Filomat 38:6 (2024), 1947–1961
https://doi.org/10.2298/FIL2406947Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A pseudospectral method for continuous-time nonlinear fractional programming

Yin Yang[a], **M. H. Noori Skandari**[b,*], **Jiaqi Zhang**[a]

[a]*Key Labroratory for Computation and Simulation in Science and Engineering, School of Mathematics and Computational Science, Xiangtan University, Xiangtan, China.*
[b]*Faculty of Mathematical Sciences, Shahrood University of Technology, Shahrood, Iran.*

Abstract. In this paper, we focus on the continuous-time nonlinear fractional programming problems including the objective functional given by the ratio of two integrals. Since the standard continuous-time programming theory, such as optimal control theory, cannot be used directly to solve this type of problems, we propose a new numerical method. At first we convert the original problem into an equivalent continuous-time nonfractional problem which does not include integral term. Then, we utilize a Legendre pseudospectral method to discretize the gained problem. We also analyze the feasibility of the obtained discretized problem and the convergence of the method. Finally, we provide two numerical examples to demonstrate the efficiency and capability of the method.

1. Introduction

The optimization problem in which the objective function appears as a ratio of two real-valued functions is known as a fractional programming (FP) problem. Some applications of this type of optimization problem can be found in the information theory, stochastic programming, decomposition algorithm for large linear systems, etc [13, 23, 29, 35, 36]. Several theoretical and computational issue, related to the fractional programming, have presented in the last decades. Zalmai [48–51] investigated the continuous-time FP (CTFP) problems. Stancu-Minsion and Tigon [37] studied the stochastic CT linear FP (CTLFP) problems. They showed that, under some conditions, the CTLFP problems are equivalent to deterministic CTLFP problem. Wen et al. [38] used the Charnes and Cooper's transformations to develop a numerical algorithm for solving a class of CTLFP problems. Also, Wen and Wu [39] and Wen [40–42] have developed computational procedures by combining the parametric method and discrete approximation method to solve some classes of CTLFP. Moreover, Lur et al. [24] presented a hybrid of the parametric method and discretization approach for a class of CT quadratic FP problems. To study more related works, we refer the reader to [7, 44] and references therein.
In addition to the above-mentioned theoretical and computational methods, some researchers have attempted to solve the CTFP problems by using Dinkelbach approach [5, 6]. By Dinkelbach approach the

2020 *Mathematics Subject Classification*. 49MXX; 65K05; 90C30.
Keywords. Continuous-time fractional programming, Legendre pseudospectral method, Convergence analysis.
Received: 20 May 2019; Revised: 15 October 2023; Accepted: 02 November 2023
Communicated by Dragan S. Djordjević
* Corresponding author: M. H. Noori Skandari
Email addresses: yangyinxtu@xtu.edu.cn (Yin Yang), math.noori@yahoo.com (M. H. Noori Skandari), zjqss1992@163.com (Jiaqi Zhang)

CTFP is transformed into an equivalent family of problems which the ratio disappears and the objective function is given by the weighted difference of the numerator and the denominator of the ratio. For example, in [3] a Dinkelbach-type algorithm is suggested to solve a class of CTLFP problems. In [43] a method is given based on the Dinkelback approach for solving a special class of the CT fractional programming problems, including affine integrals and linear dynamics. Enkhbat and Zhou [33] presented a global optimization appreach for fractional optimal control problems based on the Dinkhelbach algorithm.
Despite the existence of some methods for special classes of CTFP problems, it seems that there does not exist a practical powerful method to solve the CT nonlinear FP (CTNFP) problems. Hence, the main purpose of this paper is to develop a pseudospectral method to solve CT fractional programming problems. The pseudospectral methods [2, 8, 10–12, 15, 16, 18, 26] are one of the best numerical methods to solve CT problems, since they have high accuracy (or exponential convergence) and easily application. Some new applications of these methods are suggested in the works [17, 19–22, 25, 27, 28, 30, 31, 45–47]. Before applying pseudospectral method, we present a new technique to convert the CTNLFP problem into a nonfractional one.
The paper is organized as follows. In Section 2, we consider a CTNFP problem. In Section 3, we suggest a technique to convert the CTNFP problem into an equivalent CT nonfractional programming problem. In Section 4, we suggest a Legendre pseudospectral method to discretize the obtained CT nonfractional programming problem. In Section 5, we analyze the feasibility of discretized problem. In Section 6, we give the convergence analysis of the method. In Section 7, the conclusions suggestions and are given. In Section 8, we solve two numerical examples to show the efficiency of method.

2. Problem statement

In this paper, we centralize on the following class of CTNFP problems

$$\text{Minimize} \quad J(x,u) = \frac{\int_0^T f\Big(t, x(t), u(t)\Big)dt}{\int_0^T g\Big(t, x(t), u(t)\Big)dt} \tag{1}$$

$$\text{subject to} \quad \dot{x}(t) = h\Big(t, x(t), u(t)\Big), \quad 0 \le t \le T, \tag{2}$$

$$x(t_0) = \alpha, \tag{3}$$

where $x : [0, T] \to \mathbb{R}^n$ and $u : [0, T] \to \mathbb{R}^m$ are the state and control variables, respectively, and $\alpha \in \mathbb{R}^n$. Also, we assume that functions $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $g : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $h : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are nonlinear, continuous and they have Lipschitz property on set $\Omega = [0, T] \times \mathbb{R}^n \times \mathbb{R}^m$. Moreover, we assume that the denominator of the fraction in objective functional is strictly positive, i.e,

$$\int_0^T g\Big(t, x(t), u(t)\Big)dt \ge \varepsilon > 0. \tag{4}$$

We suppose that the CTNFP problem (1)-(3) has at least one optimal solution. Up to now, many works are done on the CTNFP problem (1)-(3) and the most of them are based on the following CT nonfractional programming problem

$$\text{Minimize} \quad J_p(x,u) = \int_0^T \Big(f(t,x,u) - pg(t,x,u)\Big)dt \tag{5}$$

$$\text{subject to} \quad \dot{x}(t) = h(t,x,u), \quad 0 \le t \le T, \tag{6}$$

$$x(0) = \alpha, \tag{7}$$

where p is a parameter. Define $\Omega = \{(x,u) : \dot{x} = h(t,x,u), x(0) = \alpha\}$ and

$$F(p) = \underset{(x,u)\in\Omega}{\text{Minimum}} \; J_p(x,u). \tag{8}$$

It can be proved that for any $p \in \mathbb{R}$, there is an optimal solution for the problem (5)-(7). Some properties of function $F(.)$ defined by (8) are given below.

Lemma 2.1. *(see [23]) Function F defined by* (8) *satisfies the following properties:*
i) The function F is concave on $\mathbb{R}$.
ii) The function F is strictly decreasing i.e., for any p_1 *and* p_2 *in* $\mathbb{R}$, $p_1 < p_2$ *implies* $F(p_1) > F(p_2)$.
iii) Equation $F(p) = 0$ *has an unique real solution.*
iv) For any $(x, u) \in \Omega$, *if*

$$p = \frac{\int_0^T f(t, x, u)dt}{\int_0^T g(t, x, u)dt},$$

Then $F(p) \leq 0$.

The relation between function $F(.)$ and optimal solution of the original problem (1)-(3) is given in the following theorem [23].

Theorem 2.2. *If* $F(p^*) = J_{p^*}(x^*, u^*) = 0$. *Then*

$$p^* = \frac{\int_0^T f(t, x^*, u^*)dt}{\int_0^T g(t, x^*, u^*)dt} = \underset{(x,u)\in\Omega}{\text{Minimum}} \ \frac{\int_0^T f(t, x, u)dt}{\int_0^T g(t, x, u)dt}$$

and (x^*, u^*) *is an optimal solution of problem* (1)-(3).

By Theorem 1, solving problem (1)-(3) is equivalent to determining the root of the equation $F(p) = 0$, where $F(.)$ is defined by (8). Hence several researchers attempted to solve CTNFP problem (1)-(3) based on the function $F(.)$. But, the more works are for special class of CTNFP problems, such as problems with linear or affine integrals and linear dynamics with respect to the state and control variables. Moreover, the presented methods at previous, are dependent on the parameter p. Also, the standard theory of CT optimization (or optimal control theory) cannot directly be used to solve CTNFP problems. Hence, in this paper, we apply a pseudospectral method for solving the CTNFP problem (1)-(3), which is one of the best numerical methods for solving CT problems. Here, we first show that the CTNFP problem (1)-(3) can be converted into an equivalent nonfractional problem and the pseudospectral method can be applied to solve it, numerically. We then will analyze the convergence of the method.

