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Norm attaining and absolutely norm attaining of $*$-paranormal operators

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Abstract. Author in [10] gave a characterization for norm attaining and absolutely norm attaining paranormal operators. In the present paper, we generalize these results for $*$-paranormal operators. We prove that elements of this class of operators admit invariant subspaces as an extension of Istratescu result [9]. Compacity and conditions for normality of norm attaning $*$-paranormal operators are also established.

Introduction and Background

One of the consequences of the famous Hahn-Banach theorem, is that for every vector x in a Banach space $\mathcal{X}$, there exists a functional f in the dual space $\mathcal{X}'$ with $\|f\| = 1$ for which $f(x) = \|x\|$. Hence, if $x_0 \in \mathcal{X}$ is a unit vector, then there exists $f \in \mathcal{X}'$ satisfying $\|f\| = 1 = f(x_0)$. That is, on each Banach space, there exist linear bounded functionals that attain their norms. Author in [16] showed that the set of norm attaining operators on Banach spaces is not dense by giving an example of an operator $T_0 : L^1_{[0,1]} \to C([0,1])$ for which if T is a contraction satisfying $\|T - T_0\| \leq \frac{1}{2}$, then T is not norm attaining. Also, in [5], it's proved that if $\mathcal{Y}$ is a Banach space of dimension 1, then the same set is norm dense in the Banach algebra $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ for any Banach space $\mathcal{X}$. Throughout this paper, H denotes a separable complex Hilbert space. An operator A is said to be paranormal if $\|Ax\|^2 \leq \|A^2x\|\|x\|$ for each vector $x \in H$, and $*$-paranormal if $\|A^*x\|^2 \leq \|A^2x\|\|x\|$, $x \in H$, i.e., the operator $A^{*2}A^2 - 2rAA^* + r^2$ is nonnegative for all $r > 0$. Properties of these classes of operators are given in [1, 2] and [12]. Also, A is said to be norm attaining, if there exists a unit vector $x \in H$ for which $\|Ax\| = \|A\|$, and absolutely norm attaining, if the restriction $A\big|_M$ is norm attaining for any closed subspace $M \subset H$ (see [10]). According to [10], the operator A is norm attaining if and only if $\|A\|^2$ is an eigenvalue of A^*A. For an operator $A \in B(H)$, the range of A and the null space are denoted by $R(A)$ and $N(A)$ respectively.

In [10], are given a characterization and some properties of norm attaining paranormal operators. In the present paper, we extend these results for $*$-paranormal operators. We give a nontrivial invariant subspace for elements of such a class of operators, and we show that this subspace becomes reducing under a certain condition. Next, a structure theorem of the given operators and their normality are also established. Other results on compact $*$-paranormal operators are showed at the end of this work.

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1. Main results

Definition 1.1. *[6] An operator $A \in B(H)$ is said to be norm attaining if there exists a unit vector $x \in H$ satisfying $\|Ax\| = \|A\|$.*

Example 1.2. *Let S be the unilateral right shift on ℓ_2 defined for each integer n, $(n \geq 1)$ by $Se_n = e_{n+1}$, where $(e_n)_{n\geq 1}$ denotes the standard orthonormal basis of ℓ_2. Then, S attains its norm since $\|Se_1\| = 1 = \|S\|$.*

Example 1.3. *[10] If S_α is the operator defined on ℓ_2 by $S_\alpha x = (\alpha_n x_n)_n$, $x = (x_n)_{n\geq 1} \in \ell_2$, where $\alpha = (\alpha_n)_{n\geq 1}$ is a real increasing sequence, then*

$$\|S_\alpha x\|_2^2 < \left(\sup_{n\geq 1}\|\alpha_n\|\right)^2 \|x\|_2^2,\ x \in \ell_2 \tag{1}$$

Hence, $\|S_\alpha\| \leq \sup_{n\geq 1}\|\alpha_n\|$. *Next,*

$$\|S_\alpha\| \geq \sup_{n\geq 1}\|Se_n\| = \sup_{n\geq 1}\|\alpha_n\|$$

Thus, $\|S_\alpha\| = \sup_{n\geq 1}\|\alpha_n\|$. *Inequality (1) shows that S_α is not norm attaining.*

