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Fixed point theorems involving $\mathcal{FZ}$-ϑ_f-contractions in GV-fuzzy metrics

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Abstract. In this study, we introduce two new types of fuzzy contraction principles, namely: fuzzy $\mathcal{FZ}$-ϑ_f-contraction and modified fuzzy $\mathcal{FZ}$-ϑ_f-contraction mappings, and utilize the same to establish some fixed point results in the framework of complete GV-fuzzy metric spaces. To further demonstrate the accuracy of the established concepts and results, we provide some examples along with some illustrative corollaries and remarks. The presented results unify, generalize, and improve various existing results in the literature.

1. Introduction

One of the most increasingly prominent study areas in nonlinear functional analysis is fixed point (FP) theory, which contains a wide variety of mathematical tools for addressing a multitude of issues resulting from other fields of mathematics. The Banach contraction result is the first relevant metric FP theorem, this innovative principle has been extended, developed, generalized, and refined in the setting of numerous abstract spaces. By coining the concept of simulation functions, Khojasteh *et al*. [15] pioneered a new framework for the study of FP theorems. Many researchers have continued to improve the idea of simulation functions in different approaches (see e.g. [16, 18–20, 24, 25, 28]).

In 1965, L.A. Zadeh [1] offered the fuzzy set concept as a novel mathematical approach to dealing with ambiguity and vagueness in practical applications. The conception of fuzzy sets has evolved into a valuable and significant modeling tool. Finding a suitable and consistent definition of fuzzy metric is one of the difficult issues in fuzzy topology. Many researchers have addressed at this concern in a variety of approaches (see e.g [2, 3]). Kramosil and Michalek [4] established fuzzy metric (KM-fuzzy metric) by generalizing the idea of probabilistic metric to the fuzzy backdrop. Thereafter, George and Veeramani [6] revised Kramosil and Michalek's definition of fuzzy metric space in order to gain a Hausdorff topology for this class (GV-fuzzy metric), which has crucial applications in quantum mechanics, notably in the context of string and $\epsilon^{(\infty)}$ theory [31]. Previous research has demonstrated that fuzzy metrics are effective in a broad spectrum of applications, including color picture filtering, where fuzzy metrics have recently been used to improve some filters by substituting traditional metrics [17].

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Over the last years, there has been an intense interest in studying the FP theory in fuzzy metric spaces. In this direction, Gregori and Sapena [7] defined fuzzy contractive mappings and obtained some FP results. Later on, Mihet [12] proposed the class of fuzzy ψ-contractive mappings. Wardowski [13] presented and studied the concept of fuzzy $\mathcal{H}$-contractive mappings. Inspired by the work of Khojasteh *et al.* [15], Melliani and Moussaoui ([21, 22]) initiated a simulation function approach to the study of FP theory in fuzzy metric spaces and proposed the concept of $\mathcal{FZ}$-contractions. Recently, Saleh *et al.* [29] brought in the concept of fuzzy ϑ_f-contractive mappings, which was inspired by the results of Jleli *et al.* [26], by employing an auxiliary function $\vartheta_f : (0,1) \to (0,1)$ fulfilling suitable conditions. For further insight into current advancements in fuzzy metric FP theory and related approaches, we refer the reader to [5, 7, 13, 14, 21–23, 25, 27, 30, 32–34]

We introduce two types of new fuzzy contractions called $\mathcal{FZ}$-ϑ_f-contraction and modified $\mathcal{FZ}$-ϑ_f-contraction mappings by using the auxiliary function ϑ_f embedded in $\mathcal{FZ}$-simulation function. We prove some FP results in the context of complete fuzzy metric spaces for such contractions. The presented results unify, generalize, and improve on various existing concepts in the literature.

2. Preliminaries

In this section, we cover some central concepts and results that will be beneficial in the next section.

Definition 2.1. *[11] An operation* $* : [0,1]^2 \to [0,1]$ *is a continuous t-norm if* $([0,1], *)$ *is an Abelien topological monoid with unit* 1 *such that* $\jmath_1 * \jmath_2 \geq \jmath_3 * \jmath_4$ *whenever* $\jmath_1 \geq \jmath_3$ *and* $\jmath_2 \geq \jmath_4$, *for all* $\jmath_i \in [0,1], i = 1,2,3,4.$

Example 2.2. i) $\jmath_1 *_m \jmath_2 = \min\{\jmath_1, \jmath_2\}$,

ii) $\jmath_1 *_L \jmath_2 = \max[0, \jmath_1 + \jmath_2 - 1]$,

iii) $\jmath_1 *_p \jmath_2 = \jmath_1 . \jmath_2$.

Definition 2.3. *[6] The 3-tuple* $(\mathcal{E}, \omega, *)$ *is said to be a GV-fuzzy metric space (GV-FMS) if* $\mathcal{E}$ *is an arbitrary set,* $*$ *is a continuous t-norm and* ω *is a fuzzy set on* $\mathcal{E}^2 \times (0, +\infty)$ *satisfying:*

$(\mathcal{GV}1)$ $\omega(\phi, \varphi, \gamma) > 0,$

$(\mathcal{GV}2)$ $\omega(\phi, \varphi, \gamma) = 1$ *iff* $\phi = \varphi,$

$(\mathcal{GV}3)$ $\omega(\phi, \varphi, \gamma) = \omega(\varphi, \phi, \gamma),$

$(\mathcal{GV}4)$ $\omega(\phi, \chi, \gamma + \delta) \geq \omega(\phi, \varphi, \gamma) * \omega(\varphi, \chi, \delta),$

$(\mathcal{GV}5)$ $\omega(\phi, \varphi, .) : (0, +\infty) \to [0,1]$ *is continuous.*

for all $\phi, \varphi, \chi \in \mathcal{E}$ *and* $\gamma, \delta > 0.$

In the next examples, assume that d is a metric on $\mathcal{E}$ and $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function.

Example 2.4. *[9] Define the mapping* ω *by*

$$\omega(\phi, \varphi, \gamma) = \frac{h(\gamma)}{h(\gamma) + md(\phi, \varphi)} \text{ for all } \phi, \varphi \in \mathcal{E}, \gamma > 0 \text{ with } m > 0. \tag{1}$$

Then $(\mathcal{E}, \omega, *_p)$ *is a GV-FMS. In particular, if we set* $h(\gamma) = \gamma^n$, $n \in \mathbb{N}$ *and* $m = 1$, *then* (1) *yields*

$$\omega(\phi, \varphi, \gamma) = \frac{\gamma^n}{\gamma^n + d(\phi, \varphi)} \text{ for all } \phi, \varphi \in \mathcal{E}, \gamma > 0.$$

Then $(\mathcal{E}, \omega, *_m)$ *is GV-FMS, as shown in [8].*

Example 2.5. *[9] Define the mapping* ω *by*

$$\omega(\phi,\varphi,\gamma)=\exp(\frac{-d(\phi,\varphi)}{h(\gamma)})\ \text{for all}\ \phi,\varphi\in\mathcal{E},\gamma>0. \tag{2}$$

Then $(\mathcal{E},\omega,*_p)$ *is an FMS. More notably, if we take h as the identity mapping, then (2) gives*

$$\omega(\phi,\varphi,\gamma)=\exp(\frac{-d(\phi,\varphi)}{\gamma})\ \text{for all}\ \phi,\varphi\in\mathcal{E},\gamma>0.$$

In this instance, $(\mathcal{E},\omega,*_m)$ *is a GV-FMS [6].*

Lemma 2.6. *[5]* $\omega(\phi,\varphi,.)$ *is nondecreasing function for all* ϕ,φ *in* $\mathcal{E}$.