3. Converting the CTNFP problem into an equivalent nonfractional form

We define the new state variables $y(.)$ and $z(.)$ for CTNFP problem (1)-(3) as follows:

$$y(t) = \int_0^t f(t, x, u)dt, \quad z(t) = \int_0^t g(t, x, u)dt, \quad 0 \leq t \leq T. \tag{9}$$

By (9), we get

$$\begin{cases} \dot{y}(t) = f(t, x, u), & y(0) = 0, \\ \dot{z}(t) = g(t, x, u), & z(0) = 0. \end{cases}$$

Hence, we can write the CTNFP problem (1)-(3) as the following equivalent problem

$$\text{Minimize} \quad J = \frac{y(T)}{z(T)}, \tag{10}$$

$$\text{subject to} \quad \dot{x} = h\big(t, x, u\big), \quad 0 \leq t \leq T, \tag{11}$$

$$\dot{y} = f\big(t, x, u\big), \quad 0 \leq t \leq T, \tag{12}$$

$$\dot{z} = g\big(t, x, u\big), \quad 0 \leq t \leq T. \tag{13}$$

$$x(0) = x_0, \quad y(0) = 0, \quad z(0) = 0 \tag{14}$$

Now, define $\lambda = \frac{y(T)}{z(T)}$, where $z(T) > 0$. By this we can achieve the following CT nonlinear nonfractional programming problem:

$$\text{Minimize} \quad J = \lambda \tag{15}$$

$$\text{subject to} \quad \dot{x} = h\Big(t, x, u\Big), \quad 0 \le t \le T, \tag{16}$$

$$\dot{y} = f\Big(t, x, u\Big), \quad 0 \le t \le T, \tag{17}$$

$$\dot{z} = g\Big(t, x, u\Big), \quad 0 \le t \le T, \tag{18}$$

$$y(T) - \lambda z(T) = 0, \tag{19}$$

$$x(0) = \alpha, \; y(0) = 0, \; z(0) = 0. \tag{20}$$

Theorem 3.1. *If* $(x^*, y^*, z^*, u^*, \lambda^*)$ *is an optimal solution for the nonfractional problem* (15)-(20)*, then* (x^*, u^*) *is an optimal solution for the fractional problem* (1)-(3).

Proof. Assume that (x^*, u^*) is not optimal for the CTNFP problem (1)-(3). So there exists a feasible solution $(\bar{x}, \bar{u}) \in \Omega$ for problem (1)-(3) such that $J(x^*, u^*) > J(\bar{x}, \bar{u})$. Now, define

$$\bar{y}(t) = \int_0^t f(s, \bar{x}, \bar{u}) ds, \quad \bar{z}(t) = \int_0^t g(s, \bar{x}, \bar{u}) ds, \quad \bar{\lambda} = \frac{\bar{y}(T)}{\bar{z}(T)}.$$

So we get

$$\dot{\bar{y}} = f(t, \bar{x}, \bar{u}), \quad \dot{\bar{z}}(t) = g(t, \bar{x}, \bar{u}), \quad \bar{y}(0) = 0, \quad \bar{z}(0) = 0.$$

Hence, $(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{\lambda})$ is a feasible solution for problem (15)-(20) and moreover,

$$\lambda^* = \frac{y^*(T)}{z^*(T)} = J(x^*, u^*) > J(\bar{x}, \bar{u}) = \frac{\bar{y}(T)}{\bar{z}(T)} = \bar{\lambda},$$

which is a contradiction with optimality of $(x^*, y^*, z^*, u^*, \lambda^*)$ for problem (15)-(20). □

In next section, we extend a Legendre pseudospectral method to discretize the nonfractional problem (15)-(20).

4. Legendre pseudospectral method

Before applying Legendre pseudospectral method, we utilize the time transformation $t = \frac{T}{2}(\tau + 1), -1 \le \tau \le 1$ and define

$$X(\tau) = x\left(\frac{T}{2}(\tau + 1)\right), \quad Y(\tau) = y\left(\frac{T}{2}(\tau + 1)\right),$$
$$Z(\tau) = z\left(\frac{T}{2}(\tau + 1)\right), \quad U(\tau) = u\left(\frac{T}{2}(\tau + 1)\right).$$

By these definitions, the problem (15)-(20) converted into the following problem

$$\text{Minimize} \quad J = \lambda \tag{21}$$

$$\text{subject to} \quad \dot{X}(\tau) = H\Big(\tau, X(\tau), U(\tau)\Big), \quad -1 \le \tau \le 1, \tag{22}$$

$$\dot{Y}(\tau) = F\Big(\tau, X(\tau), U(\tau)\Big), \quad -1 \le \tau \le 1, \tag{23}$$

$$\dot{Z}(\tau) = G\Big(\tau, X(\tau), U(\tau)\Big), \quad -1 \le \tau \le 1, \tag{24}$$

$$Y(1) - \lambda Z(1) = 0, \tag{25}$$

$$X(-1) = \alpha, \; Y(-1) = 0, \; Z(-1) = 0, \tag{26}$$

where

$$H\Big(\tau, X(\tau), U(\tau)\Big) = \frac{T}{2}h\Big(\frac{T}{2}(\tau+1), x(\frac{T}{2}(\tau+1)), u(\frac{T}{2}(\tau+1))\Big),$$
$$F\Big(\tau, X(\tau), U(\tau)\Big) = \frac{T}{2}f\Big(\frac{T}{2}(\tau+1), x(\frac{T}{2}(\tau+1)), u(\frac{T}{2}(\tau+1))\Big),$$
$$G\Big(\tau, X(\tau), U(\tau)\Big) = \frac{T}{2}g\Big(\frac{T}{2}(\tau+1), x(\frac{T}{2}(\tau+1)), u(\frac{T}{2}(\tau+1))\Big).$$

Now, assume that $p_j(.)$ is the Legendre polynomial of degree j. This polynomial can be obtained with the following recurrence formula

$$\begin{cases} p_{j+1}(\tau) = \dfrac{2j+1}{j+1}\tau p_j(\tau) - \dfrac{j}{j+1}p_{j-1}(\tau), \quad -1 \le \tau \le 1 \\ p_0(\tau) = 1, \quad p_1(\tau) = \tau. \end{cases}$$

Suppose that $\{\tau_k\}_{k=0}^{N}$ are the roots of polynomial $(1-\tau^2)p'_N(\tau)$. These roots are called the Legendre-Gauss-Lobato points or nodes. For interpolating the variables of problem (21)-(26) we need the Lagrange polynomial

$$L_i(\tau) = \prod_{j=0, i\neq N}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad -1 \le \tau \le 1.$$

Now, we approximate the optimal solutions of the problem (21)-(26) by the Lagrange interpolations as follow

$$\begin{cases} X(\tau) \approx X_N(\tau) = \displaystyle\sum_{l=0}^{N} \bar{x}_l L_l(\tau), \quad U(\tau) \approx U_N(\tau) = \sum_{l=0}^{N} \bar{u}_l L_l(\tau), \\ Y(\tau) \approx Y_N(\tau) = \displaystyle\sum_{l=0}^{N} \bar{y}_l L_l(\tau), \quad Z(\tau) \approx Z_N(\tau) = \sum_{l=0}^{N} \bar{z}_l L_l(\tau), \end{cases} \tag{27}$$

where $(\bar{x}_l, \bar{y}_l, \bar{z}_l, \bar{u}_l)$ for $l = 0, 1, ..., N$ are unknown coefficients. Since $L_k(\tau_k) = 1, k = 0, 1, ..., N$ and $L_k(\tau_j) = 0$, for all $j \neq k$, we have (for all $k = 0, 1, ..., N$)

$$\begin{cases} X(\tau_k) \approx X_N(\tau_k) = \bar{x}_k, \quad U(\tau_k) \approx U_N(\tau_k) = \bar{u}_k, \\ Y(\tau_k) \approx Y_N(\tau_k) = \bar{y}_k, \quad Z(\tau_k) \approx Z_N(\tau_k) = \bar{z}_k, \end{cases} \tag{28}$$

