Filomat 38:7 (2024), 2217–2226
https://doi.org/10.2298/FIL2407217L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Characterization of linear mappings on Banach algebras

Shan Li[a], Jiankui Li[b,*], Kaijia Luo[b]

[a]*Department of Mathematics, Jiangsu University of Technology, Changzhou 213000, China*
[b]*School of Mathematics, East China University of Science and Technology, Shanghai 200237, China*

Abstract. Let $\mathcal{A}$ be a unital Banach algebra, $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule, and W be a separating point of $\mathcal{M}$. We show that if linear mappings δ and τ from $\mathcal{A}$ into $\mathcal{M}$ satisfy $\delta(AB) = \delta(A)B + A\tau(B)$ for each A, B in $\mathcal{A}$ with $AB = W$, then τ is a Jordan derivation and δ is a generalized Jordan derivation. Based on this result, if linear mappings δ and τ from a unital semisimple Banach algebra $\mathcal{A}$ into itself satisfy $\delta(W) = \delta(A)B + A\tau(B)$ for each $A, B \in \mathcal{A}$ with $AB = W$, then τ is a Jordan derivation and $\delta(A) = \tau(A) + \delta(I)A$ for every A in $\mathcal{A}$. As an application, we present a characterization of linear mappings δ and τ on a unital semisimple Banach $*$-algebra $\mathcal{A}$ satisfying $\delta(W) = \delta(A)B^* + A\tau(B)^*$ for each $A, B \in \mathcal{A}$ with $AB^* = W$.

1. Introduction

Let $\mathcal{A}$ be a unital Banach algebra over the complex field $\mathbb{C}$, $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule, and δ be a linear mapping from $\mathcal{A}$ into $\mathcal{M}$. Recall that δ is called a *derivation* if $\delta(AB) = \delta(A)B + A\delta(B)$ for each $A, B \in \mathcal{A}$; δ is called a *Jordan derivation* if $\delta(A^2) = \delta(A)A + A\delta(A)$ for each $A \in \mathcal{A}$.

Let W be a fixed element of $\mathcal{A}$. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is said to be a *derivable mapping at* W if

$$\delta(W) = \delta(A)B + A\delta(B)$$

for every A, B in $\mathcal{A}$ with $AB = W$. There are a wide number of scholars investigating the conditions under which mappings on algebras are thoroughly determined by their action on fixed products. Several previous papers study derivable mappings at zero on various algebras [1, 9–11]. Several people consider derivable mappings at a non-zero element under certain conditions [4, 12, 14, 16, 18].

We recall that a $*$-*algebra* is an algebra $\mathcal{A}$ equipped with an involution that is a mapping $*$ from $\mathcal{A}$ into itself, such that

$$(\lambda A + \mu B)^* = \overline{\lambda}A^* + \overline{\mu}B^*, \quad (AB)^* = B^*A^* \quad \text{and} \quad (A^*)^* = A,$$

2020 *Mathematics Subject Classification*. Primary 47B47; Secondary 16W25.
Keywords. Banach algebra; Derivation; Jordan derivation; Separating point.
Received: 27 August 2021; Accepted: 18 October 2023
Communicated by Dragan S. Djordjević
The first author (Shan Li) was supported by Changzhou Sci&Tech Program (Grant No. CJ20235062), the second author (Jiankui Li) was supported by the National Natural Science Foundation of China (Grant No.11871021).
* Corresponding author: Jiankui Li
Email addresses: lishan_math@163.com (Shan Li), jkli@ecust.edu.cn (Jiankui Li), kaijia_luo@163.com (Kaijia Luo)

whenever A, B in $\mathcal{A}$, λ, μ in $\mathbb{C}$, where $\overline{\lambda}$, $\overline{\mu}$ denote conjugate complex numbers. Let $\mathcal{A}$ be a $*$-algebra, then an $\mathcal{A}$-bimodule $\mathcal{M}$ is a *$*$-$\mathcal{A}$-bimodule* if $\mathcal{M}$ equipped with a $*$-mapping from $\mathcal{M}$ into itself, such that

$$(\lambda M + \mu N)^* = \overline{\lambda}M^* + \overline{\mu}N^* \quad (AM)^* = M^*A^* \quad (MA)^* = A^*M^* \text{ and } (M^*)^* = M,$$

whenever A, B in $\mathcal{A}$, M, N in $\mathcal{M}$ and λ, μ in $\mathbb{C}$.

Kishimoto [13] studies $*$-derivations. Let $\mathcal{A}$ be a $*$-algebra and $\mathcal{M}$ be a $*$-$\mathcal{A}$-bimodule. A derivation δ from $\mathcal{A}$ into $\mathcal{M}$ is a *$*$-derivation* if $\delta(A^*) = \delta(A)^*$ for each A in $\mathcal{A}$. In fact, every derivation is a linear combination of two $*$-derivations. We can define a linear mapping $\widehat{\delta}$ from $\mathcal{A}$ into $\mathcal{M}$ by $\widehat{\delta}(A) = \delta(A^*)^*$ for every A in $\mathcal{A}$. Therefore $\delta = \delta_1 + i\delta_2$, where $\delta_1 = \frac{1}{2}(\delta + \widehat{\delta})$ and $\delta_2 = \frac{1}{2i}(\delta - \widehat{\delta})$. Hence, δ_1 and δ_2 are both $*$-derivations.