Definition 2.7. *[6] Let* $(\mathcal{E},\omega,*)$ *be a GV-FMS.*

1. *A sequence* $\{\phi_n\}\subseteq\mathcal{E}$ *is said to be convergent or converges to* $\phi\in\mathcal{E}$ *if* $\lim_{n\to\infty}\omega(\phi_n,\phi,\gamma)=1$ *for all* $\gamma>0$.
2. *A sequence* $\{\phi_n\}\subseteq\mathcal{E}$ *is said to be an Cauchy sequence if for all* $\varepsilon\in(0,1)$ *and* $\gamma>0$, *there exists* $n_0\in\mathbb{N}$ *such that* $\omega(\phi_n,\phi_m,\gamma)>1-\varepsilon$ *for all* $n,m\geq n_0$.
3. *A GV-FMS in which each Cauchy sequence is convergent is called a complete GV-FMS.*

Definition 2.8. *[7] Let* $(\mathcal{E},\omega,*)$ *be a GV-FMS. A mapping* $\mathcal{T}:\mathcal{E}\to\mathcal{E}$ *is said to be a fuzzy contractive mapping if there exists* $k\in(0,1)$ *such that*

$$\frac{1}{\omega(\mathcal{T}\phi,\mathcal{T}\varphi,\gamma)}-1\leq k\left(\frac{1}{\omega(\phi,\varphi,\gamma)}-1\right),$$

for all $\phi,\varphi\in\mathcal{E}$ *and* $\gamma>0$.

Definition 2.9. *[7] A sequence* $\{x_n\}$ *in a GV-FMS* $(\mathcal{E},\omega,*)$ *is said to be fuzzy contractive if there exists* $k\in(0,1)$ *such that*

$$\frac{1}{\omega(\phi_{n+1},\phi_{n+2},\gamma)}-1\leq k\left(\frac{1}{\omega(\phi_n,\phi_{n+1},\gamma)}-1\right),$$

for all $n\in\mathbb{N}$ *and* $\gamma>0$.

Theorem 2.10. *[7] Let* $(\mathcal{E},\omega,*)$ *be a complete GV-FMS in which fuzzy contractive sequences are Cauchy. If* $\mathcal{T}:\mathcal{E}\to\mathcal{E}$ *is a fuzzy contractive mapping then* $\mathcal{T}$ *has a unique fixed point.*

As a result of his study the following theorem was established by Tirado [10].

Theorem 2.11. *[10] Let* $(\mathcal{E},\omega,*_L)$ *be a complete GV-FMS and* $\mathcal{T}:\mathcal{E}\to\mathcal{E}$ *be a mapping such that*

$$1-\omega(\mathcal{T}\phi,\mathcal{T}\varphi,\gamma)\leq k\Big(1-\omega(\phi,\varphi,\gamma)\Big),$$

for all $\phi,\varphi\in\mathcal{E}$, $\gamma>0$ *and for some* $k\in(0,1)$. *Then* $\mathcal{T}$ *has a unique fixed point.*

Recently, Hayel *et al*. [29] introduced the following class of functions. Let Ω be the class of functions $\vartheta_f:(0,1)\longrightarrow(0,1)$ such that

(Ω_1) ϑ_f is non-decreasing,

(Ω_2) ϑ_f is continuous,

(Ω_3) $\lim_{n\to+\infty}\vartheta_f(\beta_n)=1$ if and only if $\lim_{n\to+\infty}\beta_n=1$, where $\{\beta_n\}$ is a sequence in $(0,1)$.

Definition 2.12. *[29] Let $(\mathcal{E}, \omega, *)$ be a GV-FMS. A mapping $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ is said to be a fuzzy ϑ_f-contractive mapping w.r.t $\vartheta_f \in \Omega$ if there exists $k \in (0,1)$ such that*

$$\omega(\mathcal{T}\phi, \mathcal{T}\varphi, \gamma) < 1 \Rightarrow \vartheta_f(\omega(\mathcal{T}\phi, \mathcal{T}\varphi, \gamma)) \geq \left[\vartheta_f(\omega(\phi, \varphi, \gamma))\right]^k,$$

for all $\phi, \varphi \in \mathcal{E}$ and $\gamma > 0$.

Theorem 2.13. *[29] Let $(\mathcal{E}, \omega, *)$ be a complete GV-FMS and $\mathcal{T} : \mathcal{E} \longrightarrow \mathcal{E}$ be a fuzzy ϑ_f-contractive mapping, then $\mathcal{T}$ has a unique FP.*

Definition 2.14. *([21],[22]) The function $\Xi : (0,1] \times (0,1] \longrightarrow \mathbb{R}$ is an $\mathcal{FZ}$-simulation function if:*

(Ξ1) $\Xi(1,1) = 0$,

(Ξ2) $\Xi(a,b) < \frac{1}{b} - \frac{1}{a}$ *for all* $a, b \in (0,1)$,

(Ξ3) *if* $\{a_n\}, \{b_n\}$ *are sequences in* $(0,1]$ *such that* $\lim_{n\to\infty} a_n = \lim_{n\to+\infty} a_n < 1$, *then* $\lim_{n\to+\infty} \sup \Xi(a_n, b_n) < 0$.

The collection of all $\mathcal{FZ}$-simulation functions is denoted by $\mathcal{FZ}$.

Definition 2.15. *([21, 22]) Let $(\mathcal{E}, \omega, *)$ be a GV-FMS, $\mathcal{T} : \vartheta \longrightarrow \vartheta$ be a mapping and $\Xi \in \mathcal{FZ}$. Then $\mathcal{T}$ is called an $\mathcal{FZ}$-contraction w.r.t Ξ if*

$$\Xi(\omega(\mathcal{T}\phi, \mathcal{T}\varphi, \gamma), \omega(\phi, \varphi, \gamma)) \geq 0 \text{ for all } \phi, \varphi \in \mathcal{E}, \gamma > 0.$$

Example 2.16. *If $k \in (0,1)$ and we define $\Xi_{GS} : (0,1] \times (0,1] \to \mathbb{R}$ by*

$$\Xi_{GS}(a,b) = k\left(\frac{1}{b} - 1\right) - \frac{1}{a} + 1 \text{ for all } a, b \in (0,1],$$

then Ξ_{GS} is an $\mathcal{FZ}$-simulation function.