Also,

$$\begin{cases} X'(\tau_k) \approx X'_N(\tau_k) = \displaystyle\sum_{l=0}^{N} \bar{x}_l D_{kl}, \quad Y'(\tau_k) \approx Y'_N(\tau_k) = \sum_{l=0}^{N} \bar{y}_l D_{kl}, \\ Z'(\tau_k) \approx Z'_N(\tau_k) = \displaystyle\sum_{l=0}^{N} \bar{z}_l D_{kl}, \quad k = 0, 1, ..., N, \end{cases} \tag{29}$$

where

$$D_{kl} = L'_l(\tau_k) \begin{cases} -\dfrac{N(N+1)}{4}, & k = l = 0, \\ \dfrac{p_N(\tau_k)}{(\tau_k - \tau_l)p_N(\tau_l)}, & k \neq l, \quad 0 \le k, l \le N, \\ \dfrac{N(N+1)}{4}, & k = l = N, \\ 0, & 1 \le k = l \le N-1, \end{cases} \tag{30}$$

and $D = (D_{kl})$ is called matrix differentiation. Using (27), (28) and (29), we can approximate the CT problem (21)-(26) by the following discrete-time (DT) problem:

$$\text{Minimize} \quad J_N = \lambda \tag{31}$$

$$\text{subject to} \quad \sum_{l=0}^{N} \bar{x}_l D_{kl} = H\Big(\tau_k, \bar{x}_k, \bar{u}_k\Big), \quad k = 0, 1, ..., N \tag{32}$$

$$\sum_{l=0}^{N} \bar{y}_l D_{kl} = F\Big(\tau_k, \bar{x}_k, \bar{u}_k\Big), \quad k = 0, 1, ..., N \tag{33}$$

$$\sum_{l=0}^{N} \bar{z}_l D_{kl} = G\Big(\tau_k, \bar{x}_k, \bar{u}_k\Big), \quad k = 0, 1, ..., N \tag{34}$$

$$\bar{y}_N - \lambda \bar{z}_N = 0, \ \bar{x}_0 = \alpha, \ \bar{y}_0 = 0, \ \bar{z}_0 = 0 \tag{35}$$

where $(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \lambda) = (\bar{x}_0, ..., \bar{x}_N, \bar{y}_0, ..., \bar{y}_N, \bar{z}_0, ..., \bar{z}_N, \bar{u}_0..., \bar{u}_N, \lambda)$ is the variable of the problem. In next section, we analyze the feasibility of DT problem (31)-(35).

5. Analyzing the feasibility of obtained discrete-time problem

Assume that $W^{m,p}(m \geq 2, p \geq 1)$ is Sobolov space on $[-1, 1]$, that consists of all functions $\varphi : [-1, 1] \to \mathbb{R}^n$ such that $\varphi^{(j)}(.)$, $j = 0, 1, 2, ..., m$ lie in L^p space with the following norm

$$\|\phi\|_{W^{m,p}} = \sum_{j=1}^{m} \left(\int_{-1}^{1} \|\varphi^{(j)}(\tau)\|_p^p dt \right)^{\frac{1}{p}}.$$

In this section, we require the following Lemma.

Lemma 5.1. *(see[4]) For any given function $\varphi \in W^{m,p}$ on $[-1, 1]$ there is a polynomial $p_N(.)$ of degree N or less such that*

$$\|\phi(\tau) - p_N(\tau)\|_\infty \leq c c_0 N^{-m}, \quad -1 \leq \tau \leq 1,$$

where c is a constant independent of N and $c_0 = \|\varphi\|_{W^{m,p}}$.

Now, to guarantee the feasibility of DT problem (31)-(35), we relax its constraints and rewrite the problem as follows:

$$\text{Minimize} \quad J_N = \lambda \tag{36}$$

$$\text{subject to} \quad \left\| \sum_{l=0}^{N} \bar{x}_l D_{kl} - H(\tau_k, \bar{x}_k, \bar{u}_k) \right\|_\infty \leq (N-1)^{\frac{3}{2}-m}, \quad k = 0, 1, ..., N, \tag{37}$$

$$\left| \sum_{l=0}^{N} \bar{y}_l D_{kl} - F(\tau_k, \bar{x}_k, \bar{u}_k) \right| \leq (N-1)^{\frac{3}{2}-m}, \quad k = 0, 1, ..., N, \tag{38}$$

$$\left| \sum_{l=0}^{N} \bar{z}_l D_{kl} - G(\tau_k, \bar{x}_k, \bar{u}_k) \right| \leq (N-1)^{\frac{3}{2}-m}, \quad k = 0, 1, ..., N, \tag{39}$$

$$\Big|\bar{y}_N - \lambda \bar{z}_N\Big| \leq (N-1)^{\frac{3}{2}-m}, \quad \|\bar{x}_0 - \alpha\| \leq (N-1)^{\frac{3}{2}-m}, \tag{40}$$

$$\Big|\bar{y}_0\Big| \leq (N-1)^{\frac{3}{2}-m}, \quad |\bar{z}_0| \leq (N-1)^{\frac{3}{2}-m}, \tag{41}$$

where $m \geq 2$. The above relaxation is based on the Polak's theory of consistent approximation [32]. We note that when N tends to infinite, there is no different between constraints of problems (31)-(35) and (36)-(41).

Remark 5.2. *Since functions F, G and H are Lipschitz property, there are constants M_1, M_2 and M_3 such that for all $(\overline{X}, \overline{U})$ and $(\widetilde{X}, \widetilde{U})$ we have*

$$
\begin{aligned}
|F(t, \bar{X}, \bar{U}) - F(t, \widetilde{X}, \widetilde{U})| &\le M_1(\|\bar{X} - \widetilde{X}\| + \|\bar{U} - \widetilde{U}\|)\\
|G(t, \bar{X}, \bar{U}) - G(t, \widetilde{X}, \widetilde{U})| &\le M_2(\|\bar{X} - \widetilde{X}\| + \|\bar{U} - \widetilde{U}\|)\\
|H(t, \bar{X}, \bar{U}) - H(t, \widetilde{X}, \widetilde{U})| &\le M_3(\|\bar{X} - \widetilde{X}\| + \|\bar{U} - \widetilde{U}\|)
\end{aligned}
$$

Theorem 5.3. *(Feasibility) Let $(X^*, Y^*, Z^*, U^*, \lambda^*)$ be an optimal solution for the CT problem* (21)-(26). *Then there exists a positive integer K such that for any $N \ge K$, the DT problem* (36)-(41) *has a feasible solution $(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{\lambda}) = (\bar{x}_0, ..., \bar{x}_N, \bar{y}_0, ..., \bar{y}_N, \bar{z}_0, ..., \bar{z}_N, \bar{u}_0 ..., \bar{u}_N, \bar{\lambda})$ such that $\bar{u}_k = U^*(\tau_k)$ and*

$$\|X^*(\tau_k) - \bar{x}_k\|_\infty \le L_1(N-1)^{1-m}, \quad k = 0, 1, 2..., N, \tag{42}$$

$$|Y^*(\tau_k) - \bar{y}_k| \le L_2(N-1)^{1-m}, \quad k = 0, 1, 2..., N, \tag{43}$$

$$|Z^*(\tau_k) - \bar{z}_k| \le L_3(N-1)^{1-m}, \quad k = 0, 1, 2..., N, \tag{44}$$

where L_1, L_2 and L_3 are positive constants independent of N.

