbb{I}\}$. Then for all $f \in \mathcal{H}$ we have

$$\sum_{i\in\mathbb{I},j\in\mathbb{J}_i} \|\Lambda_{ij}T_iS_i(f)\|^2 \leqslant \sum_{i\in\mathbb{I}} \mu_i\|T_iS_i(f)\|^2 \leqslant \beta\mu\|f\|^2.$$

Also we have

$$\sum_{i\in\mathbb{I},j\in\mathbb{J}_i}\|\Lambda_{ij}T_iS_i(f)\|^2 \geqslant \sum_{i\in\mathbb{I}}\lambda_i\|T_iS_i(f)\|^2 \geqslant \alpha\lambda\|f\|^2.$$

To derive the first statement from the second, assume that $\{\Lambda_{ij}T_iS_i \in \mathcal{B}(\mathcal{H}, V_{ij}) : i \in \mathbb{I}, j \in \mathbb{J}_i\}$ is an $(\alpha, \beta)$-g-frame for $\mathcal{H}$ with respect to $\{V_{ij} : i \in \mathbb{I}, j \in \mathbb{J}_i\}$. Since $T_iS_i(f) \in \mathcal{K}_i$ for each $f \in \mathcal{H}$, we have

$$\sum_{i\in\mathbb{I}}\|T_iS_i(f)\|^2 \leqslant \sum_{i\in\mathbb{I}}\frac{1}{\lambda_i}\sum_{j\in\mathbb{J}_i}\|\Lambda_{ij}T_iS_i(f)\|^2 \leqslant \frac{\beta}{\lambda}\|f\|^2.$$

Also

$$\sum_{i\in\mathbb{I}}\|T_iS_i(f)\|^2 \geqslant \sum_{i\in\mathbb{I}}\frac{1}{\mu_i}\sum_{j\in\mathbb{J}_i}\|\Lambda_{ij}T_iS_i(f)\|^2 \geqslant \frac{\alpha}{\mu}\|f\|^2.$$

This implies the first statement of the theorem. □

Now we consider (non-orthogonal) fusion frames instead of $(\lambda_i, \mu_i)$-g-frames and we have the following corollary.

**Corollary 2.19.** *Let $\{W_i\}_{i\in\mathbb{I}} \sqsubset \mathcal{H}$ and $T_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i)$, $i \in \mathbb{I}$. Let $\{(V_{ij}, v_{ij})\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ be a $(\lambda_i, \mu_i)$-(non-orthogonal) fusion frame for each $\mathcal{K}_i$ and suppose that they are $(\lambda, \mu)$-bounded. Then the following conditions are equivalent.*

(i) *$\{T_i\pi_{W_i} \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i) : i \in \mathbb{I}\}$ is a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i : i \in \mathbb{I}\}$.*

(ii) *$\{(W_i, V_{ij}, T_i, v_{ij})\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ is a (non-orthogonal) r-fusion frame for $\mathcal{H}$.*

*Moreover, if $\{(V_{ij}, v_{ij})\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ is a Parseval fusion frame for $\mathcal{K}_i$ for all $i \in \mathbb{I}$, then g-frame bound for $\{T_i\pi_{W_i} \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i) : i \in \mathbb{I}\}$ and r-fusion frame bound for $\{(W_i, V_{ij}, T_i, v_{ij})\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ are the same.*

*Proof.* To prove this corollary, just mimic the proof of the Theorem 2.18. □

**Theorem 2.20.** *Let $\{(W_i, w_i)\}_{i\in\mathbb{I}}$ be an $(\alpha, \beta)$-fusion frame for $\mathcal{H}$ and $\{(P_{V_{ij}}, v_{ij})\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ be a $(\lambda_i, \mu_i)$-non-orthogonal fusion frame for each $W_i$ which are $(\lambda, \mu)$-bounded. Then $\{(P_{V_{ij}}\pi_{W_i}, v_{ij}w_i)\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ is an $(\alpha\lambda, \beta\mu)$-non-orthogonal fusion frame for $\mathcal{H}$.*

*Proof.* We first note that

$$P_{V_{ij}}\pi_{W_i}P_{V_{ij}}\pi_{W_i} = P_{V_{ij}}\pi_{W_i}.$$

Therefore, $P_{V_{ij}}\pi_{W_i}$ is a non-orthogonal projection from $\mathcal{H}$ onto $V_{ij}$ for each $i \in \mathbb{I}$, $j \in \mathbb{J}_i$. Let $f$ be an arbitrary element of $\mathcal{H}$. Then we have

$$\sum_{i\in\mathbb{I},j\in\mathbb{J}_i}\left\|v_{ij}w_iP_{V_{ij}}\pi_{W_i}(f)\right\|^2 \geqslant \sum_{i\in\mathbb{I}}\lambda_i\left\|w_i\pi_{W_i}(f)\right\|^2 \geqslant \lambda\sum_{i\in\mathbb{I}}w_i^2\left\|\pi_{W_i}(f)\right\|^2 \geqslant \alpha\lambda\|f\|^2.$$

Similarly we can prove that

$$\sum_{i\in\mathbb{I},j\in\mathbb{J}_i}\left\|v_{ij}w_iP_{V_{ij}}\pi_{W_i}(f)\right\|^2 \leqslant \beta\mu\|f\|^2.$$

□

The following theorem now gives an explicit characterization concerning the existence of Parseval r-fusion frame consisting of two initial subspaces, which also indicates how to construct such special Parseval r-fusion frame. For a characterization of general Parseval r-fusion frame we refer to [15, Theorem 2.15].