Let W be a fixed element in $\mathcal{A}$. A linear mapping δ from a $*$-algebra $\mathcal{A}$ into its $*$-$\mathcal{A}$-bimodule $\mathcal{M}$ is a *$*$-derivable mapping at W* if

$$\delta(W) = A\delta(B)^* + \delta(A)B^*$$

for each A, B in $\mathcal{A}$ with $AB^* = W$. A number of papers are devoted to characterizing $*$-derivable mappings at zero on C^*-algebras, zero product determined algebras, group algebras $L^1(G)$ and standard operator algebras [6, 7, 10].

In this paper, we study a natural problem of characterizing linear mappings δ and τ from a unital Banach algebra $\mathcal{A}$ into its unital $\mathcal{A}$-bimodule $\mathcal{M}$ satisfying

$$\delta(W) = \delta(A)B + A\tau(B) \qquad (F_W)$$

for each A, B in $\mathcal{A}$ with $AB = W$, where W is a fixed element of $\mathcal{A}$. The forerunners in this line can be attributed to Ghahramani, Benkovic, etc [3, 5, 8] where these authors consider linear mappings δ and τ from a zero product determined algebra (ring) or standard operator algebra into its bimodule satisfying (F_W), where W is zero. However, according to our knowledge, δ and τ satisfying (F_W), where W is a non-zero element, have not been deeply studied.

We also study the problem of characterizing linear mappings δ and τ from a Banach $*$-algebra $\mathcal{A}$ into its $*$-$\mathcal{A}$-bimodule $\mathcal{M}$ satisfying

$$\delta(W) = \delta(A)B^* + A\tau(B)^* \qquad (F_W^*)$$

for each A, B in $\mathcal{A}$ with $AB^* = W$, where W is a fixed element of $\mathcal{A}$. If we assume that $\delta = \tau$ in (F_W^*), then the $*$-derivable mapping at W is obtained.

This paper is organized as follows. In Section 2, we give a completely characterization of linear mappings δ and τ from a Banach algebra $\mathcal{A}$ with unit I into its unital $\mathcal{A}$-bimodule $\mathcal{M}$ satisfying (F_I) (Theorem 2.3). In this section, we also study linear mappings δ and τ from a unital Banach $*$-algebra $\mathcal{A}$ into its unital $*$-$\mathcal{A}$-bimodule $\mathcal{M}$ satisfying (F_I^*) (Theorem 2.7).

In Section 3, we establish some theorems about linear mappings satisfying F_W, where W is a separating point. One of the main results (Theorem 3.1) states that if δ and τ are linear mappings from a unital Banach algebra $\mathcal{A}$ into its unital $\mathcal{A}$-bimodule $\mathcal{M}$ satisfying (F_W), then τ is a Jordan derivation and δ is a generalized Jordan derivation.

Based on this result, we consider a special class of algebra: semisimple algebra. We prove that linear mappings δ and τ on a unital semisimple Banach algebra $\mathcal{A}$ satisfy (F_W) if and only if τ is a derivation and $\delta(A) = \tau(A) + \delta(I)A$ for each A in $\mathcal{A}$. Moreover, δ and τ satisfy $\delta(WA) = \delta(W)A + W\tau(A)$ and $\tau(AW) = A\tau(W) + \delta(A)W - \delta(I)AW$ for each A in $\mathcal{A}$ (Theorem 3.3). As an application, we present a characterization of linear mappings δ and τ on a unital semisimple Banach $*$-algebra $\mathcal{A}$ satisfying (F_W^*) (Theorem 3.4).

2. Unit element

This section is devoted to characterizing linear mappings δ and τ from a unital Banach algebra $\mathcal{A}$ into its unital $\mathcal{A}$-bimodule $\mathcal{M}$ satisfying F_I or F_I^*, where I is the unit element of $\mathcal{A}$.

For this purpose, it is worth to recall some notions. Let $\mathcal{A}$ be a unital algebra and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *generalized Jordan derivation* if $\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B) - A\delta(I)B - B\delta(I)A$ for each $A, B \in \mathcal{A}$, where $A \circ B = AB + BA$ on $\mathcal{A}$.

Lemma 2.1. *Let $\mathcal{A}$ be a unital algebra, $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule, and δ be a linear mapping from $\mathcal{A}$ into $\mathcal{M}$. The following are equivalent:*

1. *δ is a generalized Jordan derivation;*
2. *there is a Jordan derivation τ from $\mathcal{A}$ into $\mathcal{M}$ such that $\delta(A \circ B) = \delta(A)B + \delta(B)A + A\tau(B) + B\tau(A)$ for each A, B in $\mathcal{A}$;*
3. *there is a Jordan derivation τ from $\mathcal{A}$ into $\mathcal{M}$ such that $\delta(A \circ B) = \tau(A)B + \tau(B)A + A\delta(B) + B\delta(A)$ for each A, B in $\mathcal{A}$;*
4. *there is a Jordan derivation τ from $\mathcal{A}$ into $\mathcal{M}$ such that $\delta(A^2) = \delta(A)A + A\tau(A)$ for each A in $\mathcal{A}$;*
5. *there is a Jordan derivation τ from $\mathcal{A}$ into $\mathcal{M}$ such that $\delta(A^2) = \tau(A)A + A\delta(A)$ for each A in $\mathcal{A}$;*
6. *there is a Jordan derivation τ from $\mathcal{A}$ into $\mathcal{M}$ such that $\delta(A) = \tau(A) + \delta(I)A$ for each A in $\mathcal{A}$;*
7. *there is a Jordan derivation τ from $\mathcal{A}$ into $\mathcal{M}$ such that $\delta(A) = \tau(A) + A\delta(I)$ for each A in $\mathcal{A}$.*