Proof. Let $P(.), Q(.)$ and $R(.)$ be $(N-1)$th order best polynomial approximations of $\dot{X}^*(.), \dot{Y}^*(.)$ and $\dot{Z}^*(.)$, respectively. By Lemma 5, there are the constants c_1, c_2 and c_3 independent of N such that

$$\|\dot{X}^*(\tau) - P(\tau)\|_\infty \le c_1(N-1)^{1-m}, \tag{45}$$

$$|\dot{Y}^*(\tau) - Q(\tau)| \le c_2(N-1)^{1-m}, \tag{46}$$

$$|\dot{Z}^*(\tau) - R(\tau)| \le c_3(N-1)^{1-m}. \tag{47}$$

Define

$$X_N(\tau) = X^*(-1) + \int_{-1}^{\tau} P(s)ds,\ Y_N(\tau) = \int_{-1}^{\tau} Q(s)ds,\ Z_N(\tau) = \int_{-1}^{\tau} R(s)ds, \tag{48}$$

$$\bar{\lambda} = \lambda^*,\ \bar{x}_k = X_N(\tau_k),\ \bar{y}_k = Y_N(\tau_k),\ \bar{z}_k = Z_N(\tau_k),\ \bar{u}_k = U_N(\tau_k),\ k = 0, 1, ..., N. \tag{49}$$

We first show that $(\bar{x}, \bar{u})$ satisfies relation (37). By (45), (48) and (49) we get (for all $\tau \in [-1, 1]$)

$$
\begin{aligned}
\|X^*(\tau) - X^N(\tau)\|_\infty &= \left\| \int_{-1}^{\tau} (\dot{X}^*(s) - P(s))ds \right\|_\infty \le \int_{-1}^{\tau} \left\| (\dot{X}^*(s) - P(s)) \right\|_\infty ds\\
&\le c_1(N-1)^{1-m} \int_{-1}^{\tau} ds \le 2c_1(N-1)^{1-m}.
\end{aligned} \tag{50}
$$

By a similar procedure, we have

$$|Y^*(\tau) - Y^N(\tau)| \le 2c_2(N-1)^{1-m}, \tag{51}$$

$$|Z^*(\tau) - Z^N(\tau)| \le 2c_3(N-1)^{1-m}. \tag{52}$$

Moreover, since $x^*(-1) = \alpha$, we get

$$\|\bar{x}_0 - \alpha\| - \|X^*(-1) - \alpha\| \le \|(X^*(-1) - \alpha) - (\bar{x}_0 - \alpha)\| = \|X^*(-1) - \bar{x}_0\| \le 2c_1(N-1)^{1-m}.$$

So

$$\|\bar{x}_0 - \alpha\| \le \|X^*(-1) - \alpha\| + 2c_1(N-1)^{1-m} = 2c_1(N-1)^{1-m}. \tag{53}$$

Similarly, we can show that

$$|\bar{y}_0| \le 2c_2(N-1)^{1-m}, \quad |\bar{z}_0| \le 2c_3(N-1)^{1-m}. \tag{54}$$

Also, since $\bar{\lambda} = \lambda^*$ and $Y^*(1) - \lambda^* Z^*(1) = 0$, we get

$$\begin{aligned} |Y_N(1) - \bar{\lambda} Z_N(1)| &= \left| \left(Y^*(1) - \bar{\lambda} Z^*(1) \right) - \left(Y_N(1) - \bar{\lambda} Z_N(1) \right) \right| \\ &= \left| Y^*(1) - Y_N(1) - \bar{\lambda}\left(Z^*(1) - Z_N(1) \right) \right| \\ &\leq |Y^*(1) - Y_N(1)| + |\bar{\lambda}|\, |Z^*(1) - Z_N(1)| \\ &\leq \left(2c_2 + \frac{|Y^*(1)|}{|Z^*(1)|} 2c_3 \right) (N-1)^{1-m}. \end{aligned} \tag{55}$$

So, if we select K such that for any $N \geq K$,

$$\max \left\{ 2c_1, 2c_2, 2c_3, 2c_2(N-1) + 2c_3 \frac{|Y^*(1)|}{|Z^*(1)|} \right\} \leq (N-1)^{\frac{1}{2}},$$

then $(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{\lambda})$ satisfies the constraints (37)-(41). Hence, we can imply that $(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{\lambda})$ is a feasible solution for DT problem (36)-(41). Finally, by selecting $L_1 = 2c_1, L_2 = 2c_2$ and $L_3 = 2c_3$, we can achieve (42)-(44) from (50)-(52). □

Remark 5.4. *We note that in the proof of Theorem 3, by attention to the relation* (4)*, we have* $Z^*(1) = \int_{-1}^{1} G\left(\tau, X(\tau), U(\tau)\right) dt = \frac{2}{T} \int_0^T g\left(t, x(t), u(t)\right) dt \geq \varepsilon > 0.$

6. Convergence of the method

In this section, we show that the sequence of optimal solutions of DT problem (36)-(41) is convergent to the optimal solution of the CT problem (21)-(26). The method is a generalization of convergence analysis given in [23] which is based on Polak's theory of consistent approximation.
Let $(\bar{x}_k^*, \bar{y}_k^*, \bar{z}_k^*, \bar{u}_k^*, \lambda_N^*), k = 0, 1, ..., N$ be an optimal solution to the DT problem (36)-(41). Define

$$X_N^*(\tau) = \sum_{k=0}^{N} \bar{x}_k^* L_k(\tau), \quad Y_N^*(\tau) = \sum_{k=0}^{N} \bar{y}_k^* L_k(\tau), \quad Z_N^*(\tau) = \sum_{k=0}^{N} \bar{z}_k^* L_k(\tau), \tag{56}$$

$$U_N^*(\tau) = \sum_{k=0}^{N} \bar{u}_k^* L_k(\tau), \quad -1 \leq \tau \leq 1, \tag{57}$$

where $L_k(.), k = 0, 1, ..., N$ are the Lagrange polynomials. Thus, we have a sequence of optimal solutions and their sequence of interpolating polynomials as follows respectively

$$\left\{ (\bar{x}_k^*, \bar{y}_k^*, \bar{z}_k^*, \bar{u}_k^*, \lambda_N^*); k = 0, 1, ..., N \right\}_{N=N_1}^{\infty}, \ \left\{ \left(X_N^*(.), Y_N^*(.), Z_N^*(.), U_N^*(.) \right) \right\}_{N=N_1}^{\infty}.$$

Assumption 1. We assume that the sequence $\left\{ \left(\bar{x}_0^*, \bar{y}_0^*, \bar{z}_0^*, X_N'^*(.), Y_N'^*(.), Z_N'^*(.), U_N^*(.), \lambda_N^* \right) \right\}_{N=N_1}^{\infty}$ has a subsequence that uniformly converges to $\left(x_0^{\infty}, y_0^{\infty}, z_0^{\infty}, q_1(.), q_2(.), q_3(.), U^*(.), \lambda^* \right)$ where $q_i(.), i = 1, 2, 3$ are continuous functions.

Theorem 6.1. *(Convergence) Assume that interpolating polynomials* (56) *and* (57) *satisfy Assumption 1. Then*

$\phi^*(.) = (X^*(.), Y^*(.), Z^*(.), U^*(.), \lambda^*)$ is an optimal solution for the CT problem (21)-(26) *where*

$$X^*(\tau) = x_0^\infty + \int_{-1}^{\tau} q_1(s)ds, -1 \le \tau \le 1, \tag{58}$$

$$Y^*(\tau) = \int_{-1}^{\tau} q_2(s)ds, -1 \le \tau \le 1, \tag{59}$$

$$Z^*(\tau) = \int_{-1}^{\tau} q_3(s)ds, -1 \le \tau \le 1, \tag{60}$$

$$\lambda^* = \lim_{N\to\infty} \lambda_N^*. \tag{61}$$

Proof. By Assumption 1, there exists a subsequence $N_i \in \{1, 2, ...\}$ with $\lim_{i\to\infty} N_i = \infty$ such that

$$\lim_{i\to\infty} X'^*_{N_i}(.) = q_1(.),\ \lim_{i\to\infty} Y'^*_{N_i}(.) = q_2(.),\ \lim_{i\to\infty} Z^*_{N_i}(.) = q_3(.),\ \lim_{i\to\infty} \lambda^*_{N_i} = \lambda^*. \tag{62}$$

Hence, by (58)-(60) and (62) we get

$$\lim_{i\to\infty} X^*_{N_i}(.) = X^*(.), \quad \lim_{i\to\infty} Y^*_{N_i}(.) = Y^*(.), \quad \lim_{i\to\infty} Z^*_{N_i}(.) = Z^*(.). \tag{63}$$

We now follow the proof in three steps. In step 1, we show that $\phi^*(.)$ is a feasible solution to the CT problem (21)-(26). In step 2, we prove the convergence of $J^*_{N_i}$ to J^*, where $J^*_{N_i}$ and J^* are the optimal values of objective functionals of the problem (36)-(41) and (21)-(26), respectively. Finally, we show that $\phi^*(.)$ is an optimal solution for the CT problem (21)-(26).