Example 2.17. *Let $\psi : (0,1] \longrightarrow (0,1]$ be a nondecreasing and continuous mapping such that $\psi(c) > c$, for all $c \in (0,1)$, then*

$$\Xi_M(a,b) = \frac{1}{\psi(b)} - \frac{1}{a} \text{ for all } a, b \in (0,1],$$

is an $\mathcal{FZ}$-simulation function.

Example 2.18. *If $\eta : (0,1] \longrightarrow [0, +\infty)$ is a strictly decreasing function and transforms $(0,1]$ onto $[0, +\infty)$, and we define $\Xi_W : (0,1] \times (0,1] \to \mathbb{R}$ by*

$$\Xi_W(a,b) = \frac{1}{\eta^{-1}(k.\eta(b))} - \frac{1}{a} \text{ for all } a, b \in (0,1],$$

where $k \in (0,1)$, then Ξ_W is an $\mathcal{FZ}$-simulation function.

3. Main results

In this section, we initiate a very general type of fuzzy contractions on FMSs, and we establish related existence and uniqueness FP theorems.

Let $O_{\mathcal{FZ}}$ be the collection of all functions $\Xi : (0,1] \times (0,1] \longrightarrow \mathbb{R}$ satisfying (Ξ1), (Ξ3) and (Ξ'2) (instead of (Ξ2)):

$$(\Xi'2) : \Xi(a,b) \leq \frac{1}{b} - \frac{1}{a} \text{ for all } a, b \in (0,1).$$

Definition 3.1. *Let $(\mathcal{E},\omega,*)$ be a GV-FMS, $\mathcal{T}:\mathcal{E}\longrightarrow\mathcal{E}$ be a mapping and $\Xi\in\mathcal{O}_{\mathcal{FZ}}$. Then $\mathcal{T}$ is called an $\mathcal{FZ}-\vartheta_f$-contraction w.r.t Ξ and $\vartheta_f\in\Omega$ if there exists $k\in(0,1)$ such that*

$$\omega(\mathcal{T}\phi,\mathcal{T}\varphi,\gamma)<1 \text{ implies } \Xi\Big(\vartheta_f(\omega(\mathcal{T}\phi,\mathcal{T}\varphi,\gamma)),(\vartheta_f(\omega(\phi,\varphi,\gamma)))^k\Big)\geq 0, \tag{3}$$

for all $\phi,\varphi\in\mathcal{E}$ and $\gamma>0$.

Example 3.2. *Every fuzzy contractive mapping is an $\mathcal{FZ}$-ϑ_f-contraction w.r.t $\Xi_{\mathcal{M}}\in\mathcal{O}_{\mathcal{FZ}}$ and $\vartheta_f\in\Omega$ defined by $\Xi_{\mathcal{AB}}(a,b)=\frac{1}{b}-\frac{1}{a}$ for all $a,b\in(0,1)$ and $\vartheta_f(\beta)=e^{1-\frac{1}{\beta}}$ for all $\beta\in(0,1)$.*

Example 3.3. *Every Tirado contraction is an $\mathcal{FZ}$-ϑ_f-contraction w.r.t $\Xi\in\mathcal{O}_{\mathcal{FZ}}$ and $\vartheta_f\in\Omega$ defined by $\Xi=\Xi_{\mathcal{AB}}$ and $\vartheta_f(\beta)=e^{\beta-1}$ for all $\beta\in(0,1)$.*

Example 3.4. *Let $\mathcal{E}=[0,1]$ be endowed with the a GV-fuzzy metric ω given by $\omega(\phi,\varphi,\gamma))=\frac{\gamma}{\gamma+d(\phi,\varphi)}$ for all $\phi,\varphi\in\mathcal{E}$, $\gamma>0$, where d is the usual metric. Then, $(\mathcal{E},\omega,*_p)$ is a GV-FMS,. Consider the mapping $\mathcal{T}:\mathcal{E}\to\mathcal{E}$ given by $\mathcal{T}\phi=\frac{\phi}{\phi+1}$, for all $\phi\in\mathcal{E}$. Define the control function $\Xi:(0,1]\times(0,1]\longrightarrow\mathbb{R}$ by $\Xi(a,b)=\frac{1}{b}-\frac{1}{a}$ for all $a,b\in(0,1)$. For all $\phi,\varphi\in\mathcal{E}$, we have*

$$\begin{aligned}\frac{1}{4}(1-\frac{1}{\omega(\phi,\varphi,\gamma)})&<\frac{1}{(\phi+1)(\varphi+1)}(1-\frac{1}{\omega(\phi,\varphi,\gamma)})\\&=\frac{1}{(\phi+1)(\varphi+1)}(-\frac{d(\phi,\varphi)}{\gamma})\\&=-\frac{d(\mathcal{T}\phi,\mathcal{T}\varphi)}{\gamma}\\&=1-\frac{1}{\omega(\mathcal{T}\phi,\mathcal{T}\varphi,\gamma)}.\end{aligned}$$

Consider the function $\vartheta_f\in\Omega$ defined by $\vartheta_f(\beta)=e^{1-\frac{1}{\beta}}$ for all $\beta\in(0,1)$ with $k=\frac{1}{4}$. Then, we derive that

$$\frac{1}{\Big(\vartheta_f(\omega(\phi,\varphi,\gamma))\Big)^k}=\frac{1}{\Big(e^{1-\frac{1}{\omega(\phi,\varphi,\gamma)}}\Big)^k}>\frac{1}{e^{1-\frac{1}{\omega(\mathcal{T}\phi,\mathcal{T}\varphi,\gamma)}}}=\frac{1}{\vartheta_f(\omega(\mathcal{T}\phi,\mathcal{T}\varphi,\gamma))}.$$

Hence

$$\begin{aligned}\Xi\Big(\vartheta_f(\omega(\mathcal{T}\phi,\mathcal{T}\varphi,\gamma)),(\vartheta_f(\omega(\phi,\varphi,\gamma)))^{\mathcal{K}}\Big)&=\frac{1}{\Big(\vartheta_f(\omega(\phi,\varphi,\gamma))\Big)^k}-\frac{1}{\vartheta_f(\omega(\mathcal{T}\phi,\mathcal{T}\varphi,\gamma))}\\&=\frac{1}{\Big(e^{1-\frac{1}{\omega(\phi,\varphi,\gamma)}}\Big)^k}-\frac{1}{e^{1-\frac{1}{\omega(\mathcal{T}\phi,\mathcal{T}\varphi,\gamma)}}}\\&>0.\end{aligned}$$

Therefore $\mathcal{T}$ is an $\mathcal{FZ}$-ϑ_f-contraction w.r.t $\Xi\in\mathcal{O}_{\mathcal{FZ}}$ and $\vartheta_f\in\Omega$.