**Theorem 2.21.** *Let $W_1, W_2$ be closed non-trivial subspaces of $\mathcal{H}$ and $V_1, V_2$ be closed non-trivial subspaces of $\mathcal{K}$. Suppose that $T_1 : W_1 \mapsto V_1$, $T_2 : W_2 \mapsto V_2$ are unitary operators and $v_1, v_2 > 0$. The following conditions are equivalent.*

(i) *$\{(W_1, V_1, T_1, v_1), (W_2, V_2, T_2, v_2)\}$ is a Parseval r-fusion frame for $\mathcal{H}$.*
(ii) *Either we have $V_1 \perp V_2$ and $v_1 = v_2 = 1$ or we have $W_1 = W_2 = \mathcal{H}$ and $v_1^2 + v_2^2 = 1$.*

*Proof.* (i) ⇒ (ii): First we assume that $W_2 \neq \mathcal{H}$. Fix some $g \perp W_2$. Then, by (i),

$$\begin{aligned}\|g\|^2 &= v_1^2\|\tau_{V_1}T_1\pi_{W_1}(g)\|^2 + v_2^2\|\tau_{V_2}T_2\pi_{W_2}(g)\|^2\\ &= v_1^2\|\pi_{W_1}(g)\|^2 + v_2^2\|\pi_{W_2}(g)\|^2\\ &= v_1^2\|\pi_{W_1}(g)\|^2\\ &\leqslant v_1^2\|g\|^2,\end{aligned}$$

hence $v_1^2 \geqslant 1$. On the other hand for all $f \in W_1$, we have

$$\begin{aligned}\|f\|^2 &= v_1^2\|\tau_{V_1}T_1\pi_{W_1}(f)\|^2 + v_2^2\|\tau_{V_2}T_2\pi_{W_2}(f)\|^2\\ &= v_1^2\|f\|^2 + v_2^2\|\pi_{W_2}(f)\|^2\\ &\geqslant v_1^2\|f\|^2.\end{aligned}$$

This implies $v_1^2 \leqslant 1$ and therefore $v_1 = 1$. Now for all $f \in W_1$,

$$\|f\|^2 = \|f\|^2 + v_2^2\|\pi_{W_2}(f)\|^2.$$

This shows that $W_1 \perp W_2$ and $V_1 \perp V_2$ follows immediately. Now $v_2 = 1$ follows from

$$\|f\|^2 = \|(\pi_{W_1} + \pi_{W_2})f\|^2 = \|\pi_{W_1}(f)\|^2 + v_2^2\|\pi_{W_2}(f)\|^2, \forall f \in \mathcal{H}.$$

If $W_2 = \mathcal{H}$, towards a contradiction assume that $W_1 \neq \mathcal{H}$. Fix $g \perp W_1$. Then

$$\|g\|^2 = v_1^2\|\pi_{W_1}(g)\|^2 + v_2^2\|\pi_{W_2}(g)\|^2 = v_2^2\|g\|^2,$$

hence $v_2^2 = 1$. Now for $g \in W_1$, we obtain

$$\|g\|^2 = v_1^2\|\pi_{W_1}(g)\|^2 + v_2^2\|\pi_{W_2}(g)\|^2 = (v_1^2 + 1)\|g\|^2.$$

But this can only be true if $v_1 = 0$, a contradiction. Thus $W_1 = \mathcal{H}$. Now for all $f \in \mathcal{H}$,

$$\|f\|^2 = v_1^2\|f\|^2 + v_2^2\|\pi_{W_2}(f)\|^2 = (v_1^2 + v_2^2)\|f\|^2,$$

so $v_1^2 + v_2^2 = 1$.
(ii) ⇒ (i): This is obvious. □

The following proposition is an r-fusion frame version of a result due to Asgari and Khosravi [1, Theorem 2.8].

**Proposition 2.22.** *Let $\{M_i\}_{i\in\mathbb{I}} \sqsubset \mathcal{H}$, $S_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i)$, $\{U_{ij}\}_{j\in\mathbb{J}_i} \sqsubset \mathcal{K}_i$ for each $i \in \mathbb{I}$ such that $\{(M_i, U_{ij}, S_i, u_{ij})\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ is r-complete and let $\{(W_i, V_{ij}, T_i, v_{ij})\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ be an $(\alpha, \beta)$-r-fusion frame. If $\Psi : \mathcal{H} \mapsto \mathcal{H}$ define by*

$$\Psi(f) = \sum_{i\in\mathbb{I},j\in\mathbb{J}_i} (u_{ij}^2\pi_{M_i}S_i^*\tau_{U_{ij}}S_i\pi_{M_i} - v_{ij}^2\pi_{W_i}T_i^*\tau_{V_{ij}}T_i\pi_{W_i})(f)$$

*is a compact operator, then $\{(M_i, U_{ij}, S_i, u_{ij})\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ is an r-fusion frame.*

*Proof.* Let $\Phi : \mathcal{H} \mapsto \mathcal{H}$ be an operator defined by $\Phi = S_{\mathcal{R}} + \Psi$. Since $\Psi$ is bounded and self-adjoint, a simple computation shows that $\Phi$ is a bounded and self-adjoint operator. Now for all $f \in \mathcal{H}$ we have

$$\|\Phi(f)\| = \|S_{\mathcal{R}}(f) + \Psi(f)\| \leqslant (\beta + \|\Psi\|)\|f\|.$$

Hence

$$\sum_{i\in\mathbb{I}, j\in\mathbb{J}_i} u_{ij}^2 \|\tau_{U_{ij}} S_i \pi_{M_i}(f)\|^2 = \langle \Phi(f), f\rangle \leqslant (\beta + \|\Psi\|)\|f\|^2. \tag{6}$$