Step 1. We will show that $\phi^*(.)$ satisfies the constraints (22)-(26). Assume that $\phi^*(.)$ is not a solution of the equation (22). Then, there is a time $\bar{\tau} \in [-1, 1]$ such that

$$\dot{X}(\bar{\tau}) - H\Big(\bar{\tau}, X(\bar{\tau}), U(\bar{\tau})\Big) \ne 0. \tag{64}$$

Since the LGL points are dense in $[-1, 1]$ (see [14]), there is a sequence $\{k_{N_i}\}_{i=1}^\infty$ such that $0 < k_{N_i} < N_i$ and $\lim_{i\to\infty} \tau_{k_{N_i}} = \bar{\tau}$. Thus,

$$\lim_{i\to\infty} \Big(X'^*_{N_i}(\tau_{k_{N_i}}) - H(\tau_{k_{N_i}}, X(\tau_{k_{N_i}}), U(\tau_{k_{N_i}})\Big) = X^*(\bar{\tau}) - H\Big(\bar{\tau}, X^*(\bar{\tau}), U^*(\bar{\tau})\Big) \ne 0. \tag{65}$$

Now, since $\lim_{i\to\infty}(N_i - 1)^{\frac{3}{2}-m} = 0$, by constraint (22) we get

$$\lim_{i\to\infty} \Big(X'^*_{N_i}(\tau_{k_{N_i}}) - H(\tau_{k_{N_i}}, X(\tau_{k_{N_i}}), U(\tau_{k_{N_i}})\Big) = 0,$$

which is a contradiction with (65). Thus $\phi^*(.)$ satisfies the equation (22). By a similar process we can see that $\phi^*(.)$ satisfies equations (23) and (24). Now, we show that $\phi^*(.)$ satisfies constraint (25). We have

$$\begin{aligned} 0 &\le \Big| Y^*(1) - \lambda^* Z^*(1)| = |\lim_{i\to\infty} \Big(Y^*_{N_i}(1) - \lambda^*_{N_i} Z^*_{N_i}(1)\Big)\Big| \\ &= \lim_{i\to\infty} |Y^*_{N_i}(1) - \lambda^*_{N_i} Z^*_{N_i}(1)| = \lim_{i\to\infty} |\bar{y}^*_{N_i} - \lambda^*_{N_i} \bar{z}^*_{N_i}| \\ &\le \lim_{i\to\infty}(N_i - 1)^{\frac{3}{2}-m} = 0 \end{aligned}$$

So $Y^*(1) - \lambda^* Z^*(1) = 0$. Also, by a similar procedure we get $X^*(-1) = \alpha$. Moreover, it is trivial that $X^*(.)$ and $Y^*(.)$ satisfy relations $Y(-1) = 0$ and $Z(-1) = 0$.

Step 2. By relations (31) and (61) we get

$$J^* = \lambda^* = \lim_{i\to\infty} \lambda^*_{N_i} = \lim_{i\to\infty} J_{N_i}.$$

Step 3. Assume that $\widetilde{\phi}(.) = \left(\widetilde{X}(.), \widetilde{Y}(.), \widetilde{Z}(.), \widetilde{U}(.), \widetilde{\lambda}\right)$ is an optimal solution of the CT problem (21)-(26). By using the same discussion in Theorem 1, there exists a sequence of feasible solutions $\{\tilde{x}, \tilde{y}, \tilde{z}, \tilde{u}, \tilde{\lambda}_N\}_{N=N_1}^{\infty}$, for DT problem (36)-(41), such that its corresponding interpolating polynomial converges uniformly to $\widetilde{\phi}(.)$. Now, by optimality of $\widetilde{\phi}(.)$ we get

$$\widetilde{J} = \widetilde{\lambda} \leq J^* = \lambda^* = \lim_{i\to\infty} \lambda^*_{N_i} \leq \lim_{i\to\infty} \widetilde{\lambda}_{N_i} = \widetilde{J}.$$

So $\lambda^* = \widetilde{\lambda}$. Hence, $\phi^*(.)$ is a feasible solution which achieves the optimal value of functional J. □

7. Numerical examples

In this section, we apply our proposed method to solve two CTNFP problems. We utilize the FMINCON command in MATLAB software to solve the corresponding DT problem (31)-(35). Moreover, we calculate the absolute error of approximate optimal solutions by the following relations

$$e_x(t_k) = |x^*(t_k) - \bar{x}_k|, \; e_u(t_k) = |u^*(t_k) - \bar{u}_k|, \quad k = 0, 1, 2, ..., N,$$

where $\left(x^*(.), u^*(.)\right)$ and $\left(x(.), u(.)\right)$ are the exact and approximate optimal solutions, respectively.

Exercise 7.1. *Consider the following CTNFP problem*

$$\text{Minimize} \quad J = \frac{\int_0^1 \left(t^2(x(t) - e^t)^2 + 1\right)dt}{\int_0^1 \left(1 - (x(t) - e^t)^2 - (u(t) - t)^2\right)dt} \tag{66}$$

$$\text{subject to} \quad \dot{x}(t) = x(t)\left(u^2(t)e^t - t^2 x(t) + 1\right), \quad 0 \leq t \leq 1, \tag{67}$$

$$x(0) = 1. \tag{68}$$

The exact optimal solution of this problem is $(x^*(.), u^*(.)) = (e^t, t)$. *Also, the optimal value of objective functional is* $J^* = 1$. *The corresponding DT problem* (31)-(35) *for* (66)-(68) *is as follows*

$$\text{Minimize} \quad J_N = \lambda$$

$$\text{subject to} \quad \sum_{j=0}^{N} \bar{x}_j D_{kj} = \frac{1}{2}\bar{x}_k\left(\bar{u}_k^2 e^{\frac{1}{2}(\tau_k+1)} - \left(\frac{1}{2}(\tau_k+1)\right)^2 \bar{x}_k + 1\right), \; k = 0, 1, 2, ..., N,$$

$$\sum_{j=0}^{N} \bar{y}_j D_{kj} = \frac{1}{2}\left(\left(\frac{1}{2}(\tau_k+1)\right)^2 \left(\bar{x}_k - e^{\frac{1}{2}(\tau_k+1)}\right)^2 + 1\right), \; k = 0, 1, 2, ..., N,$$

$$\sum_{j=0}^{N} \bar{z}_j D_{kj} = \frac{1}{2}\left(1 - \left(\bar{x}_k - e^{\frac{1}{2}(\tau_k+1)}\right)^2 - \left(\bar{u}_k - \frac{1}{2}(\tau_k+1)\right)\right), \; k = 0, 1, 2, ..., N,$$

$$\bar{y}_N - \lambda \bar{z}_N = 0, \quad \bar{x}_0 = 1, \quad \bar{y}_0 = \bar{z}_0 = 0,$$

where $\bar{x} = (\bar{x}_0, \bar{x}_1, ..., \bar{x}_N), \bar{y} = (\bar{y}_0, \bar{y}_1, ..., \bar{y}_N), \bar{z} = (\bar{z}_0, \bar{z}_1, ..., \bar{z}_N)$ *and* $\bar{u} = (\bar{u}_0, \bar{u}_1, ..., \bar{u}_N)$ *are the variables of the problem. Having solved this DT problem for* $N = 5$, *we achieve the approximate solution* $(\bar{x}, \bar{u})$ *for the CTNFP problem* (66)-(68), *which is illustrated in Figures (1) and (2), Also, the errors of approximate solutions for the state and control are illustrated in Figure (3) and (4). Her,e we get the approximate optimal value* $\lambda^* = J_5^* = 1 + 2.11 \times 10^{-11}$. *Results show that the presented approach has a good accuracy.*