If δ has one of the properties (2)-(7), then δ is called a *generalized Jordan derivation*, and τ the *relating Jordan derivation*

Recall that a linear mapping γ from $\mathcal{A}$ into $\mathcal{M}$ is called a *left Jordan centralizer* if $\gamma(A^2) = \gamma(A)A$ holds for each A in $\mathcal{A}$.

Lemma 2.2. *Let $\mathcal{A}$ be an algebra. Then δ is a generalized Jordan derivation if and only if $\delta = \tau + \gamma$, where τ is a Jordan derivation of $\mathcal{A}$ and γ is a left Jordan centralizer of $\mathcal{A}$.*

Proof. Firstly, suppose that δ is a generalized Jordan derivation of $\mathcal{A}$, then there exists a Jordan derivation τ such that $\delta(A^2) = \delta(A)A + A\tau(A)$ and $\tau(A^2) = \tau(A)A + A\tau(A)$ for each $A \in \mathcal{A}$. Thus $(\delta - \tau)(A^2) = (\delta - \tau)(A)A$ for each A in $\mathcal{A}$. This implies $\gamma = \delta - \tau$ is a left Jordan centralizer of $\mathcal{A}$ and $\delta = \tau + \gamma$, as desired.

Conversely, by direct calculation

$$\begin{aligned}\delta(A^2) &= \tau(A^2) + \gamma(A^2)\\ &= \tau(A)A + A\tau(A) + \gamma(A)A\\ &= (\tau + \gamma)(A)A + A\tau(A)\\ &= \delta(A)A + A\tau(A)\end{aligned}$$

for each A in $\mathcal{A}$. Thus δ is a generalized Jordan derivation. □

We now explore our first conclusion about linear mappings satisfying F_I.

Theorem 2.3. *Let $\mathcal{A}$ be a unital Banach algebra and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. If δ and τ are linear mappings from $\mathcal{A}$ into $\mathcal{M}$, then the following are equivalent:*
(i) $\delta(I) = \delta(A)A^{-1} + A\tau(A^{-1})$ for each invertible element A in $\mathcal{A}$;
(ii) $\delta(I) = \delta(A)B + A\tau(B)$ for each $A, B \in \mathcal{A}$ with $AB = I$;
(iii) τ is a Jordan derivation and $\delta(A) = \tau(A) + \delta(I)A$ for each A in $\mathcal{A}$.

Proof. It is clear that (ii) implies (i). First we prove that (i) implies (iii). By assumption $\delta(I) = \delta(I) + \tau(I)$, we have $\tau(I) = 0$. Let T be an invertible element in $\mathcal{A}$. By (i),

$$\delta(I) = \delta(TT^{-1}) = \delta(T)T^{-1} + T\tau(T^{-1}).$$

It follows that

$$\delta(T)T^{-1} = \delta(I) - T\tau(T^{-1}), \tag{2.1}$$

$$T\tau(T^{-1}) = \delta(I) - \delta(T)T^{-1}. \tag{2.2}$$

Let $A \in \mathcal{A}$, $n \in \mathbb{N}$ with $n \geq \|A\| + 2$, and $B = nI + A$. Then B and $I - B$ are both invertible in $\mathcal{A}$. By (2.1) and (2.2), we obtain that

$$\begin{aligned}
\delta(B)B^{-1} &= \delta(I) - B\tau(B^{-1}) \\
&= \delta(I) - B\tau(B^{-1}(I-B) + I) \\
&= \delta(I) - B\tau(B^{-1}(I-B)) \\
&= \delta(I) - (I-B)[(I-B)^{-1}B\tau(B^{-1}(I-B))] \\
&= \delta(I) - (I-B)[\delta(I) - \delta((I-B)^{-1}B)B^{-1}(I-B)] \\
&= \delta(I) - (I-B)\delta(I) + (I-B)\delta((I-B)^{-1}B)B^{-1}(I-B) \\
&= B\delta(I) + (I-B)\delta((I-B)^{-1} - I)B^{-1}(I-B) \\
&= B\delta(I) + (I-B)\delta((I-B)^{-1})(I-B)B^{-1} - (I-B)\delta(I)B^{-1}(I-B) \\
&= B\delta(I) + (I-B)[\delta(I) - (I-B)^{-1}\tau(I-B)]B^{-1} - (I-B)\delta(I)B^{-1}(I-B) \\
&= B\delta(I) + (I-B)\delta(I)B^{-1} - \tau(I-B)B^{-1} - (I-B)\delta(I)B^{-1}(I-B) \\
&= \delta(I) + \tau(B)B^{-1}.
\end{aligned}$$

Thus $\delta(B)B^{-1} = \delta(I) + \tau(B)B^{-1}$, i.e., $\delta(B) = \delta(I)B + \tau(B)$. Since $B = nI + A$, we have

$$\delta(A) = \tau(A) + \delta(I)A \tag{2.3}$$

for each A in $\mathcal{A}$.