Definition 3.5. *Let $(\mathcal{E},\omega,*)$ be a GV-FMS,, $\mathcal{T}:\mathcal{E}\longrightarrow\mathcal{E}$ a mapping and $\Xi\in\mathcal{O}_{\mathcal{FZ}}$. Then $\mathcal{T}$ is said to be a modified $\mathcal{FZ}$-ϑ_f-contraction w.r.t Ξ and $\vartheta_f\in\Omega$ if there exists $k\in(0,1)$ such that*

$$\omega(\mathcal{T}\phi,\mathcal{T}\varphi,\gamma)<1 \text{ implies } \Xi\Big(\vartheta_f(\omega(\mathcal{T}\phi,\mathcal{T}\varphi,\gamma)),(\vartheta_f(\Gamma(\phi,\varphi,\gamma)))^k\Big)\geq 0, \tag{4}$$

for all $\phi,\varphi\in\vartheta$ and $\gamma>0$, where $\Gamma(\phi,\varphi,\gamma))=\min\{\omega(\phi,\varphi,\gamma),\omega(\phi,\mathcal{T}\phi,\gamma),\omega(\varphi,\mathcal{T}\varphi,\gamma)\}$

Example 3.6. *Let $\mathcal{E} = [0,1]$ endowed with the fuzzy metric ω given by $\omega(\phi,\varphi,\gamma)) = \frac{\gamma}{\gamma+d(\phi,\varphi)}$ for all $\phi,\varphi \in \mathcal{E}, \gamma > 0$, where d is the usual metric. Then, $(\mathcal{E},\omega,*_p)$ is a GV-FMS,. Let $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ be defined by*

$$\mathcal{T}\phi = \begin{cases} \frac{1}{4} & \text{if } \phi = 1, \\ \\ \frac{1}{2} & \text{if } 0 \leq \phi < 1. \end{cases}$$

and $\vartheta_f(\beta) = e^{1-\frac{1}{\beta}}$ for all $\beta \in (0,1)$. We consider two cases:

. If $\phi \in [0, \frac{1}{4})$ and $\varphi = 1$, then

$$\begin{aligned} \Gamma(\phi,1,\gamma) &= \min\{\omega(\phi,1,\gamma), \omega(\phi,\mathcal{T}\phi,\gamma), \omega(1,\mathcal{T}1,\gamma)\} \\ &= \omega(\phi,1,\gamma) \\ &= \frac{\gamma}{\gamma+ \mid \phi - 1 \mid}. \end{aligned}$$

. If $\phi \in [\frac{1}{4}, 1)$ and $\varphi = 1$, then

$$\begin{aligned} \Gamma(\phi,1,\gamma) &= \min\{\omega(\phi,1,\gamma), \omega(\phi,\mathcal{T}\phi,\gamma), \omega(1,\mathcal{T}1,\gamma)\} \\ &= \omega(1,\mathcal{T}1,\gamma) \\ &= \frac{\gamma}{\gamma + \frac{3}{4}}. \end{aligned}$$

Thus, in all cases if we take $\vartheta_f(\beta) = e^{1-\frac{1}{\beta}}$ for all $\beta \in (0,1)$, $k = \frac{1}{3}$ and $\Xi = \Xi_{\mathcal{AB}}$, then we obtain that $\Xi\Big(\vartheta_f(\omega(\mathcal{T}\phi,\mathcal{T}\varphi,\gamma)), (\vartheta_f(\omega(\phi,\varphi,\gamma)))^k\Big) \geq 0$, that is, $\mathcal{T}$ is an $\mathcal{FZ}$-ϑ_f-contraction w.r.t Ξ and ϑ_f.

Theorem 3.7. *Let $(\mathcal{E},\omega,*)$ be a complete GV-FMS, and $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ be an $\mathcal{FZ}$-ϑ_f-contraction w.r.t Ξ. Then $\mathcal{T}$ has a unique FP.*

Proof. Define $\{\phi_n\}$ in $\mathcal{E}$ by

$$\mathcal{T}\phi_n = \phi_{n+1},$$

for all $n \geq 0$. If there exists $m \in \mathbb{N}$ such that $\phi_{n_0} = \phi_{n_0+1}$, then it follows that ϕ_{n_0} is a FP of $\mathcal{T}$. Assume that $\phi_n \neq \phi_{n+1}$ for all $n \in \mathbb{N}$. Then $\omega(\phi_n,\phi_{n+1},\gamma) < 1$ for all $n \in \mathbb{N}$ and $\gamma > 0$. From (3), we get

$$0 \leq \Xi\Big(\vartheta_f(\omega(\mathcal{T}\phi_n,\mathcal{T}\phi_{n-1},\gamma)), (\vartheta_f(\omega(\phi_n,\phi_{n-1},\gamma)))^k\Big) \tag{5}$$

$$\leq \frac{1}{(\vartheta_f(\omega(\phi_n,\phi_{n-1},\gamma)))^k} - \frac{1}{\vartheta_f(\omega(\mathcal{T}\phi_n,\mathcal{T}\phi_{n-1},\gamma))}. \tag{6}$$

Consequently,

$$\vartheta_f(\omega(\phi_n,\phi_{n-1},\gamma)) < (\vartheta_f(\omega(\phi_n,\phi_{n-1},\gamma)))^k \leq \vartheta_f(\omega(\mathcal{T}\phi_n,\mathcal{T}\phi_{n-1},\gamma)),$$

which means that

$$\omega(\phi_n,\phi_{n-1},\gamma) < \omega(\phi_{n+1},\phi_n,\gamma). \tag{7}$$

It follows that $\{\omega(\phi_n,\phi_{n-1},\gamma)\}$ is a nondecreasing sequence. Thus there exists $l(\gamma) \geq 1$ such that $\lim_{n\to+\infty} \omega(\phi_n,\phi_{n-1},\gamma) = l(\gamma)$ for all $\gamma > 0$. We shall prove that $l(\gamma) = 1$. Suppose that $l(\gamma_0) < 1$ for some $\gamma_0 > 0$. Take the sequences $\{\omega(\phi_{n+1},\phi_n,\gamma_0)$ and $\{\omega(\phi_n,\phi_{n-1},\gamma_0)\}$ and considering (Ξ3), we get

$$0 \leq \lim_{n\to+\infty} \sup \Xi\Big(\vartheta_f(\omega(\phi_{n+1},\phi_n,\gamma_0)), (\vartheta_f(\omega(\phi_n,\phi_{n-1},\gamma_0)))^k\Big) < 0,$$

a contradiction. Therefore,

$$\lim_{n\to+\infty}\omega(\phi_n,\phi_{n-1},\gamma)=1 \text{ for all } \gamma>0. \tag{8}$$