On the other hand, since $\Psi$ is a compact operator, $\Psi S_{\mathcal{R}}^{-1}$ is also a compact operator on $\mathcal{H}$. Thus $\Phi$ has closed range. Now we show that $\Phi$ is injective. If $f \in \mathcal{H}$ and $\Phi(f) = 0$, then

$$\sum_{i\in\mathbb{I}, j\in\mathbb{J}_i} u_{ij}^2 \|\tau_{U_{ij}} S_i \pi_{M_i}(f)\|^2 = \langle \Phi(f), f\rangle = 0.$$

Hence $u_{ij}\tau_{U_{ij}} S_i \pi_{M_i}(f) = 0$ for each $i \in \mathbb{I}, j \in \mathbb{J}_i$. Since $\{(M_i, U_{ij}, S_i, u_{ij})\}_{i\in\mathbb{I}, j\in\mathbb{J}_i}$ is r-complete, we have $f = 0$. Moreover, we have

$$\text{ran }\Phi = (\ker \Phi^*)^\perp = (\ker \Phi)^\perp = \mathcal{H}.$$

Hence $\Phi$ is surjective and thus invertible on $\mathcal{H}$. Now, by using the Cauchy-Schwartz inequality and (6), we compute

$$\begin{aligned}
\|\Phi(f)\|^4 &= \Big(\Big\langle \sum_{i\in\mathbb{I}, j\in\mathbb{J}_i} u_{ij}^2 \pi_{M_i} S_i^* \tau_{U_{ij}} S_i \pi_{M_i}(f), \Phi(f)\Big\rangle\Big)^2\\
&= \Big(\sum_{i\in\mathbb{I}, j\in\mathbb{J}_i} u_{ij}^2 \Big\langle \tau_{U_{ij}} S_i \pi_{M_i}(f), \tau_{U_{ij}} S_i \pi_{M_i}\Phi(f)\Big\rangle\Big)^2\\
&\leqslant \Big(\sum_{i\in\mathbb{I}, j\in\mathbb{J}_i} u_{ij}^2 \Big\|\tau_{U_{ij}} S_i \pi_{M_i}(f)\Big\|\Big\|\tau_{U_{ij}} S_i \pi_{M_i}\Phi(f)\Big\|\Big)^2\\
&\leqslant \Big(\sum_{i\in\mathbb{I}, j\in\mathbb{J}_i} u_{ij}^2 \Big\|\tau_{U_{ij}} S_i \pi_{M_i}(f)\Big\|^2\Big)\Big(\sum_{i\in\mathbb{I}, j\in\mathbb{J}_i} u_{ij}^2 \Big\|\tau_{U_{ij}} S_i \pi_{M_i}\Phi(f)\Big\|^2\Big)\\
&\leqslant (\beta + \|\Psi\|)\|\Phi(f)\|^2 \Big(\sum_{i\in\mathbb{I}, j\in\mathbb{J}_i} u_{ij}^2 \Big\|\tau_{U_{ij}} S_i \pi_{M_i}(f)\Big\|^2\Big).
\end{aligned}$$

Altogether, we obtain

$$\sum_{i\in\mathbb{I}, j\in\mathbb{J}_i} u_{ij}^2 \|\tau_{U_{ij}} S_i \pi_{M_i}(f)\|^2 \geqslant \frac{\|f\|^2}{(\beta + \|\Psi\|)\|\Phi^{-1}\|^2}.$$

□

We close this section with the following example, which shows that sparsity of the fusion frame operator naturally exists by applying the structure of r-fusion frames.

**Example 2.23.** *As we know, the standard fusion frame operator $S_{\mathcal{W}}$ is always non-sparse with an extremely high probability [10]. Whereas, the effectiveness of fusion frame applications in distributed systems is reflected in the efficiency of the end fusion process. This in turn is reflected in the efficiency of the inversion of $S_{\mathcal{W}}$, which in turn is heavily dependent on the sparsity of $S_{\mathcal{W}}$. The lack of sparsity of $S_{\mathcal{W}}$ is a significant hinderance in computing $S_{\mathcal{W}}$ and its inverse, which is necessary to apply the theory. So the central issue in the effective application of fusion frames is to have sparsity for $S_{\mathcal{W}}$—preferably for it to be a diagonal operator. We now show that sparsity of the fusion frame operator naturally exists by applying the structure of r-fusion frames.*

*Let $\mathcal{W} = \{(W_i, v_i)\}_{i\in\mathbb{I}}$ be a fusion frame for $\mathcal{H}$. Then the fusion frame operator for $\mathcal{W}$ is given by*

$$S_{\mathcal{W}} = \sum_{i\in\mathbb{I}} v_i^2 \pi_{W_i}.$$

*Diagonalization of $S_{\mathcal{W}}$ involves a unitary operator $T$ such that*

$$S_{\mathcal{R}} = T^H S_{\mathcal{W}} T = T^H\Big(\sum_{i\in\mathbb{I}} v_i^2 \pi_{W_i}\Big)T = \sum_{i\in\mathbb{I}} v_i^2 T^H(\tilde{X}_i^H X_i)T,$$

*where $X_i$ and $\tilde{X}_i$ are frame matrices with columns being the frame elements $\{w_{ij}\}_j$ and its dual $\{\tilde{w}_{ij}\}_j$, cf. [10, Section 6.2].*