In operator theory, it's well known that the existence of nontrivial invariant subspaces for bounded linear operators on separable Hilbert spaces is one of the hard open problems. It dates back to the 1950s. Positive solutions of this problem are showed for special types of nonnormal operators [13, 14]. Istratescu in [9, Lemma 2.3] gave an invariant subspace for paranormal operators. In the sequel, we extend the Istratescu's result for $*$-paranormal operators as follows

Theorem 1.4. *Let $A \in B(H)$ be a $*$-paranormal operator with $\|A\| = 1$. Then, the subspace $N(I - AA^*)$ is invariant for A.*

Proof. Note first that for a unit vector $x \in N(I - AA^*)$, $\|A^*x\| = 1$ and by the $*$-paranormality of A,

$$\|A^*x\|^2 = 1 \leq \|A^2x\|$$

Hence,

$$\begin{aligned}
\|A^*Ax - x\|^2 = \|A^*Ax\|^2 - 2\|Ax\|^2 + 1 &= \|A^*Ax\|^2 - 2\|A^2A^*x\|^2 + 1 \\
&= \|A^*Ax\|^2 - 2\left\|A^2\frac{A^*x}{\|A^*x\|}\right\|^2\|A^*x\|^2 + 1 \\
&\leq \|A^*Ax\|^2 - 2 + 1 \\
&\leq 0
\end{aligned}$$

Thus, $x \in N(I - A^*A)$. Therefore, $N(I - AA^*) \subset N(I - A^*A)$. Furthermore,

$$(AA^*)Ax = A(A^*Ax) = Ax$$

This shows that $Ax \in N(I - AA^*)$. The result is proved. □

Corollary 1.5. *Let $A \in B(H)$ be $*$-paranormal. Then, $M = N(\|A\|^2I - AA^*)$ is an invariant subspace for A.*

We can easily prove the following result

Lemma 1.6. *For an operator $A \in B(H)$, $M = N(\|A\|^2I - AA^*) = N(|A^*| - \|A\|I)$. Furthermore, if A^* is norm attaining, then $M \neq \{0\}$.*

Lemma 1.7. *[15] Let $T \in B(H)$. The following statements are equivalent*

1. *T is norm attaining.*
2. *T^* is norm attaining .*
3. *$\|T\|$ is an eigenvalue of $|T|$.*
4. *$\||T|\|$ is an eigenvalue of $|T|$.*
5. *$|T|$ is norm attaining.*
6. *$|T^*|$ is norm attaining.*
7. *$|T|^2$ is norm attaining.*
8. *$|T^*|^2$ is norm attaining.*
9. *$\|T\|$ is an eigenvalue of $|T^*|$.*

Corollary 1.8. *Let $A \in B(H)$ be $*$-paranormal. If A is norm attaining, then $\|A\|$ is an eigenvalue of $|A|$.*

Proof. By Corollary 1.5 and Lemma 1.6, the operator $|A^*| - \|A\|I$ is not injective. The result follows then by applying Lemma 1.7. □

Corollary 1.9. *If both A and A^* in $B(H)$ are $*$-paranormal, then $M = N(\|A\|^2 I - AA^*)$ is a reducing subspace for A*

Proof. By Corollary 1.5, M is invariant for A, and $M \subset N(\|A\|^2 I - A^*A)$ according to the proof of Theorem 1.4. Since A^* is also $*$-paranormal, $N(\|A\|^2 I - A^*A) \subset M$ is an invariant subspace for A^*. Thus, $M = N(\|A\|^2 I - A^*A)$ is a reducing subspace for A. □

Theorem 1.10. *Let $A \in B(H)$ be a $*$-paranormal such that A is norm attaining. If $M = N(|A^*| - \|A\|I)$ is finite dimensional subspace, then*