Next, we prove the Cauchyness of the sequence $\{\phi_n\}$. Suppose that $\{\phi_n\}$ is not a Cauchy sequence. Then there exists $\epsilon\in(0,1)$, $\gamma_0>0$ and two subsequences $\{\phi_{n_k}\}$ and $\{\phi_{m_k}\}$ of $\{\phi_n\}$ with $m_k>n_k\geq k$ for all $k\in\mathbb{N}$ such that

$$\omega(\phi_{m_k},\phi_{n_k},\gamma_0)\leq 1-\epsilon. \tag{9}$$

From Lemma 2.6, we derive that

$$\omega(\phi_{m_k},\phi_{n_k},\frac{\gamma_0}{2})\leq 1-\epsilon. \tag{10}$$

By considering n_k as the lowest value fulfilling (10), we get

$$\omega(\phi_{m_k-1},\phi_{n_k},\frac{\gamma_0}{2})>1-\epsilon. \tag{11}$$

Using (3) with $\phi=\phi_{m_k-1}$ and $\varphi=\phi_{n_k-1}$, we obtain

$$\begin{aligned}0&\leq\Xi\Big(\vartheta_f(\omega(\mathcal{T}\phi_{m_k-1},\mathcal{T}\phi_{n_k-1},\gamma_0)),(\vartheta_f(\omega(\phi_{m_k-1},\phi_{n_k-1},\gamma_0)))^k\Big)\\&=\Xi\Big(\vartheta_f(\omega(\phi_{m_k},\phi_{n_k},\gamma_0)),(\vartheta_f(\omega(\phi_{m_k-1},\phi_{n_k-1},\gamma_0))^k\Big)\\&\leq\frac{1}{(\vartheta_f(\omega(\phi_{m_k},\phi_{n_k},\gamma_0)))^k}-\frac{1}{\vartheta_f(\omega(\phi_{m_k-1},\phi_{n_k-1},\gamma_0))},\end{aligned} \tag{12}$$

which implies that

$$\vartheta_f(\omega(\phi_{m_k-1},\phi_{n_k-1},\gamma_0)<(\vartheta_f(\omega(\phi_{m_k-1},\phi_{n_k-1},\gamma_0)))^k\leq\vartheta_f(\omega(\phi_{m_k},\phi_{n_k},\gamma_0)).$$

Since ϑ_f is nondecreasing, we have

$$\omega(\phi_{m_k-1},\phi_{n_k-1},\gamma_0)<\omega(\phi_{m_k},\phi_{n_k},\gamma_0) \tag{13}$$

On account of (9),(11),(13) and the condition ($\mathcal{GV}4$) of fuzzy metric, we have

$$\begin{aligned}1-\epsilon&\geq\omega(\phi_{m_k},\phi_{n_k},\gamma_0)\\&>\omega(\phi_{m_k-1},\phi_{n_k-1},\gamma_0)\\&\geq\omega(\phi_{m_k-1},\phi_{n_k},\frac{\gamma_0}{2})*\omega(\phi_{n_k-1},\phi_{n_k},\frac{\gamma_0}{2})\\&>(1-\epsilon)*\omega(\phi_{n_k-1},\phi_{n_k},\frac{\gamma_0}{2}).\end{aligned}$$

Taking the limit as $k\to+\infty$ in both sides of the last inequality, by (8), we derive

$$\lim_{k\to+\infty}\omega(\phi_{m_k},\phi_{n_k},\gamma_0)=\lim_{k\to+\infty}\omega(\phi_{m_k-1},\phi_{n_k-1},\gamma_0)=1-\epsilon. \tag{14}$$

Passing to the limit as $k\to+\infty$ in (12), using (14) and the continuity of ϑ_f, we get

$$\Big(\vartheta_f(1-\epsilon)\Big)^k\leq\vartheta_f(1-\epsilon),$$

a contradiction. Therefore, $\{\phi_n\}$ is a Cauchy sequence. As $(\mathcal{E},\omega,*)$ is a complete GV-FMS, there exists $\phi\in\mathcal{E}$ such that $\phi_n\to\phi$ as $n\to+\infty$, that is,

$$\lim_{n\to+\infty}\omega(\phi_n,\phi,\gamma)=1. \tag{15}$$

By the continuity of $\mathcal{T}$ and (15), we obtain

$$\lim_{n\to+\infty}\omega(\phi_{n+1},\mathcal{T}\phi,\gamma)=\lim_{n\to+\infty}\omega(\mathcal{T}\phi_n,\mathcal{T}\phi,\gamma)=1.$$

Given that the limit is unique, it follows that $\mathcal{T}\phi=\phi$, thus ϕ is a FP of $\mathcal{T}$. Next, we prove the uniqueness of the FP of $\mathcal{T}$. Suppose that $\phi,\phi^*\in\mathcal{E}$ are two distinct FPs of the mapping $\mathcal{T}$. Then, $\mathcal{T}\phi=\phi$ and $\mathcal{T}\phi^*=\phi^*$ with $\phi\neq\phi^*$. By (3) we have

$$\begin{aligned}0&\leq\Xi\Big(\vartheta_f(\omega(\mathcal{T}\phi,\mathcal{T}\phi^*,\gamma)),(\vartheta_f(\omega(\phi,\phi^*,\gamma)))^k\Big)\\&=\Xi\Big(\vartheta_f(\omega(\phi,\phi^*,\gamma)),(\vartheta_f(\omega(\phi,\phi^*,\gamma)))^k\Big)\\&\leq\frac{1}{(\vartheta_f(\omega(\phi,\phi^*,\gamma)))^k}-\frac{1}{\vartheta_f(\omega(\phi,\phi^*,\gamma))}.\end{aligned}\tag{16}$$

Consequently

$$(\vartheta_f(\omega(\phi,\phi^*,\gamma)))^k\leq\vartheta_f(\omega(\phi,\phi^*,\gamma))\text{ for all }\gamma>0,$$

a contradiction. Thus, the FP is unique. □

Remark 3.8. *Note that if we take $\Xi=\Xi_{\mathcal{AB}}$ and $\vartheta_f(\beta)=e^{1-\frac{1}{\beta}}$ for all $\beta\in(0,1)$ then Theorem 3.7 reduces to Theorem 2.10 due to Gregori and Sapena. Moreover, if we consider $\zeta_{\mathcal{AB}}$ with $\vartheta_f(\beta)=e^{\beta-1}$ for all $\beta\in(0,1)$, Theorem 3.7 reduces to Theorem 2.11 due to Tirado.*