It remains to show that τ is a Jordan derivation. For each invertible element $T \in \mathcal{A}$, by (2.3) and (i), we have

$$\begin{aligned}
\tau(T^{-1})T + T^{-1}\tau(T) &= [\delta(T^{-1}) - \delta(I)T^{-1}]T + T^{-1}\tau(T) \\
&= \delta(T^{-1})T + T^{-1}\tau(T) - \delta(I) \\
&= 0.
\end{aligned}$$

It follows from [16, Lemma 2.1], τ is a Jordan derivation.

Finally, we prove that (iii) implies (ii). For each $A, B \in \mathcal{A}$ with $AB = I$, since τ is a Jordan derivation,

$$\begin{aligned}
\delta(A) = \delta(ABA) &= \tau(ABA) + \delta(I)ABA \\
&= \tau(A)BA + A\tau(B)A + AB\tau(A) + \delta(I)ABA \\
&= (\tau(A) + \delta(I)A)BA + A\tau(B)A + \tau(A) \\
&= \delta(A)BA + A\tau(B)A + \delta(A) - \delta(I)A.
\end{aligned}$$

That is, $0 = [\delta(A)B + A\tau(B) - \delta(I)]A$. Hence $\delta(I) = \delta(A)B + A\tau(B)$ for each $A, B \in \mathcal{A}$ with $AB = I$. □

Remark 2.4. *(i) Let us point out here that one can also show that if (ii) is replaced with $\delta(I) = \tau(A)B + A\delta(B)$ for each $A, B \in \mathcal{A}$ with $AB = I$ in the above theorem, the conclusion still holds.*
(ii) In fact, Theorem 2.3 is an extension of [15, Theorem 2]. If δ and τ are linear mappings from $\mathcal{A}$ into $\mathcal{M}$ with $\delta(I) = 0$, then the following are equivalent:

1. $0 = \delta(A)A^{-1} + A\tau(A^{-1})$ *for each invertible element A in $\mathcal{A}$;*
2. $0 = \delta(A)B + A\tau(B)$ *for each $A, B \in \mathcal{A}$ with $AB = I$;*

3. *δ and τ are Jordan derivations and $\delta(A) = \tau(A)$ for each A in $\mathcal{A}$.*

The next corollary follows immediately from Theorem 2.3.

Corollary 2.5. *Let $\mathcal{A}$ be a unital Banach algebra and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. If δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$, then the following are equivalent:*
(i) $\delta(I) = \delta(A)A^{-1} + A\delta(A^{-1}) - A\delta(I)A^{-1}$ for each inverible $A \in \mathcal{A}$;
(ii) $\delta(I) = \delta(A)B + A\delta(B) - A\delta(I)B$ for each $A, B \in \mathcal{A}$ with $AB = I$;
(iii) δ is a generalized Jordan derivation.

Let W be a fixed element of $\mathcal{A}$. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *generalized derivable mapping at W* if $\delta(AB) = \delta(A)B + A\delta(B) - A\delta(I)B$ for each $A, B \in \mathcal{A}$ with $AB = W$. It is worth noting in Corollary 2.5 that every generalized derivable mapping at I from a unital Banach algebra $\mathcal{A}$ into its unital $\mathcal{A}$-bimodule $\mathcal{M}$ is a generalized Jordan derivation.

Corollary 2.6. *Let $\mathcal{A}$ be a unital Banach algebra and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. If δ is a generalized derivable mapping at X and Y from $\mathcal{A}$ into $\mathcal{M}$ with $X + Y = I$, then δ is a generalized Jordan derivation.*

Proof. For arbitary $A, B \in \mathcal{A}$ with $AB = I$, we have $ABX = X$ and $ABY = Y$. Since δ is a generalized derivable mapping at X and Y, it follows that

$$\delta(X) = \delta(ABX) = \delta(A)BX + A\delta(BX) - A\delta(I)BX \tag{2.4}$$

and

$$\delta(Y) = \delta(ABY) = \delta(A)BY + A\delta(BY) - A\delta(I)BY. \tag{2.5}$$

Combining (2.4) and (2.5), we obtain

$$\delta(I) = \delta(AB(X + Y)) = \delta(A)B + A\delta(B) - A\delta(I)B,$$

i.e., δ is a generalized derivable mapping at I. On account of Corollary 2.5, δ is a generalized Jordan derivation. □

Our following result explores the connection between linear mappings satisfying (F_I) on a Banach algebra and $*$-linear mappings satisfying (F_I^*) on a Banach $*$-algebra.

Theorem 2.7. *Let $\mathcal{A}$ be a unital Banach $*$-algebra, $\mathcal{M}$ be a unital $*$-$\mathcal{A}$-bimodule, and δ, τ be linear mappings from $\mathcal{A}$ into $\mathcal{M}$. Then the following are equivalent:*
(i) δ and τ satisfy

$$\delta(A)B^* + A\tau(B)^* = \delta(I)$$

for each $A, B \in \mathcal{A}$ with $AB^* = I$;
(ii) $\tau(A^*)^*$ is a Jordan derivation and $\delta(A) = \tau(A^*)^* + \delta(I)A$ for each A in $\mathcal{A}$.