1. *M is a reducing subspace for A.*
2. *$A\big|_{M^\perp}$ is also $*$-paranormal.*

Proof. 1. M is an invariant subspace for A by Corollary 1.5, and $\frac{A^*}{\|A\|}$ is an isometry operator on M. Since M is of finite dimension, $\frac{A^*}{\|A\|}$ is unitary on M. Under the decomposition $H = M \oplus M^\perp$, we can write

$$A = \begin{pmatrix} A\big|_M & S \\ 0 & T \end{pmatrix}$$

where $S \in B(M^\perp, M)$ and $T \in B(M^\perp, M^\perp)$. Since A is $*$-paranormal,

$$0 \le A^{*2}A^2 - 2rAA^* + r^2 = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}$$

for all $r > 0$, where

$$X = (A\big|_M)^{*2}(A\big|_M)^2 - 2r\left(A\big|_M(A\big|_M)^* + SS^*\right) + r^2 I$$
$$Y = (A\big|_M)^{*2}(A\big|_M S + ST) - 2rST^*$$
$$Z = (A\big|_M S + ST)^*(A\big|_M S + ST) + T^{*2}T^2 - 2rTT^* + r^2 I$$

By [7, Theorem 6], we get $X \ge 0$ and $Z \ge 0$. Hence,

$$(A\big|_M)^{*2}\left(A\big|_M)^2 - 2r(A\big|_M(A\big|_M)^* + SS^*\right) + r^2 I \ge 0$$

for all $r > 0$. Thus, for $r = 1$, and since $(A\big|_M)^*$ is unitary, $SS^* \le 0$. Implying $S = 0$.
2. Follows from the fact that $Z \ge 0$ and $S = 0$. □

Corollary 1.11. *Let $A \in B(H)$ be a compact $*$-paranormal operator. Then, $M = N(\|A\|^2 I - AA^*)$ reduces A.*

Proof. Since A is compact, so is AA^*. Then, $M \neq \{0\}$, and M is finite dimensional by Fredholm Alternative. The rest of the proof follows by applying Theorem 1.10. □

Definition 1.12. *[6] An operator $A \in B(H)$ is said to be absolutely norm attaining, brievly $\mathcal{AN}$-operator, if the restriction $A\big|_M$ is norm attaining for any closed subspace $M \subset H$, that is, there exists a unit vector $x \in M$ for which*

$$\|A\big|_M x\| = \|Ax\| = \|A\big|_M\|$$

Example 1.13. *[15] The operator A on ℓ_2 defined by $Ae_1 = \frac{1}{2}e_1$ and $Ae_n = e_n$, $(n \geq 2)$ is in $\mathcal{AN}(\ell_2)$, where $(e_n)_n$ is the standard basis of ℓ_2.*

An $\mathcal{AN}$-operator is norm attaining. Also, both of isometries and compact operators are $\mathcal{AN}$-operators [6]. If A is $\mathcal{AN}$-operator, then A^* may not be $\mathcal{AN}$-operator [6, 15].

Lemma 1.14. *[11] The restriction of a $*$-paranormal operator on a closed invariant subspace is also $*$-paranormal.*

Theorem 1.15. *Let $A \in B(H)$ be a $*$-paranormal operator. Suppose that $A^* \in \mathcal{AN}(H)$ is also $*$-paranormal. Put $\Lambda = \sigma(|A^*|)$. Then, there exists a sequence $(H_\alpha, V_\alpha)_{\alpha\in\Lambda}$ where*

1. *H_α is a reducing subspace for A.*
2. *$V_\alpha \in B(H_\alpha)$ is an isometry.*

for which

a. $H = \bigoplus_{\alpha\in\Lambda} H_\alpha$

b. $A^* = \bigoplus_{\alpha\in\Lambda} \alpha V_\alpha$

c. $\sigma(A^*) \subseteq \bigoplus_{\alpha\in\Lambda} \alpha\mathbb{T}$ *with $\mathbb{T}$ denotes the unit circle.*