*Obviously, a transformation in the form of $E = X_iT$ would have diagonalized $S_{\mathcal{W}}$. The new frame system $E = X_iT$ will measure signal $f$ through $E = X_iT(f)$. This can be implemented by requiring $T$ acts on $f$ before sensor $X_i$ acts on $f$ (which can be designed so at the sensor manufacturing stage). Concurrently, the new frame system $E = X_iT$ has the structure of r-fusion frames. This is why r-fusion frame is a more natural tool to realize the sparsity of the fusion frame operator. The reader should be aware that subspace transformation is not feasible under the mechanism of standard fusion frames.*

## 3. Bessel r-fusion sequences and alternate dual r-fusion frames

In this section, we consider two Bessel r-fusion sequences $\mathcal{R}_V = \{(W_i, V_{ij}, T_i, v_{ij})\}_{i\in\mathbb{I}, j\in\mathbb{J}_i}$ with Bessel bound $\beta_1$ and $\mathcal{R}_U = \{(M_i, U_{ij}, S_i, u_{ij})\}_{i\in\mathbb{I}, j\in\mathbb{J}_i}$ with Bessel bound $\beta_2$. We introduce the operator

$$S_{VU}(f) = \sum_{i\in\mathbb{I}, j\in\mathbb{J}_i} v_{ij}u_{ij}\pi_{W_i}T_i^*\tau_{V_{ij}}\tau_{U_{ij}}S_i\pi_{M_i}(f),\ \forall f\in\mathcal{H}.$$

By [14], it follows that series converges unconditionally. We have also

$$\langle S_{VU}(f), g\rangle = \sum_{i\in\mathbb{I}, j\in\mathbb{J}_i} v_{ij}u_{ij}\Big\langle \tau_{U_{ij}}S_i\pi_{M_i}(f), \tau_{V_{ij}}T_i\pi_{W_i}(g)\Big\rangle,\ \forall f, g\in\mathcal{H}.$$

By Cauchy-Schwartz inequality, we have

$$|\langle S_{VU}(f), g\rangle| \leqslant \sqrt{\beta_1\beta_2}\|f\|\|g\|,$$

hence $S_{VU}$ is a bounded operator and $\|S_{VU}\| \leqslant \sqrt{\beta_1\beta_2}$. We also note that $S_{VU}^* = S_{UV}$ and $S_{VV} = S_{\mathcal{R}_V}$, where $S_{\mathcal{R}_V}$ is the frame operator for $\mathcal{R}_V$. We say that $\mathcal{R}_U$ is an *alternate dual* for $\mathcal{R}_V$ if we have $S_{VU} = I_{\mathcal{H}}$. Alternate duality is a symmetric relation, and we can say that $\mathcal{R}_V$ and $\mathcal{R}_U$ are alternate dual to each other.

We recall that a family of bounded operator $\{T_i\}_{i\in\mathbb{I}}$ on a Hilbert space $\mathcal{H}$ is called a *resolution of the identity* on $\mathcal{H}$ if we have

$$f = \sum_{i\in\mathbb{I}} T_i(f),\ \forall f\in\mathcal{H},$$

where the series converges unconditionally for all $f\in\mathcal{H}$.

**Theorem 3.1.** *Let $\mathcal{R}_V$ and $\mathcal{R}_U$ be Bessel r-fusion sequences as mentioned above. Then the following are equivalent.*

(i) *$S_{VU}$ is bounded below.*
(ii) *There exists an operator $A\in\mathcal{B}(\mathcal{H})$ such that $\{B_{ij}\}_{i\in\mathbb{I}, j\in\mathbb{J}_i}$ is a resolution of identity, where*

$$B_{ij} = v_{ij}u_{ij}A\pi_{W_i}T_i^*\tau_{V_{ij}}\tau_{U_{ij}}S_i\pi_{M_i},\ i\in\mathbb{I}, j\in\mathbb{J}_i.$$

*Moreover, if one of conditions holds, then both of these Bessel r-fusion sequences are r-fusion frames.*

*Proof.* (i) ⇒ (ii): If $S_{VU}$ is bounded below, then there exists $A \in \mathcal{B}(\mathcal{H})$ such that $AS_{VU} = I_{\mathcal{H}}$. It follows that

$$f = \sum_{i\in\mathbb{I},j\in\mathbb{J}_i} v_{ij}u_{ij}A\pi_{W_i}T_i^*\tau_{V_{ij}}\tau_{U_{ij}}S_i\pi_{M_i}(f),\ \forall f \in \mathcal{H}.$$

(ii) ⇒ (i): If there exists an operator $A \in \mathcal{B}(\mathcal{H})$ such that $\{B_{ij}\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ is a resolution of identity, then for each $f \in \mathcal{H}$ we have

$$f = \sum_{i\in\mathbb{I},j\in\mathbb{J}_i} v_{ij}u_{ij}A\pi_{W_i}T_i^*\tau_{V_{ij}}\tau_{U_{ij}}S_i\pi_{M_i}(f) = AS_{VU}(f).$$

Hence $I_{\mathcal{H}} = AS_{VU}$. It follows that for all $f \in \mathcal{H}$,

$$\|S_{VU}(f)\| \geqslant \frac{1}{\|A\|}\|f\|.$$

For the 'Moreover' part, we assume that $S_{VU}$ is bounded below, then there exists a number $C > 0$ such that for every $f \in \mathcal{H}$, $\|S_{VU}(f)\| \geqslant C\|f\|$. It follows that

$$\begin{aligned} C\|f\| &\leqslant \|S_{VU}(f)\| \\ &= \sup_{g\in\mathcal{H},\|g\|=1}\left|\left\langle \sum_{i\in\mathbb{I},j\in\mathbb{J}_i} v_{ij}u_{ij}\pi_{W_i}T_i^*\tau_{V_{ij}}\tau_{U_{ij}}S_i\pi_{M_i}(f), g\right\rangle\right| \\ &\leqslant \Big(\sum_{i\in\mathbb{I},j\in\mathbb{J}_i} v_{ij}^2\big\|\tau_{V_{ij}}T_i\pi_{W_i}(g)\big\|^2\Big)^{\frac{1}{2}}\Big(\sum_{i\in\mathbb{I},j\in\mathbb{J}_i} u_{ij}^2\big\|\tau_{U_{ij}}S_i\pi_{M_i}(f)\big\|^2\Big)^{\frac{1}{2}} \\ &\leqslant \sqrt{\beta_1}\Big(\sum_{i\in\mathbb{I},j\in\mathbb{J}_i} u_{ij}^2\big\|\tau_{U_{ij}}S_i\pi_{M_i}(f)\big\|^2\Big)^{\frac{1}{2}}. \end{aligned}$$