Proof. Suppose δ and τ satisfy (i), define the linear mapping $\widehat{\tau}$ from $\mathcal{A}$ into $\mathcal{M}$ by $\widehat{\tau}(A) = \tau(A^*)^*$ for each A in $\mathcal{A}$. By assumption

$$\delta(A)B + A\tau(B^*)^* = \delta(I)$$

for each $A, B \in \mathcal{A}$ with $AB = I$, that is,

$$\delta(A)B + A\hat{\tau}(B) = \delta(I)$$

for each $A, B \in \mathcal{A}$ with $AB = I$. This shows that δ and $\widehat{\tau}$ satisfy the hypothesis in Theorem 2.3, thus $\widehat{\tau}$ is a Jordan derivation and $\delta(A) = \widehat{\tau}(A) + \delta(I)A = \tau(A^*)^* + \delta(I)A$, for each $A \in \mathcal{A}$. Therefore

$$\tau(A^*) = (\delta(A) - \delta(I)A)^*$$

for each $A \in \mathcal{A}$.

Conversely, since $\tau(A^*)^*$ is a Jordan derivation, for each $A, B \in \mathcal{A}$ with $AB^* = I$, we have

$$\begin{aligned}\delta(B^*) &= \tau((B^*)^*)^* + \delta(I)B^* = \tau((B^*AB^*)^*)^* + \delta(I)B^* \\ &= \tau((B^*)^*)^*AB^* + B^*\tau((A)^*)^*B^* + B^*A\tau((B^*)^*)^* + \delta(I)B^* \\ &= \tau(B)^* + B^*\tau(A^*)^*B^* + B^*A\tau(B)^* + \delta(I)B^*.\end{aligned}$$

It follows from $\tau(B)^* = \delta(B^*) - B^*\tau(A^*)^*B^* - B^*A\tau(B)^* - \delta(I)B^*$ that

$$\begin{aligned}\delta(A)B^* + A\tau(B)^* &= \delta(A)B^* + A[\delta(B^*) - B^*\tau(A^*)^*B^* - B^*A\tau(B)^* - \delta(I)B^*] \\ &= \delta(A)B^* + A\delta(B^*) - \tau(A^*)^*B^* - A\tau(B)^* - A\delta(I)B^* \\ &= [\delta(A) - \tau(A^*)^*]B^* + A[\delta(B^*) - \tau(B)^* - \delta(I)B^*] \\ &= \delta(I).\end{aligned}$$

This completes the proof. □

3. Separating points

For an algebra $\mathcal{A}$ and an $\mathcal{A}$-bimodule $\mathcal{M}$, an element W in $\mathcal{A}$ is a *left (or right) separating point* of $\mathcal{M}$ if $WM = 0$ (or $MW = 0$) implies $M = 0$ for each $M \in \mathcal{M}$. W is called a *separating point* if W is both a left separating point and a right separating point. It is easy to see that left (right) invertible elements in $\mathcal{A}$ are left (right) separating points of $\mathcal{M}$, and invertible elements in $\mathcal{A}$ are separating points of $\mathcal{M}$.

We are interested in the properties of linear mappings satisfying F_W and F_W^*, where W is a separating point of $\mathcal{M}$. A celebrated result is established that linear mappings δ and τ from a unital Banach algebra $\mathcal{A}$ into a unital $\mathcal{A}$-bimodule $\mathcal{M}$ satisfying F_I, where I is the unit of $\mathcal{A}$, if and only if τ is a Jordan derivation and $\delta(A) = \tau(A) + \delta(I)A$ for each A in $\mathcal{A}$. At this point the reader should be tempted to ask if a similar situation exists for linear mappings satisfying F_W, where W is a separating point of $\mathcal{M}$. We see next that at least one of the directions is true.

Theorem 3.1. *Let $\mathcal{A}$ be a unital Banach algebra, $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule, and W be a separating point of $\mathcal{M}$. If linear mappings δ and τ from $\mathcal{A}$ into $\mathcal{M}$ satisfy*

$$\delta(W) = \delta(A)B + A\tau(B)$$

for each A, B in $\mathcal{A}$ with $AB = W$, then τ is a Jordan derivation and δ is a generalized Jordan derivation. Moreover δ and τ satisfy $\delta(WA) = \delta(W)A + W\tau(A)$ and $\tau(AW) = A\tau(W) + \delta(A)W - \delta(I)AW$ for each A in $\mathcal{A}$.

Proof. By $WI = W$, it follows that $\delta(W) = \delta(W) + W\tau(I)$. Since W is a separating point of $\mathcal{M}$, we have $\tau(I) = 0$.