Example 3.9. *Let $\mathcal{E}=\{3^{3^\ell}:\ell\in\mathbb{N}\}\cup\{3\}$ be endowed with the fuzzy metric ω given by*

$$\omega(\phi,\varphi,\gamma)=\begin{cases}\frac{\phi}{\varphi} & \text{if } \phi\leq\varphi,\\ \frac{\varphi}{\phi} & \text{if } \varphi\leq\phi.\end{cases}$$

Then, $(\mathcal{E},\omega,*_p)$ is a complete GV-FMS. Now, consider the mapping $\mathcal{T}:\mathcal{E}\to\mathcal{E}$ defined by

$$\mathcal{T}\phi=\begin{cases}3^{3^{\ell-1}} & \text{if } \phi=3^{3^\ell},\ell\in\mathbb{N}\\ 3 & \text{if } \phi=3.\end{cases}$$

Also define $\Xi:(0,1]\times(0,1]\to\mathbb{R}$ by

$$\Xi(a,b)=\frac{1}{b}-\frac{1}{a}\text{ for all }a,b\in(0,1).$$

Take $\Xi_f(\beta)=\beta$ for all $\beta\in(0,1)$ and $k=\frac{1}{3}$. Then, for all $\ell,p\in\mathbb{N}$ with $\ell>p$, and all $\gamma>0$, we have

$$\begin{aligned}\vartheta_f(\omega(\mathcal{T}3^{3^p},\mathcal{T}3^{3^\ell},\gamma))&=\omega(\mathcal{T}3^{3^p},\mathcal{T}3^{3^\ell},\gamma)\\&=\frac{3^{3^{p-1}}}{3^{3^{\ell-1}}}\\&=\Big(\vartheta_f(\omega(3^{3^p},3^{3^\ell},\gamma))\Big)^{\frac{1}{3}}=\Big(\vartheta_f(\omega(3^{3^p},3^{3^\ell},\gamma))\Big)^k.\end{aligned}$$

It follows that

$$\begin{aligned}&\Xi\Big(\vartheta_f(\omega(\mathcal{T}(3^{3^p}),\mathcal{T}(3^{3^\ell}),\gamma)),(\vartheta_f(\omega(3^{3^p},3^{3^\ell},\gamma)))^k\Big)\\&=\Xi\Big(\omega(\mathcal{T}(3^{3^p}),\mathcal{T}(3^{3^\ell}),\gamma),(\omega(3^{3^p},3^{3^\ell},\gamma))^k\Big)\\&=\frac{1}{(\omega(3^{3^p},3^{3^\ell},\gamma))^{\frac{1}{3}}}-\frac{1}{\omega(\mathcal{T}(3^{3^p}),\mathcal{T}(3^{3^\ell}),\gamma)}\\&\geq 0.\end{aligned}$$

Also, we have

$$\begin{aligned}\vartheta_f(\omega(\mathcal{T}3,\mathcal{T}3^{3^\ell},\gamma)) &= \omega(\mathcal{T}3,\mathcal{T}3^{3^\ell},\gamma)\\ &= \frac{3}{3^{3^{\ell-1}}}\\ &> \Big(\frac{3}{3^{3^\ell}}\Big)^{\frac{1}{3}} = \Big(\vartheta_f(\omega(3,3^{3^\ell},\gamma))\Big)^{\frac{1}{3}}.\end{aligned}$$

Hence

$$\begin{aligned}&\Xi\Big(\vartheta_f\Big(\omega(\mathcal{T}3,\mathcal{T}3^{3^\ell},\gamma)\Big),\Big(\vartheta_f(\omega(3,3^{3^\ell},\gamma))\Big)^k\Big)\\ &= \Xi\Big(\omega(\mathcal{T}3,\mathcal{T}3^{3^\ell},\gamma),(\omega(3,3^{3^\ell},\gamma))^{\frac{1}{3}}\Big)\\ &= \frac{1}{(\omega(3,3^{3^\ell},\gamma))^{\frac{1}{3}}} - \frac{1}{\omega(\mathcal{T}3,\mathcal{T}3^{3^\ell},\gamma)}\\ &= \frac{1}{(\frac{3}{3^{3^\ell}})^{\frac{1}{3}}} - \frac{1}{\frac{3}{3^{3^{\ell-1}}}}\\ &\geq 0.\end{aligned}$$

Thus, in all cases (3) is satisfied. Therefore, $\mathcal{T}$ is an $\mathcal{FZ}$-ϑ_f-contraction w.r.t Ξ and $\mathcal{T}$ possesses a unique FP, namely, $\phi = 3$.

Corollary 3.10. *[29] Let $(\mathcal{E},\omega,*)$ be a complete GV-FMS, and $\mathcal{T}:\mathcal{E}\longrightarrow\mathcal{E}$ be a mapping satisfying:*

$$\omega(\mathcal{T}\phi,\mathcal{T}\varphi,\gamma) < 1 \text{ implies } \vartheta_f(\omega(\mathcal{T}\phi,\mathcal{T}\varphi,\gamma)) \geq \Big(\vartheta_f(\omega(\phi,\varphi,\gamma))\Big)^k,$$

for all $\phi,\varphi\in\mathcal{E}$ and $\gamma > 0$. Then $\mathcal{T}$ has a unique FP.

Proof. The proof follows from Theorem 3.7 by considering $\Xi(a,b) = \frac{1}{b} - \frac{1}{a}$ for all $a,b\in(0,1]$. □

Corollary 3.11. *[29] Let $(\mathcal{E},\omega,*)$ be a complete GV-FMS and $\mathcal{T}:\mathcal{E}\longrightarrow\mathcal{E}$ be a mapping satisfying:*

$$\omega(\mathcal{T}\phi,\mathcal{T}\varphi,\gamma) < 1 \Longrightarrow \Big[1+\sin\Big(\frac{\pi}{2}(\omega(\phi,\varphi,\gamma)-1)\Big)\Big]^k \leq 1+\sin\Big(\frac{\pi}{2}(\omega(\mathcal{T}\phi,\mathcal{T}\varphi,\gamma)-1)\Big),$$

for all $\phi,\varphi\in\mathcal{E}$ and $\gamma > 0$. Then $\mathcal{T}$ has a unique FP.

Proof. The proof follows from Theorem 3.7 by taking $\Xi(a,b) = \frac{1}{b} - \frac{1}{a}$ for all $a,b\in(0,1]$ with $\vartheta_f(\beta) = 1+\sin\Big(\frac{\pi}{2}(\beta-1)\Big)$ for all $\beta\in(0,1)$. □

Now, utilizing modified $\mathcal{FZ}$-ϑ_f-contraction, we prove the following more general result.