Hence

$$\frac{C^2}{\beta_1}\|f\|^2 \leqslant \sum_{i\in\mathbb{I},j\in\mathbb{J}_i} u_{ij}^2\big\|\tau_{U_{ij}}S_i\pi_{M_i}(f)\big\|^2.$$

On the other hand, since $S_{VU}^* = S_{UV}$, we can say that $S_{UV}$ is also bounded below. Now similar to the above proof we have

$$\frac{D^2}{\beta_2}\|f\|^2 \leqslant \sum_{i\in\mathbb{I},j\in\mathbb{J}_i} v_{ij}^2\big\|\tau_{V_{ij}}T_i\pi_{W_i}(f)\big\|^2.$$

□

**Theorem 3.2.** *Let $\mathcal{R}_V$ and $\mathcal{R}_U$ be Bessel r-fusion sequences as mentioned above. Assume that there exist $\lambda_1 < 1, \lambda_2 > -1$ such that*

$$\|f - S_{VU}(f)\| \leqslant \lambda_1\|f\| + \lambda_2\|S_{VU}(f)\|,\ \forall f \in \mathcal{H}. \tag{7}$$

*Then both $\mathcal{R}_V$ and $\mathcal{R}_U$ are r-fusion frames for $\mathcal{H}$ and*

$$\Big(\frac{1-\lambda_1}{1+\lambda_2}\Big)^2\frac{1}{\beta_1}\|f\|^2 \leqslant \sum_{i\in\mathbb{I},j\in\mathbb{J}_i} v_{ij}^2\big\|\tau_{V_{ij}}T_i\pi_{W_i}(f)\big\|^2,$$

$$\Big(\frac{1-\lambda_1}{1+\lambda_2}\Big)^2\frac{1}{\beta_2}\|f\|^2 \leqslant \sum_{i\in\mathbb{I},j\in\mathbb{J}_i} u_{ij}^2\big\|\tau_{U_{ij}}S_i\pi_{M_i}(f)\big\|^2.$$

*Proof.* From assumptions,

$$\|f\| - \|S_{VU}(f)\| \leqslant \lambda_1\|f\| + \lambda_2\|S_{VU}(f)\|, \ \forall f \in \mathcal{H}.$$

It follows that

$$\|S_{VU}(f)\| \geqslant \frac{1-\lambda_1}{1+\lambda_2}\|f\|, \ \forall f \in \mathcal{H}.$$

Now using Theorem 3.1 the conclusion follows. □

**Corollary 3.3.** *Let $\mathcal{R}_V$ and $\mathcal{R}_U$ be Bessel r-fusion sequences as mentioned above. Assume that there exist $\lambda_1 < 1$ such that*

$$\|f - S_{VU}(f)\| \leqslant \lambda_1\|f\|, \ \forall f \in \mathcal{H}. \tag{8}$$

*Then both $\mathcal{R}_V$ and $\mathcal{R}_U$ are r-fusion frames for $\mathcal{H}$ and*

$$\frac{(1-\lambda_1)^2}{\beta_1}\|f\|^2 \leqslant \sum_{i\in\mathbb{I}, j\in\mathbb{J}_i} v_{ij}^2 \left\|\tau_{V_{ij}} T_i \pi_{W_i}(f)\right\|^2,$$

$$\frac{(1-\lambda_1)^2}{\beta_2}\|f\|^2 \leqslant \sum_{i\in\mathbb{I}, j\in\mathbb{J}_i} u_{ij}^2 \left\|\tau_{U_{ij}} S_i \pi_{M_i}(f)\right\|^2.$$

*Proof.* It suffices to take $\lambda_2 = 0$ in Theorem 3.2. □

**Theorem 3.4.** *Let $\mathcal{R}_V$ be an $(\alpha, \beta)$-r-fusion frame and let $\mathcal{R}_U$ be a Bessel r-fusion sequence. Suppose that there exists a number $0 < \lambda < \alpha$ such that*

$$\|(S_{UV} - S_{\mathcal{R}_V})f\| \leqslant \lambda\|f\|, \forall f \in \mathcal{H}.$$

*Then both $S_{UV}$ and $S_{VU}$ are invertible and $\{(M_i, U_{ij}, S_i, u_{ij})\}_{i\in\mathbb{I}, j\in\mathbb{J}_i}$ is an r-fusion frame.*

*Proof.* Let $f \in \mathcal{H}$. Then

$$\|S_{UV}(f)\| = \|S_{UV}(f) - S_{\mathcal{R}_V}(f) + S_{\mathcal{R}_V}(f)\| \geqslant \|S_{\mathcal{R}_V}(f)\| - \|S_{UV}(f) - S_{\mathcal{R}_V}(f)\| \geqslant (\alpha - \lambda)\|f\|.$$

Therefore $S_{UV}$ is bounded below and thus injective with closed range. On the other hand, since

$$\|S_{VU} - S_{\mathcal{R}_V}\| = \|(S_{UV} - S_{\mathcal{R}_V})^*\| \leqslant \lambda,$$

we can infer that $S_{VU}$ is also injective with closed range by the above result. Hence both $S_{VU}$ and $S_{UV}$ are invertible. Now by previous Theorem 3.1, $\{(M_i, U_{ij}, S_i, u_{ij})\}_{i\in\mathbb{I}, j\in\mathbb{J}_i}$ is an r-fusion frame. □

Let us consider the case of single relay space again. The following proposition reveals the relationship between spatial complements of r-fusion frames that are alternate dual to each other.