Exercise 7.2. *Consider the following CTNFP problem*

$$\text{Minimize} \quad J = \frac{\int_0^{\pi} \left(1 + (x(t)\cos t - u(t)\sin t)^2\right)dt}{\int_0^{\pi} \left(1 - (x(t) - \sin t)^2 - (u(t) - \cos t)^2\right)dt} \tag{69}$$

$$\text{subject to} \quad \dot{x}(t) = u(t), \quad 0 \le t \le \pi, \tag{70}$$

$$x(0) = 0. \tag{71}$$

The exact optimal solution and optimal value of objective functional are $(x^*(.), u^*(.)) = (\sin t, \cos t)$ *and* $J^* = 1$*, respectively. The corresponding DT problem* (31)-(35) *for CT problem* (69)-(71) *is as follows*

$$\text{Minimize} \quad J_N = \lambda$$

$$\text{subject to} \quad \sum_{j=0}^{N} \bar{x}_j D_{kj} = \frac{\pi}{2}\bar{u}_k, \; k = 0, 1, ..., N,$$

$$\sum_{j=0}^{N} \bar{y}_j D_{kj} = \frac{\pi}{2}\left(1 + \left(\bar{x}_k \cos\left(\frac{1}{2}(\tau_k + 1)\right) - \bar{u}_k \sin\left(\frac{1}{2}(\tau_k + 1)\right)\right)^2\right), \; k = 0, 1, ..., N,$$

$$\sum_{j=0}^{N} \bar{z}_j D_{kj} = \frac{\pi}{2}\left(1 - \left(\bar{x}_k - \sin\left(\frac{1}{2}(\tau_k + 1)\right)\right)^2 - \left(\bar{u}_k - \cos\left(\frac{1}{2}(\tau_k + 1)\right)\right)^2\right), \; k = 0, 1, ..., N,$$

$$\bar{y}_N - \lambda \bar{z}_N = 0, \quad \bar{x}_0 = \bar{y}_0 = \bar{z}_0 = 0.$$

We solve this DT problem for $N = 10$ *and obtain the approximate optimal solution* $(\bar{x}, \bar{u})$*, which are given in Figures 5 and 6. Moreover in Figure 7 and 8, we illustrate the absolute errors of approximate optimal solutions. Here, we obtain* $\lambda^* = J^*_{10} = 1 + 1.88 \times 10^{-9}$ *which shows the efficiency of method.*

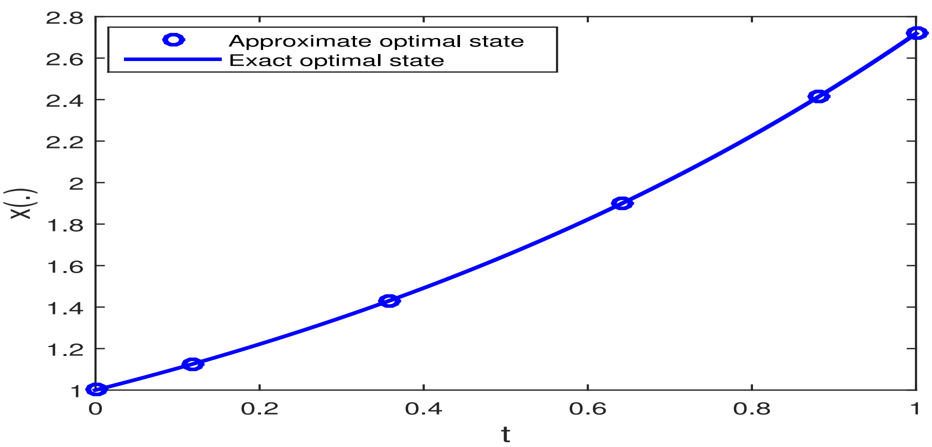

Figure 1: The exact and approximate optimal states for Example 1.

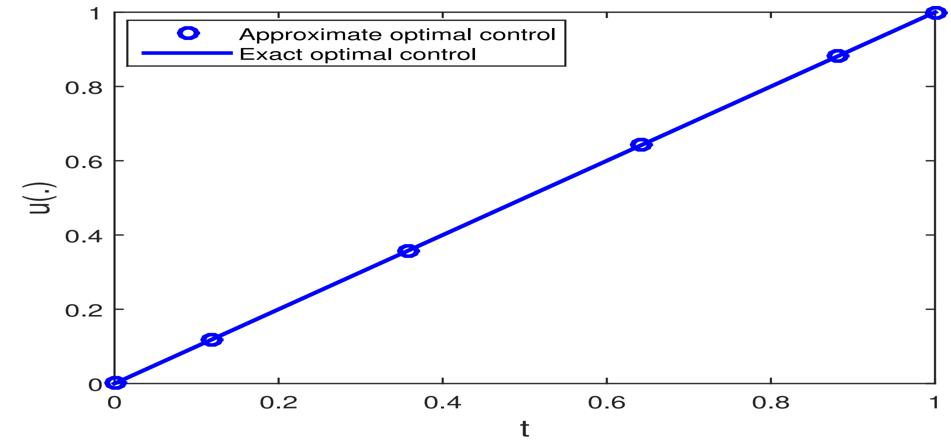

Figure 2: The exact and approximate optimal controls for Example 1.

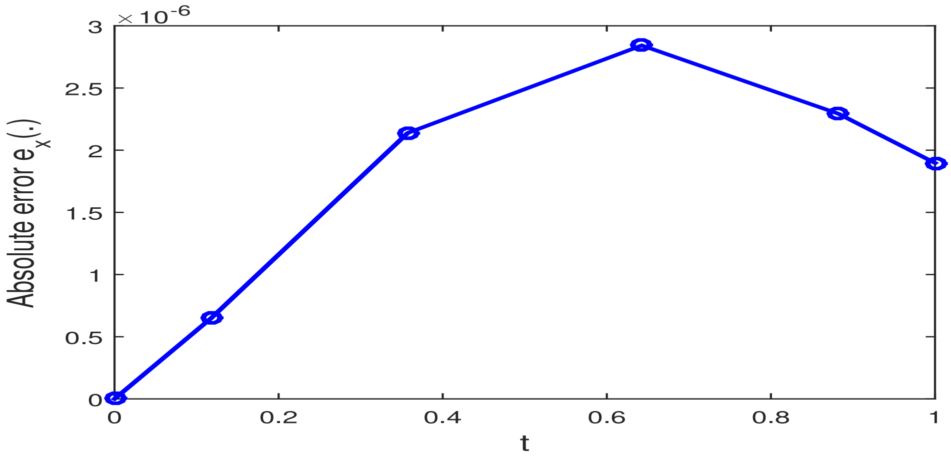

Figure 3: The absolute error $e_x(.)$ for Example 1.

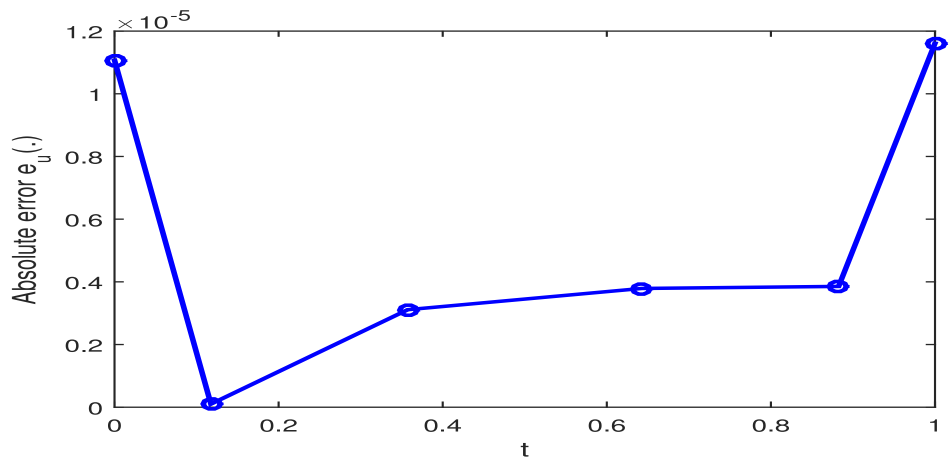

Figure 4: The absolute error $e_u(.)$ for Example 1.