Theorem 3.12. *Let $(\mathcal{E},\omega,*)$ be a complete GV-FMS and $\mathcal{T}:\mathcal{E}\longrightarrow\mathcal{E}$ be a modified $\mathcal{FZ}$-ϑ_f-contraction w.r.t Ξ. Then $\mathcal{T}$ has a unique FP.*

Proof. Define $\{\phi_n\}$ in ϑ by

$$T\phi_n = \phi_{n+1}$$

for all $n\geq 0$. If there exists $l_0\in\mathbb{N}$ such that $\phi_{l_0} = \phi_{l_0+1}$, then it follows that ϕ_{l_0} is a FP of $\mathcal{T}$. Assume that $\phi_n\neq\phi_{n+1}$ for all $n\in\mathbb{N}$. Then $\omega(\phi_n,\phi_{n+1},\gamma) < 1$ for all $n\in\mathbb{N}$ and $\gamma > 0$. From (4), we obtain

$$0 \leq \Xi\Big(\vartheta_f(\omega(\mathcal{T}\phi_{n-1},\mathcal{T}\phi_n,\gamma)),(\vartheta_f(\Gamma(\phi_{n-1},\phi_n,\gamma)))^k\Big). \tag{17}$$

Hence

$$0 \leq \frac{1}{(\vartheta_f(\Gamma(\phi_{n-1}, \phi_n, \gamma)))^k} - \frac{1}{\vartheta_f(\omega(\phi_n, \phi_{n+1}, \gamma))}.$$

Consequently,

$$(\vartheta_f(\Gamma(\phi_{n-1}, \phi_n, \gamma)))^k \leq \vartheta_f(\omega(\phi_n, \phi_{n+1}, \gamma)), \tag{18}$$

where

$$\begin{aligned}\Gamma(\phi_{n-1}, \phi_n, \gamma) &= \min\{\omega(\phi_{n-1}, \phi_n, \gamma), \omega(\phi_{n-1}, \mathcal{T}\phi_{n-1}, \gamma), \omega(\phi_n, \mathcal{T}\phi_n, \gamma)\} \\ &= \min\{\omega(\phi_{n-1}, \phi_n, \gamma), \omega(\phi_{n-1}, \phi_n, \gamma), \omega(\phi_n, \phi_{n+1}, \gamma)\} \\ &= \min\{\omega(\phi_{n-1}, \phi_n, \gamma), \omega(\phi_n, \phi_{n+1}, \gamma)\}.\end{aligned} \tag{19}$$

Now, if $\min\{\omega(\phi_{n-1}, \phi_n, \gamma), \omega(\phi_n, \phi_{n+1}, \gamma)\} = \omega(\phi_n, \phi_{n+1}, \gamma)$, it follows from (18) that

$$\omega(\phi_n, \phi_{n+1}, \gamma) < (\vartheta_f(\omega(\phi_n, \phi_{n+1}, \gamma)))^k \leq \vartheta_f(\omega(\phi_n, \phi_{n+1}, \gamma)),$$

which is a contradiction. Hence, $\min\{\omega(\phi_{n-1}, \phi_n, \gamma), \omega(\phi_n, \phi_{n+1}, \gamma)\} = \omega(\phi_{n-1}, \phi_n, \gamma)$, by (18) we have

$$\omega(\phi_{n-1}, \phi_n, \gamma) < (\vartheta_f(\omega(\phi_{n-1}, \phi_n, \gamma)))^k \leq \vartheta_f(\omega(\phi_n, \phi_{n+1}, \gamma)).$$

This means that $\{\omega(\phi_n, \phi_{n-1}, \gamma)\}$ is a nondecreasing sequence. Thus there exists $s(\gamma) \geq 1$ such that $\lim_{n\to+\infty} \omega(\phi_{n-1}, \phi_n, \gamma) = s(\gamma)$ for all $\gamma > 0$. We shall prove that $s(\gamma) = 1$. Reasoning by contradiction, suppose that $s(\gamma_0) < 1$ for some $\gamma_0 > 0$. Now, if we take the sequences $\{\omega(\phi_n, \phi_{n+1}, \gamma_0)$ and $\{\omega(\phi_{n-1}, \phi_n, \gamma_0)\}$ and considering (Ξ3), we obtain

$$0 \leq \lim_{n\to+\infty} \sup \Xi\Big(\vartheta_f(\omega(\phi_n, \phi_{n+1}, \gamma_0)), (\vartheta_f(\omega(\phi_{n-1}, \phi_n, \gamma_0)))^k\Big) < 0,$$

a contradiction. Which yields

$$\lim_{n\to+\infty} \omega(\phi_{n-1}, \phi_n, \gamma) = 1 \text{ for all } \gamma > 0. \tag{20}$$

The proof of Cauchyness of the sequence $\{\phi_n\}$ is omitted since it occurs on the same line as in the proof of Theorem 3.7. Next, since $(\mathcal{E}, \omega, *)$ is a complete GV-FMS, there exists $\phi \in \mathcal{E}$ such that $\phi_n \to \phi$ as $n \to +\infty$. Hence

$$\lim_{n\to+\infty} \omega(\phi_n, \phi, \gamma) = 1, \gamma > 0. \tag{21}$$

Now, we prove the existence of the FP. Let $S = \{n \in \mathbb{N} : \phi_{n+1} = \mathcal{T}\phi\}$. If S is infinite, then there exists $\{\phi_{n_k+1}\} \subseteq \{\phi_{n+1}\}$ such that $\lim_{k\to+\infty} \phi_{n_k+1} = \mathcal{T}\phi$, thus $\mathcal{T}\phi = \phi$. If S is finite, it follows that $\phi_{n+1} \neq \mathcal{T}\phi$ for infinitely $n \in \mathbb{N}$, then there exists $\{\phi_{n_k+1}\} \subseteq \{\phi_{n+1}\}$ with $\omega(\phi_{n_k+1}, \mathcal{T}\phi, \gamma) < 1$. From (4), we have

$$\begin{aligned}0 &\leq \Xi\Big(\vartheta_f(\omega(\phi_{n_k+1}, \mathcal{T}\phi, \gamma)), (\vartheta_f(\Gamma(\phi_{n_k}, \phi, \gamma)))^k\Big) \\ &\leq \frac{1}{(\vartheta_f(\Gamma(\phi_{n_k}, \phi, \gamma)))^k} - \frac{1}{\vartheta_f(\omega(\phi_{n_k+1}, \mathcal{T}\phi, \gamma)))},\end{aligned}$$

which means

$$(\vartheta_f(\Gamma(\phi_{n_k}, \phi, \gamma)))^k \leq \vartheta_f(\omega(\phi_{n_k+1}, \mathcal{T}\phi, \gamma)), \tag{22}$$

where

$$\Gamma(x_{n_k}, \phi, \gamma) \quad = \min\{\omega(\phi_{n_k}, \phi, \gamma), \omega(\phi_{n_k}, \phi_{n_k+1}, \gamma), \omega(\phi, \mathcal{T}\phi, \gamma)\}.$$