**Proposition 3.5.** *Let $T, S \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\{(W_i, \mathcal{K}, T, v_i)\}_{i\in\mathbb{I}}$ is an r-fusion frame for $\mathcal{H}$ with the associated spatial complement $\{(W_i^\perp, \mathcal{K}, T, v_i)\}_{i\in\mathbb{I}}$ and $\{(M_i, \mathcal{K}, S, u_i)\}_{i\in\mathbb{I}}$ is another r-fusion frame for $\mathcal{H}$ with the associated spatial complement $\{(M_i^\perp, \mathcal{K}, S, u_i)\}_{i\in\mathbb{I}}$. If $\{(M_i, \mathcal{K}, S, u_i)\}_{i\in\mathbb{I}}$ is an alternate dual of $\{(W_i, \mathcal{K}, T, v_i)\}_{i\in\mathbb{I}}$ and $\sum_{i\in\mathbb{I}} v_i u_i < \infty$, then $\{(M_i^\perp, \mathcal{K}, S, u_i)\}_{i\in\mathbb{I}}$ is an alternate dual of $\{(W_i^\perp, \mathcal{K}, T, v_i)\}_{i\in\mathbb{I}}$ if and only if*

$$\sum_{i\in\mathbb{I}} v_i u_i T^* S = \sum_{i\in\mathbb{I}} v_i u_i \pi_{W_i} T^* S + \sum_{i\in\mathbb{I}} v_i u_i T^* S \pi_{M_i},$$

*where the series converges unconditionally.*

*Proof.* Suppose that $\{(M_i,\mathcal{K},S,u_i)\}_{i\in\mathbb{I}}$ is an alternate dual r-fusion frame of $\{(W_i,\mathcal{K},T,v_i)\}_{i\in\mathbb{I}}$. Then we have

$$\sum_{i\in\mathbb{I}} v_iu_i\pi_{W_i}T^*\tau_{\mathcal{K}}\tau_{\mathcal{K}}S\pi_{M_i} = \sum_{i\in\mathbb{I}} v_iu_i\pi_{W_i}T^*S\pi_{M_i} = I_{\mathcal{H}}.$$

Observe that

$$\begin{aligned}&\sum_{i\in\mathbb{I}} v_iu_i\pi_{W_i^\perp}T^*\tau_{\mathcal{K}}\tau_{\mathcal{K}}S\pi_{M_i^\perp}\\ &=\sum_{i\in\mathbb{I}} v_iu_i\pi_{W_i^\perp}T^*S\pi_{M_i^\perp}\\ &=\sum_{i\in\mathbb{I}} v_iu_i(I_{\mathcal{H}}-\pi_{W_i})T^*S(I_{\mathcal{H}}-\pi_{M_i})\\ &=\sum_{i\in\mathbb{I}} v_iu_iT^*S-\sum_{i\in\mathbb{I}} v_iu_i\pi_{W_i}T^*S-\sum_{i\in\mathbb{I}} v_iu_iT^*S\pi_{M_i}+\sum_{i\in\mathbb{I}} v_iu_i\pi_{W_i}T^*S\pi_{M_i}.\end{aligned}$$

It follows that

$$\sum_{i\in\mathbb{I}} v_iu_i\pi_{W_i^\perp}T^*\tau_{\mathcal{K}}\tau_{\mathcal{K}}S\pi_{M_i^\perp} = I_{\mathcal{H}}$$

if and only if

$$\sum_{i\in\mathbb{I}} v_iu_iT^*S = \sum_{i\in\mathbb{I}} v_iu_i\pi_{W_i}T^*S + \sum_{i\in\mathbb{I}} v_iu_iT^*S\pi_{M_i}.$$

The proof is completed. □

We have the following fusion frame version of Proposition 3.5 above.

**Proposition 3.6.** *Let $\{(W_i,v_i)\}_{i\in\mathbb{I}}$ be a fusion frame for $\mathcal{H}$ with the associated spatial complement $\{(W_i^\perp,v_i)\}_{i\in\mathbb{I}}$ and $\{(M_i,u_i)\}_{i\in\mathbb{I}}$ be an r-fusion frame for $\mathcal{H}$ with the associated spatial complement $\{(M_i^\perp,u_i)\}_{i\in\mathbb{I}}$. If $\{(M_i,u_i)\}_{i\in\mathbb{I}}$ is an alternate dual of $\{(W_i,v_i)\}_{i\in\mathbb{I}}$ and $\sum_{i\in\mathbb{I}} v_iu_i<\infty$, then $\{(M_i^\perp,u_i)\}_{i\in\mathbb{I}}$ is an alternate dual of $\{(W_i^\perp,v_i)\}_{i\in\mathbb{I}}$ if and only if*

$$\sum_{i\in\mathbb{I}} v_iu_iI_{\mathcal{H}} = \sum_{i\in\mathbb{I}} v_iu_i\pi_{W_i} + \sum_{i\in\mathbb{I}} v_iu_i\pi_{M_i},$$