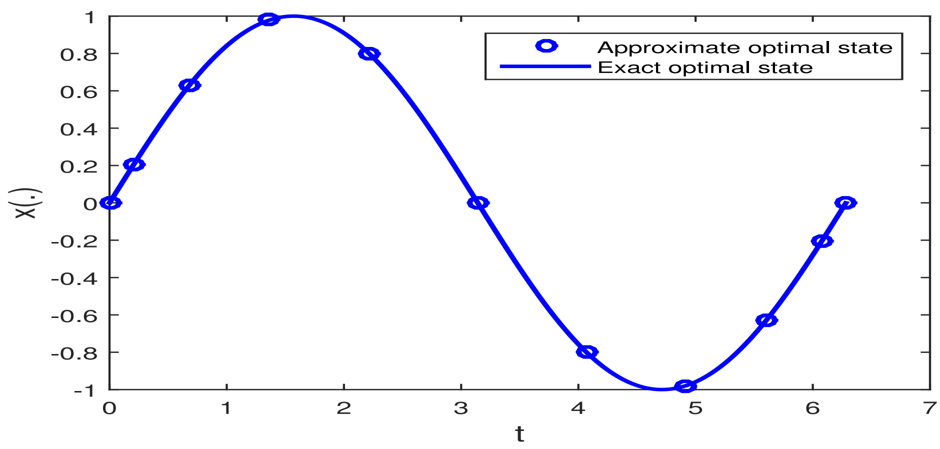

Figure 5: The exact and approximate optimal states for Example 2.

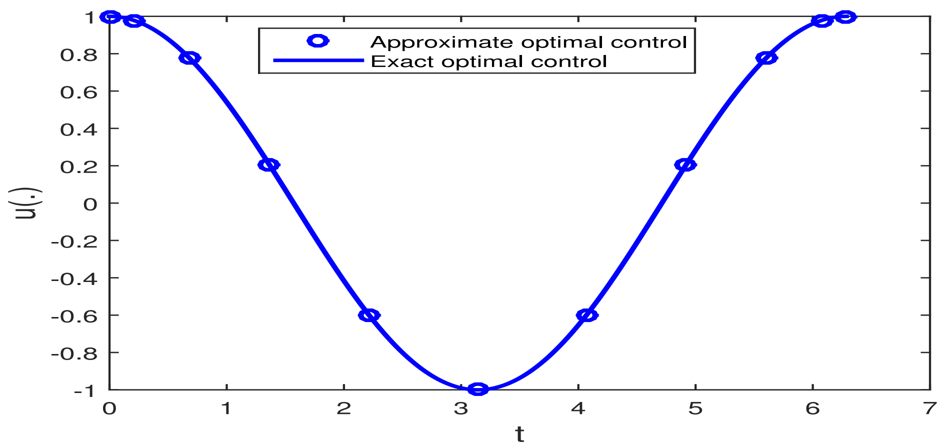

Figure 6: The exact and approximate optimal controls for Example 2.

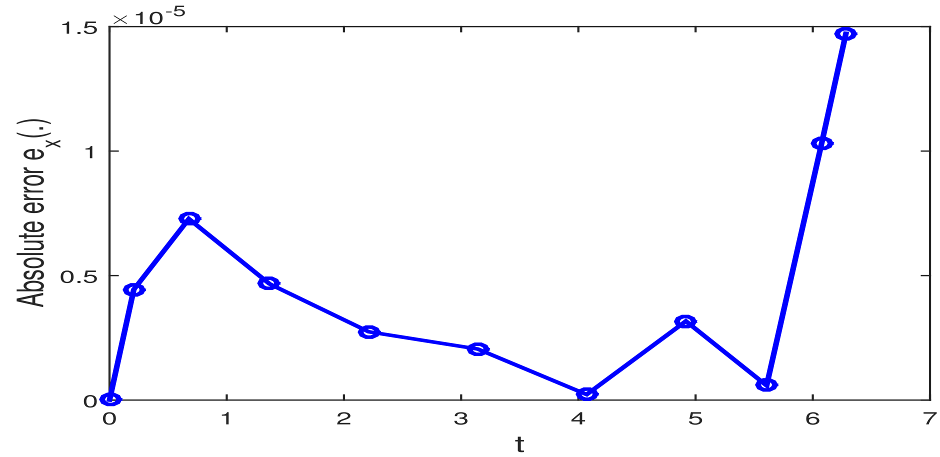

Figure 7: The absolute error $e_x(.)$ for Example 2.

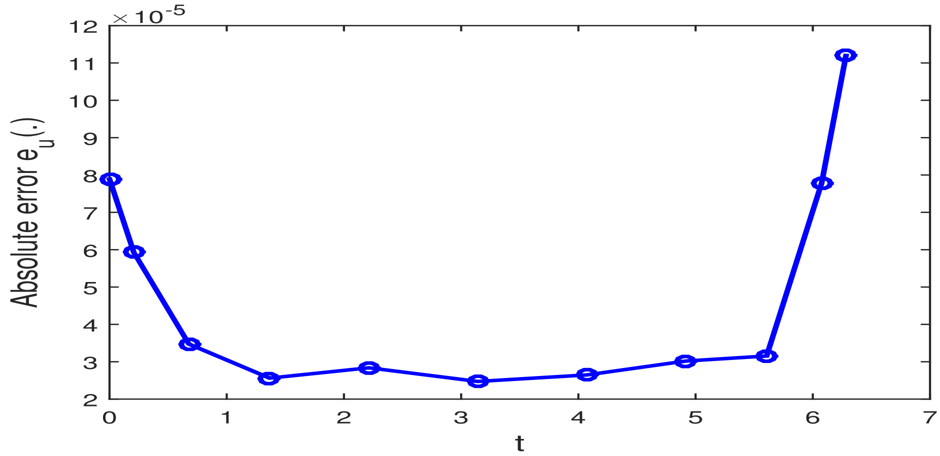

Figure 8: The absolute error $e_u(.)$ for Example 2.

8. Conclusions

In this paper, we showed that the Legendre pseudospectral method can be extended to solve the general form of continuous-time nonlinear fractional programming problems. Without using the Dinkelbach algorithm, we presented a new technique to convert the problem into an equivalent nonfractional form. Moreover, the obtained nonfractional problem can be discretized at the Legendre-Gauss-Lobatto points. We generalized the convergence analysis of pseudospectral methods to the suggested method. For future works, we will suggest the presented approach in this article to numerically solve the continuous-time fractional programming problems including the fixed-order and variable-order fractional dynamical systems [1, 9, 34].

References

[1] P. Agarwal, D. Baleanu, Y. Chen, S. Momani, J. A. Tenreiro Machado, *Fractional calculus*, In ICFDA: International Workshop on Advanced Theory and Applications of Fractional Calculus, Amman, 2019.

[2] D. A. Benson, G. T. Huntington, T. P. Thorvaldsen, A. V. Rao, *Direct trajectory optimization and costate estimation via an orthogonal collocation method*, J. Guid. Control. Dyn. **29** (2006), 1435–1440.

[3] I. Bykadorov, A. Ellero, E. Moretti, *Dinkelbach Approach to Solving a Class of Fractional Optimal Control Problems*, J. Optim. Theory Appl. **142** (2009), 55–56.

[4] C. Canuto, Y. Hussaini, A. Quarteroni, T. A. Zang, *Spectral Methods in Fluid Dynamics*, Scientific Computation, New York, Springer, 1987.

[5] J. P. Crouzeix, J. A. Ferland, S. Schaible, *Duality in generalized linear fractional programming*, Math. Program. **27** (1983), 342–354.

[6] W. Dinkelbach, *On nonlinear fractional programming*, J. Manag. Sci. **13** (1967), 492–498.

[7] A. Dolatnezhadsomarina, E. Khorrama, L. Pourkarimib, *Efficient algorithms for solving nonlinear fractional programming problems*, Filomat. **33** (2019), 2149—2179.

[8] G. Elnagar, M. A. Kazemi, M. Razzaghi, *The pseudospectral Legendre method for discretizing optimal control problems*, IEEE Trans. Automat. **40** (1995), 1793–1796.

[9] A. A. El-Sayed, P. Agarwal, *Numerical solution of multiterm variable-order fractional differential equations via shifted Legendre polynomials*, Math. Methods Appl. Sci. **42** (2019), 3978–3991.

[10] F. Fahroo, I. M. Ross, *Costate estimation by a Legendre pseudospectral method*, J. Guid. Control. Dyn. **24** (2001), 270–277.

[11] F. Fahroo, I. M. Ross, *Direct trajectory optimization by a Chebyshev pseudospectral method*, J. Guid. Control. Dyn. **25** (2002), 160–166.