Let T be an invertible element in $\mathcal{A}$. On account of

$$\delta(W) = \delta(WT^{-1}T) = \delta(WT^{-1})T + WT^{-1}\tau(T)$$

and

$$\delta(W) = \delta(TT^{-1}W) = \delta(T)T^{-1}W + T\tau(T^{-1}W),$$

we have

$$\delta(WT^{-1}) = \delta(W)T^{-1} - WT^{-1}\tau(T)T^{-1}, \tag{3.1}$$
$$WT^{-1}\tau(T) = \delta(W) - \delta(WT^{-1})T, \tag{3.2}$$
$$\delta(T)T^{-1}W = \delta(W) - T\tau(T^{-1}W), \tag{3.3}$$
$$\tau(T^{-1}W) = T^{-1}\delta(W) - T^{-1}\delta(T)T^{-1}W. \tag{3.4}$$

Let $A \in \mathcal{A}$, $n \in \mathbb{N}$ with $n \geq \|A\| + 2$, and $B = nI + A$. Then B and $I - B$ are both invertible in $\mathcal{A}$. On the one hand, it follows from (3.1) and (3.2) that

$$\begin{aligned}
WB^{-1}\tau(B) &= \delta(W) - \delta(WB^{-1})B \\
&= \delta(W) - \delta(WB^{-1}(I - B) + W)B \\
&= \delta(W)(I - B) - \delta(WB^{-1}(I - B))B \\
&= \delta(W)(I - B) - [\delta(W)B^{-1}(I - B) - WB^{-1}(I - B)\tau((I - B)^{-1}B)B^{-1}(I - B)]B \\
&= WB^{-1}(I - B)\tau((I - B)^{-1}B)(I - B) \\
&= WB^{-1}(I - B)\tau((I - B)^{-1} - I)(I - B) \\
&= WB^{-1}(I - B)\tau((I - B)^{-1})(I - B).
\end{aligned}$$

Since W is a separating point of $\mathcal{M}$, it follows that

$$B^{-1}\tau(B) = B^{-1}(I - B)\tau((I - B)^{-1})(I - B).$$

That is,

$$\tau(B) = (I - B)\tau((I - B)^{-1})(I - B).$$

Multiplying W from the left of the above equation, we have that

$$\begin{aligned}
W\tau(B) &= W(I - B)\tau((I - B)^{-1})(I - B) \\
&= [\delta(W) - \delta(W(I - B))(I - B)^{-1}](I - B) \\
&= \delta(W)(I - B) - \delta(W - WB) \\
&= \delta(WB) - \delta(W)B.
\end{aligned}$$

Thus, $\delta(WB) = \delta(W)B + W\tau(B)$. Since $B = nI + A$, we have that

$$\delta(WA) = \delta(W)A + W\tau(A) \tag{3.5}$$

for each A in $\mathcal{A}$.

On the other hand, it follows from (3.3) and (3.4) that

$$\begin{aligned}
\delta(B)B^{-1}W &= \delta(W) - B\tau(B^{-1}W) \\
&= \delta(W) - B\tau(B^{-1}(I - B)W + W) \\
&= \delta(W) - B\tau(W) - B\tau(B^{-1}(I - B)W) \\
&= \delta(W) - B\tau(W) - B[B^{-1}(I - B)\delta(W) - B^{-1}(I - B)\delta((I - B)^{-1}B)B^{-1}(I - B)W] \\
&= \delta(W) - B\tau(W) - (I - B)\delta(W) + (I - B)\delta((I - B)^{-1}B)B^{-1}(I - B)W \\
&= B(\delta(W) - \tau(W)) + (I - B)\delta((I - B)^{-1} - I)B^{-1}(I - B)W \\
&= B\delta(I)W - (I - B)\delta(I)B^{-1}(I - B)W + (I - B)\delta(I - B)^{-1}B^{-1}(I - B)W.
\end{aligned}$$

Since W is a separating point of $\mathcal{M}$, it follows that

$$\delta(B)B^{-1} = B\delta(I) - (I-B)\delta(I)B^{-1}(I-B) + (I-B)\delta((I-B)^{-1})B^{-1}(I-B).$$

That is,

$$\delta(B) = B\delta(I)B - (I-B)\delta(I)(I-B) + (I-B)\delta((I-B)^{-1})(I-B).$$

Multiplying W from the right of the above equation, we have

$$\begin{aligned}\delta(B)W &= B\delta(I)BW - (I-B)\delta(I)(I-B)W + (I-B)\delta(I-B)^{-1}(I-B)W \\ &= B\delta(I)BW - (I-B)\delta(I)(I-B)W + (I-B)[\delta(W) - (I-B)^{-1}\tau((I-B)W)] \\ &= \delta(I)BW - B\tau(W) + \tau(BW).\end{aligned}$$

Thus, $\tau(BW) = B\tau(W) + \delta(B)W - \delta(I)BW$. By $B = nI + A$, we have

$$\tau(AW) = A\tau(W) + \delta(A)W - \delta(I)AW \tag{3.6}$$

for each A in $\mathcal{A}$.