If we suppose that $\omega(\phi, \mathcal{T}\phi, \gamma) < 1$, we get

$$\begin{aligned}\lim_{k\to+\infty} \Gamma(\phi_{n_k}, \phi, \gamma) &= \min\{1, 1, \omega(\phi, \mathcal{T}\phi, \gamma)\} \\ &= \omega(\phi, \mathcal{T}\phi, \gamma)\end{aligned} \tag{23}$$

Taking the limit as $k \to +\infty$ in (22) and using (23), we derive that

$$(\vartheta_f(\Gamma(\phi, \mathcal{T}\phi, \gamma)))^k \leq \vartheta_f(\omega(\phi, \mathcal{T}\phi, \gamma)).$$

Which is a contradiction, since $k \in (0, 1)$. Therefore, $\omega(\phi, \mathcal{T}\phi, \gamma) = 1$, thus $\mathcal{T}\phi = \phi$.

Finally, we prove the uniqueness of the FP of $\mathcal{T}$. We argue by contradiction, assume that $\phi, \tilde{\phi} \in \mathcal{E}$ are two distinct FPs of $\mathcal{T}$. Applying (4), we have

$$\begin{aligned}0 &\leq \Xi\Big(\vartheta_f(\omega(\mathcal{T}\phi, \mathcal{T}\tilde{\phi}, \gamma)), (\vartheta_f(\Gamma(\phi, \tilde{\phi}, \gamma)))^k\Big) \\ &= \Xi\Big(\vartheta_f(\omega(\phi, \tilde{\phi}, \gamma)), (\vartheta_f(\Gamma(\phi, \tilde{\phi}, \gamma)))^k\Big) \\ &\leq \frac{1}{(\vartheta_f(\Gamma(\phi, \tilde{\phi}, \gamma)))^k} - \frac{1}{\vartheta_f(\omega(\phi, \tilde{\phi}, \gamma))}.\end{aligned}$$

Hence

$$(\vartheta_f(\Gamma(\phi, \tilde{\phi}, \gamma)))^k \leq \vartheta_f(\omega(\phi, \tilde{\phi}, \gamma)) \text{ for all } \gamma > 0, \tag{24}$$

where

$$\begin{aligned}\Gamma(\phi, \tilde{\phi}, \gamma) &= \min\{\omega(\phi, \tilde{\phi}, \gamma), \omega(\phi, \mathcal{T}\phi, \gamma), \omega(\tilde{\phi}, \mathcal{T}\tilde{\phi}, \gamma)\} \\ &= \min\{\omega(\phi, \tilde{\phi}, \gamma), 1, 1\} \\ &= \omega(\phi, \tilde{\phi}, \gamma).\end{aligned} \tag{25}$$

Using (24) and (25), we deduce

$$(\vartheta_f(\omega(\phi, \tilde{\phi}, \gamma)))^k \leq \vartheta_f(\omega(\phi, \tilde{\phi}, \gamma)) \text{ for all } \gamma > 0,$$

a contradiction. Hence, the FP is unique. □

Corollary 3.13. *[29] Let $(\mathcal{E}, \omega, *)$ be a complete GV-FMS and $T : \mathcal{E} \to \mathcal{E}$ be a mapping such that for all $\phi, \varphi \in \mathcal{E}$, $\gamma > 0$ and for some $k \in (0, 1)$*

$$1 - \omega(T\phi, T\varphi, \gamma) \leq k\Big(1 - \Gamma(\phi, \varphi, \gamma)\Big),$$

where $\Gamma(\phi, \varphi, \gamma)) = \min\{\omega(\phi, \varphi, \gamma), \omega(\phi, \mathcal{T}\varphi, \gamma), \omega(\varphi, \mathcal{T}\varphi, \gamma)\}$. Then T has a unique FP.

Proof. The proof follows from Theorem 3.12 by considering $\Xi = \Xi_{\mathcal{AB}}$ and $\vartheta_f(\beta) = e^{\beta-1}$ for all $\beta \in (0, 1)$. □

Corollary 3.14. *[29] Let $(\mathcal{E}, \omega, *)$ be a complete GV-FMS and $\mathcal{T} : \mathcal{E} \longrightarrow \mathcal{E}$ be a mapping such that for all $\phi, \varphi \in \mathcal{E}$, $\gamma > 0$ and for some $k \in (0, 1)$*

$$\omega(\mathcal{T}\phi, \mathcal{T}\varphi, \gamma) < 1 \text{ implies } \Big(\vartheta_f(\Gamma(\phi, \varphi, \gamma))\Big)^k \leq \vartheta_f(\omega(\mathcal{T}\phi, \mathcal{T}\varphi, \gamma)),$$

where $\Gamma(\phi, \varphi, \gamma)) = \min\{\omega(\phi, \varphi, \gamma), \omega(\phi, \mathcal{T}\phi, \gamma), \omega(\varphi, \mathcal{T}\varphi, \gamma)\}$. Then $\mathcal{T}$ has a unique FP.

Proof. The proof follows from Theorem 3.12 by considering $\Xi(a, b) = \frac{1}{b} - \frac{1}{a}$ for all $a, b \in (0, 1]$. □

Corollary 3.15. *[29] Let $(\mathcal{E}, \omega, *)$ be a complete GV-FMS and $\mathcal{T} : \mathcal{E} \longrightarrow \vartheta$ be a mapping satisfying, for all $\phi, \varphi \in \mathcal{E}$, $\gamma > 0$ and for some $k \in (0, 1)$*

$$\omega(\mathcal{T}\phi, \mathcal{T}\varphi, \gamma) < 1 \Longrightarrow \left[1 - \cos\left(\frac{\pi}{2}\Gamma(\phi, \varphi, \gamma)\right)\right]^k \leq 1 - \cos\left(\frac{\pi}{2}\omega(\mathcal{T}\phi, \mathcal{T}\varphi, \gamma)\right),$$

where $\Gamma(\phi, \varphi, \gamma)) = \min\{\omega(\phi, \varphi, \gamma), \omega(\phi, \mathcal{T}\phi, \gamma), \omega(\varphi, \mathcal{T}\varphi, \gamma)\}$. Then $\mathcal{T}$ has a unique FP.

Proof. The proof follows from Theorem 3.12 by considering $\Xi(a, b) = \frac{1}{b} - \frac{1}{a}$ for all $a, b \in (0, 1]$ with $\vartheta_f(\beta) = 1 - \cos\left(\frac{\pi}{2}(\beta)\right)$ for all $\beta \in (0, 1)$. □

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