[12] C. C. Francolin, D. A. Benson, W. W. Hager, A. V. Rao, *Costate approximation in optimal control using integral Gaussian quadrature orthogonal collocation methods*, Optim. Control Appl. Methods. **36** (2015), 381–397.

[13] H. Frenk, S. Schsible, *Fractional Programming, Encyclopedia of Optimization*, Kluwer, Dordrecht. **2** (2001), 162–172.

[14] G. Freud, *Orthogonal polynomials*, Pergamon Press, Elmsford, 1971.

[15] D. Garg, M. Patterson, W. W. Hager, A. V. Rao, D. A. Benson, G. T. Huntington, *A unified framework for the numerical solution of optimal control problems using pseudospectral methods*, Automatica (Oxf). **46** (2010), 1843–1851.

[16] D. Garg, M. A. Patterson, C. Francolin, C. L. Darby, G. T. Huntington, W. W. Hager, *Direct trajectory optimization and costate estimation of finite-horizon and infinite-horizon optimal control problems using a Radau pseudospectral method*, Comput. Optim. Appl. **49** (2011), 335–358.

[17] M. Ghaznavi, M. H. Noori Skandari, *An Efficient Pseudo-Spectral Method for Nonsmooth Dynamical Systems*, Iran J Sci Technol Trans Sci. **42**(2018), 635–646.

[18] Q. Gong, I. M. Ross, W. Kang, F. Fahroo, *Connections between the covector mapping theorem and convergence of pseudospectral methods for optimal control*, Comput. Optim. Appl. **41** (2008),307–335.

[19] M. Hedayati, H. A.Tehrani, A. Fakharzadeh J. , M. H. Noori Skandari, D. Baleanu, *A novel high accurate numerical approach for the time-delay optimal control problems with delay on both state and control variables*, AIMS Mathematics, **7**(5)(2022), 9789-9808.

[20] M. Hedayati, H. Ahsani Tehrani, A. Fakharzadeh Jahromi, M. H. Noori Skandari and D. Baleanu, *An efficient hp spectral collocation method for nonsmooth optimal control problems*, Kybernetika, **58**(6)(2022), 843-862.

[21] Y. Huang, M. H. Noori Skandari, F. Mohammadizadeh, H. Ahsani Tehrani, S. G. Georgiev, E. Tohidi and S.Shateyi, *Space–time spectral collocation method for solving Burgers equations with the convergence analysis*, Symmetry, **11**(12)(2019), 1439.

[22] H. Jafari, M. Mahmoudi, M.H. Noori Skandari, *A new numerical method to solve pantograph delay differential equations with convergence analysis*, Adv. Differ. Equ. **2021** (129)(2021).

[23] Z. A. Liang, P. M. Pardalos, H. X. Huang, *Optimality conditions and duality for a class of nonlinear fractional programming proglems*, J. Optim. Theory Appl. **110** (2001), 611–619.

[24] Y. Y. Lur, W. H. Ho, T. H. Lu, C. F. Wen, *Approximate solutions for continuous-time quadratic fractional programming problems*, Taiwan. J. Math. **18** (2014), 1791–1826.

[25] F. Mohammadizadeh H. A. Tehrani M. H. Noori Skandari, *Chebyshev pseudo-spectral method for optimal control problem of Burgers' equation*, Iranian J. Num. Anal. Optim. **9**(2)(2019), 77-102.

[26] M. H. Noori Skandari, M. Ghaznavi, *A numerical method for solving shortest path problems*, Calcolo.**55** (2018), 1–18.

[27] M. H. Noori Skandari, A. V. Kamyad and S. Effati, *Generalized Euler–Lagrange equation for nonsmooth calculus of variations*, Nonlinear Dyn. **75**(2014), 85–100.

[28] M. H Noori Skandari, M. Habibli and A. Nazemi, *A direct method based on the Clenshaw-Curtis formula for fractional optimal control problems*, Math. Contr. Related fields, **10**(2020), 171–187.

[29] P. M. Pardalos, A. Phillips, *Global optimization of fractional programs*, J. Glob. Optim. **1** (1991), 173–182.

[30] N. Peykrayegan, M. Ghovatmand, M. H. N. Skandari, *An efficient method for linear fractional delay integro-differential equations*, Comp. Appl. Math. **40**(2021), 249.

[31] N. Peykrayegan, M. Ghovatmand, M. H. Noori Skandari and D. Baleanu, *An approximate approach for fractional singular delay integro-differential equations*, AIMS Mathematics, **7**(5)(2022), 9156–9171.

[32] E. Polak, *Optimization: algorithms and consistent approximations*, Springer, heidelberg, 1997.

[33] E. Rentsen, J. Zhou, K. L. Teo, *A global optimization approach to fractional optimal control*, J. Ind. Manag. Optim. **12** (2016), 73–82.

[34] S. Salahshour, A. Ahmadian, N. Senu, D. Baleanu, P. Agarwal, *On Analytical Solutions of the Fractional Differential Equation with Uncertainty: Application to the Basset Problem*, Entropy. **17** (2015), 885–902.

[35] I. M. Stancu-Minasian, *Fractional Programming, Theory, Methods and Applications*, Kluwer, Dordrecht, 1997.

[36] S. Schaible, J. Shi, *Recent developments in fractional programming: Single-ratio and max-min case*, J. Nonlinear. Convex. Anal. (2004), 493–506.

[37] I. M. Stancu-Minasian, S. Tigan, *Continuous time linear-fractional programming: The minimum-risk approach*, RAIRO-OPER. RES. **34** (2000), 397–409.

[38] C. F. Wen, Y. Y. Lur, S. M. Guu, E. S. Lee, *On a Recurrence Alogrithm for Continuous-time linear Fractional Programming Problems*, Comput. Math. with Appl. **59** (2010), 827–852.

[39] C. F. Wen, Y. Y. Lur, S. M.Wu, *A Recurrence Method for a Special Class of Continuous Time Linear Programming Problems*, J. Global Optim. **47** (2010), 83–106.

[40] C. F. Wen, Y. Y. Lur, H. C. Lai, *Approximation Solutions and Error Bounds for a Class of Continuous Time Linear Programming Problems*, J. Optim. **61** (2012), 163–185.

[41] C. F. Wen, H. C. Wu, *Using the parametric approach to solve the continuous-time linear fractional max-min problems*, J. Global Optim. **54** (2012), 129–153.

[42] C. F. Wen, H. C. Wu, *Approximate Solutions and Duality Theorems for Continuous-time Linear Fractional Programming Problems*, Numer. Funct. Anal. Optim. **33** (2012), 80–129.

[43] C. F. Wen, H. C. Wu, *Using the Dinkelback-type algorithm to solve the continuous-time fractional programming problems*, J. Global Optim. **49** (2011), 237–263.

[44] H. C. Wu, *Numerical method for solving the continuous-time linear programming problems with time-dependent matrices and piecewise continuous functions*, AIMS Math. **5** (2020), 5572-–5627.

[45] P. Xiaobing, X. Yang, M. H. Noori Skandari, E. Tohidi, S. Shateyi, *A new high accurate approximate approach to solve optimal control problems of fractional order via efficient basis functions*, Alex. Eng. J. **61**(8)(2021), 5805-5818.

[46] Y. Yang, M. H. Noori Skandari, *Pseudospectral method for fractional infinite horizon optimal control problems*, Opti. Con. Appli. Meth. **41**(6)(2020), 2201-2212.

[47] X. Yang, Y. Yang, M. H. Noori Skandari, E. Tohidi, S. Shateyi, *A new local non-integer derivative and its application to optimal control problems*, AIMS Mathematics, **7**(9)(2022), 16692-16705.

[48] G. J. Zalmai, *Duality for a class of continuous-time fractional programming problems*, Utilitas Mathematica. **31** (1987), 209–218.

[49] G. J. Zalmai, *Optimality conditions and duality for a class of continuous-time generalized fractional programming problems*, J. Math. Anal. Appl. **153** (1990), 356–371.

[50] G. J. Zalmai, *continuous-time generalized fractional programming*, J. Optim. **36** (1996), 195–217.

[51] G. J. Zalmai, *Optimality conditions and duality for a class of nonsmooth constrained fractional optimal control problems*, J. Math. Anal. Appl. **210** (1990), 114–149.