*where the series converges unconditionally.*

**Remark 3.7.** *If relay operator $T:\mathcal{H}\mapsto\mathcal{K}$ is an isometry such that $\{(W_i,\mathcal{K},T,v_i)\}_{i\in\mathbb{I}}$ is an r-fusion frame for $\mathcal{H}$ with the associated spatial complement $\{(W_i^\perp,\mathcal{K},T,v_i)\}_{i\in\mathbb{I}}$, then $\{(W_i^\perp,\mathcal{K},T,v_i)\}_{i\in\mathbb{I}}$ can never be an alternate dual of $\{(W_i,\mathcal{K},T,v_i)\}_{i\in\mathbb{I}}$. To appreciate this, we consider equation*

$$\sum_{i\in\mathbb{I}} v_iu_i\pi_{W_i}T^*T\pi_{W_i^\perp} = 0.$$

*Therefore, in general, $M_i$ cannot be taken as $W_i^\perp$ in above propositions.*

Similarly, we have the following propostion.

**Proposition 3.8.** *Let $T,S\in\mathcal{B}(\mathcal{H},\mathcal{K})$ such that $\{(\mathcal{H},V_i,T,v_i)\}_{i\in\mathbb{I}}$ is an r-fusion frame for $\mathcal{H}$ with the associated relay spatial complement $\{(\mathcal{H},V_i^\perp,T,v_i)\}_{i\in\mathbb{I}}$ and $\{(\mathcal{H},U_i,S,u_i)\}_{i\in\mathbb{I}}$ is an r-fusion frame for $\mathcal{H}$ with the associated relay spatial complement $\{(\mathcal{H},U_i^\perp,S,u_i)\}_{i\in\mathbb{I}}$. If $\{(\mathcal{H},V_i,T,v_i)\}_{i\in\mathbb{I}}$ is an alternate dual of $\{(\mathcal{H},U_i,S,u_i)\}_{i\in\mathbb{I}}$ and $\sum_{i\in\mathbb{I}} v_iu_i<\infty$, then $\{(\mathcal{H},V_i^\perp,T,v_i)\}_{i\in\mathbb{I}}$ is an alternate dual of $\{(\mathcal{H},U_i^\perp,S,u_i)\}_{i\in\mathbb{I}}$ if and only if*

$$\sum_{i\in\mathbb{I}} v_iu_iT^*S = \sum_{i\in\mathbb{I}} v_iu_iT^*(\tau_{V_i}+\tau_{U_i})S,$$

*where the series converges unconditionally. In particular, if T and S are invertible operators, then the statement holds if and only if*

$$\sum_{i\in\mathbb{I}} v_iu_iI_{\mathcal{K}} = \sum_{i\in\mathbb{I}} v_iu_i(\tau_{V_i}+\tau_{U_i}).$$

*Proof.* The proof is similar to the proof of the Proposition 3.5. □

We end this paper with the following examples, which provide some specific scenarios of r-fusion frames. We refer the reader to [15] for more examples about r-fusion frames.

**Example 3.9.** *Consider Hilbert space* $\mathcal{H} := \{(x_1, x_2, \cdots, x_N) : x_1, x_2, \cdots, x_N \in \mathbb{R}^N\}$. *Let* $\{e_1, e_2, \cdots, e_N\}$ *and* $\{u_1, u_2, \cdots, u_N\}$ *be two orthonormal bases of* $\mathcal{H}$. *Set*

$$W_1 = span\{e_1, e_2, \cdots, e_{N-1}\}, W_2 = span\{e_2, e_3, \cdots, e_N\}, \cdots, W_N = span\{e_1, \cdots, e_{N-2}, e_N\},$$

*and*

$$V_1 = span\{u_1, u_2, \cdots, u_{N-1}\}, V_2 = span\{u_2, u_3, \cdots, u_N\}, \cdots, V_N = span\{u_1, \cdots, u_{N-2}, u_N\}.$$

*Define* $T_i : \mathcal{H} \mapsto \mathcal{H}$ *for any* $i = 1, 2, \cdots, N$ *such that for each* $f \in \mathcal{H}$,

$$T_1(f) = \sum_{i=1}^{N-1} \langle f, e_i \rangle u_i,\ T_2(f) = \sum_{i=2}^{N} \langle f, e_i \rangle u_i,\ \cdots,\ T_N(f) = \sum_{i=1}^{N-2} \langle f, e_i \rangle u_i + \langle f, e_N \rangle u_N.$$

*Assume that* $v_1 = v_2 = \cdots = v_N = 1$. *Then we get*

$$\sum_{i=1}^{N} v_i^2 \|\tau_{V_i} T_i \pi_{W_i}(f)\|^2 = \sum_{i=1}^{N} v_i^2 \|T_i \pi_{W_i}(f)\|^2 = (N-1)\|f\|^2,\ \forall f \in \mathcal{H}. \tag{9}$$

*Hence,* $\{(W_i, V_i, T_i, v_i)\}_{i \in \{1,2,\cdots,N\}}$ *is an (N-1)-tight r-fusion frame for* $\mathcal{H}$.

**Example 3.10.** *Assume that* $\{e_i\}_{i=1}^{\infty}$ *is an orthonormal basis of Hilbert space* $\mathcal{H}$ *and* $\{u_j\}_{j=1}^{\infty}$ *is an orthonormal basis of Hilbert space* $\mathcal{K}$. *Fix number* $M \in \mathbb{N}$ *and define*

$$W_i = span\{e_1, \cdots, e_i\}, 1 \leqslant i \leqslant M;$$
$$W_i = span\{e_{i-M+1}, \cdots, e_i\}, i \geqslant M.$$

*Let* $T$ *be an arbitrary isometry operator from* $\mathcal{H}$ *into* $\mathcal{K}$. *Then* $\{(W_i, \mathcal{K}, T, 1)\}_{i=1}^{\infty}$ *is an M-tight r-fusion frame for* $\mathcal{H}$.