Now we prove that τ is a Jordan derivation and δ is a generalized Jordan derivation. For each invertible element T in $\mathcal{A}$, by (3.5) and (3.6),

$$\begin{aligned}\delta(W) = \delta(WTT^{-1}) &= \delta(WT)T^{-1} + WT\tau(T^{-1}) \\ &= [\delta(W)T + W\tau(T)]T^{-1} + WT\tau(T^{-1}) \\ &= \delta(W) + W\tau(T)T^{-1} + WT\tau(T^{-1})\end{aligned}$$

and

$$\begin{aligned}\delta(W) = \delta(T^{-1}TW) &= \delta(T^{-1})TW + T^{-1}\tau(TW) \\ &= \delta(T^{-1})TW + T^{-1}[T\tau(W) + \delta(T)W - \delta(I)TW] \\ &= \delta(T^{-1})TW + \tau(W) + T^{-1}\delta(T)W - T^{-1}\delta(I)TW.\end{aligned}$$

Thus

$$W\tau(T)T^{-1} + WT\tau(T^{-1}) = 0$$

and

$$\delta(T^{-1})TW + T^{-1}\delta(T)W - T^{-1}\delta(I)TW = \delta(W) - \tau(W) = \delta(I)W.$$

Since W is a separating point,

$$\tau(T)T^{-1} + T\tau(T^{-1}) = 0$$

and

$$\delta(T^{-1})T + T^{-1}\delta(T) - T^{-1}\delta(I)T = \delta(I).$$

It follows from [16, Lemma 2.1] and Corollary 2.5 that τ is a Jordan derivation and δ is a generalized Jordan derivation. □

Remark 3.2. *(i) In Theorem 3.1, if condition (i) is changed to*

$$A, B \in \mathcal{A}, \quad AB = W \Longrightarrow, \delta(W) = \tau(A)B + A\delta(B),$$

theorem remains valid.
(ii) If I in Theorem 2.3 is replaced by W, we find that only one sufficient condition (Theorem 3.1) can be obtained. The natural question then is, does its necessity hold?

Unfortunately, we have been unable to answer the above question. In order to find a complete characterization of linear mappings satisfying F_W, we need to impose some conditions. For example, every Jordan derivation on an algebra is a derivation. In the next, assume that $\mathcal{A}$ is a *semisimple* algebra, i.e., the redical of $\mathcal{A}$, $Rad(\mathcal{A}) = \{0\}$. Our next goal is to give two characterizations of linear mappings on a unital semisimple Banach algebra which satisfy F_W and linear mappings on a unital semisimple Banach $*$-algebra which satisfy F_W^*, respectively.

Theorem 3.3. *Let $\mathcal{A}$ be a unital semisimple Banach algebra and W be a separating point of $\mathcal{A}$. If δ and τ are linear mappings from $\mathcal{A}$ into itself, then the following are equivalent:*
(i) δ and τ satisfy $\delta(W) = \delta(A)B + A\tau(B)$ for each A, B in $\mathcal{A}$ with $AB = W$;
(ii) τ is a derivation and $\delta(A) = \tau(A) + \delta(I)A$ for each A in $\mathcal{A}$.

Proof. Clearly, (ii) implies (i). Let's prove that (i) implies (ii). By the proof of Theorem 3.1, we have that τ is a Jordan derivation, δ is a generalized Jordan derivation, and

$$\tau(AW) = A\tau(W) + \delta(A)W - \delta(I)AW \tag{3.7}$$

for each A in $\mathcal{A}$. Since every Jordan derivation on a semisimple algebra is a derivation, then τ is a derivation and

$$\tau(AW) = A\tau(W) + \tau(A)W \tag{3.8}$$

for each A in $\mathcal{A}$. Comparing (3.7) and (3.8),

$$\delta(A)W - \delta(I)AW = \tau(A)W.$$

We reduce that $\delta(A) = \tau(A) + \delta(I)A$ for each A in $\mathcal{A}$ from W is a separating point of $\mathcal{A}$, as required. □

Now, we return to linear mappings satisfying F_W^* on a unital semisimple Banach $*$-algebra $\mathcal{A}$ where W is a separating point of $\mathcal{A}$.

Theorem 3.4. *Let $\mathcal{A}$ be a unital semisimple Banach $*$-algebra, δ and τ be linear mappings on $\mathcal{A}$. Then the following are equivalent:*
(i) δ and τ satisfy

$$\delta(W) = \delta(A)B^* + A\tau(B)^*$$

for each $A, B \in \mathcal{A}$ with $AB^* = W$, where W is a separating point of $\mathcal{A}$;
(ii) $\tau(A^*)^*$ is a derivation and $\delta(A) = \tau(A^*)^* + \delta(I)A$, for each A in $\mathcal{A}$.

Proof. Let's first prove that (i) implies (ii). Define the linear mapping $\widehat{\tau}$ from $\mathcal{A}$ into itself by $\widehat{\tau}(A) = \tau(A^*)^*$, for each A in $\mathcal{A}$. By assumption

$$A\hat{\tau}(B) + \delta(A)B = A\tau(B^*)^* + \delta(A)B = \delta(W)$$

for each $A, B \in \mathcal{A}$ with $AB = W$. It is easy to see that δ and $\widehat{\tau}$ coincides with the conditions in Theorem 3.3, which proves that $\widehat{\tau}$ is a derivation and $\delta(A) = \widehat{\tau}(A) + \delta(I)A = \tau(A^*)^* + \delta(I)A$, for each A in $\mathcal{A}$. Therefore we conclude that

$$\tau(A^*) = (\delta(A) - \delta(I)A)^*,$$

for each $A \in \mathcal{A}$.

Next, we prove that (ii) implies (i). Since $\tau(A^*)^*$ is a derivation, for each $A, B \in \mathcal{A}$ with $AB^* = W$,

$$\tau(W^*)^* = \tau((AB^*)^*)^* = \tau(A^*)^*B^* + A\tau(B)^*.$$

From $\delta(A) = \tau(A^*)^* + \delta(I)A$ for each A in $\mathcal{A}$, we have

$$\begin{aligned}\delta(A)B^* + A\tau(B)^* &= \tau(A^*)^*B^* + \delta(I)AB^* + A\tau(B)^* \\ &= \tau(W^*)^* + \delta(I)W \\ &= \delta(W).\end{aligned}$$

This completes the proof. □

Remark 3.5. *Let $\mathcal{A}$ be a unital Banach $*$-algebra and $\mathcal{M}$ be a unital $*$-$\mathcal{A}$-bimodule. Suppose that δ is a linear mappings from $\mathcal{A}$ into $\mathcal{M}$. By [2, Remark 1], the following conditions are not equivalent:*

$$AB^* = W \Longrightarrow \delta(W) = \delta(A)B^* + A\delta(B)^*, \tag{D1}$$

$$A^*B = W \Longrightarrow \delta(W) = \delta(A)^*B + A^*\delta(B). \tag{D2}$$

Similarly, linear mappings δ and τ from a unital semisimple Banach $*$-algebra $\mathcal{A}$ into itself, the following conditions are not equivalent:

$$AB^* = W \Longrightarrow \delta(W) = \delta(A)B^* + A\tau(B)^*, \tag{T1}$$

$$A^*B = W \Longrightarrow \delta(W) = \delta(A)^*B + A^*\tau(B). \tag{T2}$$

Hence, if linear mappings δ and τ satisfy condition (T2), where W is a separating point of $\mathcal{A}$, then by suitable modification to the proof of Theorem 3.4, we can obtain a similar result.

Corollary 3.6. *Let $\mathcal{A}$ be a unital semisimple Banach $*$-algebra and δ be a linear mapping of $\mathcal{A}$. Then δ satisfies*

$$\delta(W) = \delta(A)B^* + A\delta(B)^*$$

for each $A, B \in \mathcal{A}$ with $AB^* = W$, where W is a separating point of $\mathcal{A}$, if and only if δ is a $*$-derivation.

References

[1] J. Alaminos, M. Brešar, J. Extremera, A. R. Villena, *Maps preserving zero products*, Studia Math. **193** (2009), no. 2, 131–159.
[2] G. An, J. He, J. Li, *Characterizations of $*$ and $*$-left derivable mappings on some algebras*, Ann. Funct. Anal. **11** (2020), no. 3, 680–692.
[3] A. Barari, B. Fadaee, H. Ghahramani, *Linear maps on standard operator algebras characterized by action on zero products*, Bull. Iranian Math. Soc. **455** (2019), 1573–1583.
[4] M. Brešar, *Characterizing homomorphisms, derivations and multipliers in rings with idempotents*, Proc. Roy. Soc. Edinburgh Sect. A. **137** (2007), no. 1, 9–21.
[5] D. Benkovič, M. Grašič, *Generalized derivations on unital algebras determined by action on zero products*, Linear Algebra Appl. **445** (2014), 347–368.
[6] B. Fadaee, K. Fallahi, H. Ghahramani, *Characterization of linear mappings on (Banach) $*$-algebras by similar properties to derivations*, Math. Slovaca. **70** (2020), no. 4, 1003–1011.
[7] B. Fadaee, H. *Ghahramani, Linear maps on C^*-algebras behaving like (anti-)derivations at orthogonal elements*, Bull. Malays. Math. Sci. Soc. **43** (2020), no. 3, 2851–2859.
[8] H. Ghahramani, *On rings determined by zero products*, J. Algebra Appl. **12** (2013), no. 8, 1350058, 15 pp.
[9] H. Ghahramani, *On derivations and Jordan derivations through zero products*, Oper. Matrices. **8** (2014), no. 3, 759–771.
[10] H. Ghahramani, *Linear maps on group algebras determined by the action of the derivations or anti-derivations on a set of orthogonal elements*, Results Math. **73** (2018), no. 4, Paper No. 133, 14 pp.
[11] H. Ghahramani, Z. Pan, *Linear maps on $*$-algebras acting on orthogonal elements like derivations or anti-derivations*, Filomat. **32** (2018), no. 13, 4543–4554.
[12] J. He, J. Li, W. Qian, *Characterizations of centralizers and derivations on some algebras*, J. Korean Math. Soc. **54** (2017), no. 2, 685–696.
[13] A. Kishimoto, *Dissipations and derivations*, Comm. Math. Phys. **47** (1976), no. 1, 25–32.
[14] F. Lu, *The Jordan structure of CSL algebras*, Studia Math. **190** (2009), no. 3, 283–299.
[15] J. Li, S. Li, *Linear mappings characterized by action on zero products or unit products*, Bull. Iran. Math. Soc. **48** (2022), no. 1, 31–40.
[16] J. Li, J. Zhou, *Characterizations of Jordan derivations and Jordan homomorphisms*, Linear Multilinear Algebra. **59** (2011), no. 2, 193–204.
[17] A. M. Sinclair, *Jordan homomorphisms and derivations on semisimple Banach algebras*, Proc. Amer. Math. Soc. **24** (1970), 209–214.
[18] J. Zhu, C. Xiong, *Derivable mappings at unit operator on nest algebras*, Linear Algebra Appl. **422** (2007), no. 2–3, 721–735.