**Example 3.11.** *Let* $\mathcal{H}$ *and* $\mathcal{K}$ *be two separable Hilbert spaces such that* $\mathcal{H} = W_1 \oplus W_2$ *and* $\mathcal{K} = V_1 \oplus V_2$. *Here we denote by* $\oplus$ *the direct sum of orthogonal subspaces. Let relay operator* $T$ *be a unitary operator from* $\mathcal{H}$ *onto* $\mathcal{K}$ *so that* $TW_1 = V_1$ *and* $TW_2 = V_2$. *Assume that* $v_1 = v_2 = 1$. *Then*

$$\{(W_i, V_i, T, v_i)\}_{i \in \{1,2\}};\ \{(\mathcal{H}, V_i, T, v_i)\}_{i \in \{1,2\}};\ \{(W_i, \mathcal{K}, T, v_i)\}_{i \in \{1,2\}};$$
$$\{(W_i^{\perp}, \mathcal{K}, T, v_i)\}_{i \in \{1,2\}};\ \{(\mathcal{H}, V_i^{\perp}, T, v_i)\}_{i \in \{1,2\}};\ \{(W_i^{\perp}, V_i^{\perp}, T, v_i)\}_{i \in \{1,2\}}$$

*are all Parseval r-fusion frames for* $\mathcal{H}$. *Further,*
(a) *R-fusion frames* $\{(W_i^{\perp}, \mathcal{K}, T, v_i)\}_{i \in \{1,2\}}$ *and* $\{(W_i, \mathcal{K}, T, v_i)\}_{i \in \{1,2\}}$ *are spatial complements to each other.*
(b) *R-fusion frames* $\{(\mathcal{H}, V_i, T, v_i)\}_{i \in \{1,2\}}$ *and* $\{(\mathcal{H}, V_i^{\perp}, T, v_i)\}_{i \in \{1,2\}}$ *are relay spatial complements to each other.*
(c) *R-fusion frames* $\{(W_i, V_i, T, v_i)\}_{i \in \{1,2\}}$ *and* $\{(W_i^{\perp}, V_i^{\perp}, T, v_i)\}_{i \in \{1,2\}}$ *are dual spatial complements to each other.*

**Acknowledgment**

Both authors thank the reviewer for useful suggestions that improved the presentation.

## References


[1] M. S. Asgari, A. Khosravi, *Frames and bases of subspaces in Hilbert spaces*, J. Math. Anal. Appl., **308** (2) (2005) 541-553.
[2] J. Benedetto, A. Powell, O. Yilmaz, *Sigma-Delta quantization and finite frames*, IEEE Trans. Inform. Theory, **52** (2006) 1990-2005.
[3] H. Bölcskel, F. Hlawatsch, H. G. Feichtinger, *Frame-theoretic analysis of oversampled filter banks*, IEEE Trans. Signal Process., **46** (1998), 3256-3268.
[4] R. Calderbank, P. G. Casazza, A. Heinecke, G. Kutyniok, A. Pezeshki, *Sparse fusion frames: existence and construction*, Adv. Comput. Math., **35** (1) (2011) 1-31.
[5] E. J. Candès, D. L. Donoho, *New tight frames of curvelets and optimal representations of objects with piecewise $C^2$ singularities*, Comm. Pure Appl. Math., **56** (2004) 216-266.
[6] P. G. Casazza, G. Kutyniok, *Frames of subspaces.* Contemp. Math., **345** (2004) 87-114.
[7] P. G. Casazza, G. Kutyniok, S. Li, *Fusion frames and distributed processing.* Appl. Comput. Harmon. Anal., **25** (2008) 114-132.
[8] P. G. Casazza, G. Kutyniok, S. Li, C. Rozell, *Modeling sensor networks with fusion frames*, In: Proceed. of SPIE Wavelets XII, San Diego, vol. **6701**, p. 670111910 (2007).
[9] P. G. Casazza, M. Fickus, D. Mixon, Y. Wang, Z. Zhou, *Constructing tight fusion frames*, Appl. Comput. Harmon. Anal., **30** (2) (2011) 175-187.
[10] J. Cahill, P. G. Casazza, S. Li, *Non-orthogonal fusion frames and the sparsity of fusion frame operators.*, J. Fourier Anal. Appl., **18** (2012) 287-308.
[11] O. Christensen, *An introduction to frames and riesz bases, Applied and Numerical Harmonic Analysis*, Birkhäuser, 2016.
[12] P. Găvruta, *On the duality of fusion frames*, J. Math. Anal. Appl., **333** (2) (2007) 871-879.
[13] R. W. Heath, A. J. Paulraj, *Linear dispersion codes for MIMO systems based on frame theory,* IEEE Trans. Signal Process., **50** (2002) 2429-2441.
[14] G. Q. Hong, P. T. Li, *Relay fusion frames for Hilbert spaces*, J. Inequal. Appl, **226** (2019) 23 pp.
[15] G. Q. Hong, P. T. Li, *Relay fusion frames and bridging results for fusion frames*, J. Math. Anal. Appl., **489** (1) (2020) 124124, 20 pp.
[16] A. Khosravi, K. Musazadeh, *Fusion frames and g-frames*, J. Math. Anal. Appl., **342** (2008) 1068-1083.
[17] W. Sun, *G-frames and G-Riesz bases.* J. Math. Anal. Appl., **322** (2006) 437-452.