Proof. Since $A^* \in \mathcal{AN}(H)$, $H_1 = N(AA^* - \|A\|^2 I)$ is a reducing subspace for A by Corollary 1.9. Let $A_1 = A\big|_{H_1}$. Then, $A_1^* = A^*\big|_{H_1} \in \mathcal{AN}(H_1)$, and for all $x \in H_1$, $AA^{*x} = \|A\|^2 x$. If $H_1 = H$, then $AA^* = \|A\|^2 I$. Put $\beta_1 = \|A\|$. Then, $\beta_1 \in \sigma(|A^*|)$ and $AA^* = \beta_1^2 I$, i.e., $V_1 = \frac{1}{\beta_1}A^*$ is an isometry on $H = H_1$. Hence, $A_1^* = \beta_1 V_1$.
If $H_1 \subsetneq H$, then $H = H_1 \oplus H_1^\perp$, and $A = A_1 \oplus A_2$, where $A_2 = A\big|_{H_1^\perp}$ is $*$-paranormal by Lemma 1.14. Hence,

$$A^* = \begin{pmatrix} \beta_1 V_1 & 0 \\ 0 & A_2^* \end{pmatrix}$$

Let $H_2 = H_1^\perp$. Note that $A_2^* \in \mathcal{AN}(H_2)$ is also $*$-paranormal. By the same previous process, either $A_2^* = \beta_2 V_2$ or $A_2^* = \beta_2 V_2 \oplus A_3^*$, with $\beta_2 = \|A_2\| \in \sigma(|A^*|)$. Then, A^* admits the representation

$$A^* = \begin{pmatrix} \beta_1 V_1 & 0 \\ 0 & \beta_2 V_2 \end{pmatrix} \quad \text{or} \quad A^* = \begin{pmatrix} \beta_1 V_1 & 0 & 0 \\ 0 & \beta_2 V_2 & 0 \\ 0 & 0 & A_3^* \end{pmatrix} \tag{2}$$

Continuing with the same way, we get the following cases :
Case 1. The process stops after n steps. Then, $H = \underset{1\leq k\leq n}{\oplus} H_k$ and $A^* = \underset{1\leq k\leq n}{\oplus} \beta_k V_k$, where $\beta_k \in \sigma(|A^*|)$, $k = 1, 2, .., n$

and $\beta_1 > \beta_2 > \beta_3 ... > \beta_n$. Obviously, $|A^*| = \bigoplus_{1\le k\le n} I_k$, I_k is the identity operator on H_k.
If each eigenvalue β_i, $i = 1, 2, .., n$ is of finite multiplicity, then H_i is finite dimensional, and A_i is unitary, $i = 1, 2, .., n$ and we have

$$H = \underset{1\le k\le n}{\oplus} H_k \quad \text{and} \quad A^* = \bigoplus_{1\le k\le n} \beta_k V_k \tag{3}$$

Since $|A^*| \in \mathcal{AN}(H)$, it can have atmost one eigenvalue of infinite multiplicity, say β_j. Thus,

$$H = \left(\bigoplus_{1\le k\le n, k\neq j} H_k \right) \oplus H_j \quad \text{and } A^* = \left(\bigoplus_{1\le k\le n,\ k\neq j} \beta_k V_k \right) \oplus \beta_j V_j \tag{4}$$

Note that $V_k, (1 \le k \le n, k \neq j)$ are unitary and V_j is an isometry. Clearly $\sigma(A^*) \subset \underset{1\le k\le n}{\oplus} \beta_k \mathbb{T}$.
Case 2. The process does not stop after finite steps: We get a sequence $(\beta_n, H_n, A_n^*)_n$ for which $\beta_{n+1} < \beta_n$, $A_n^* = A^*\big|_{H_n} = \beta_n V_n$, $(n \ge 1)$. Since $\beta_n \in \sigma(|A|$ and $\beta_n \ge m(A^*)$, $n \ge 1$, the sequence $\beta_n)$ converges to $\alpha = m_e(A^*)$ the essential minimum modulus of A^*. Also, there may exist at most finitely many spectral values $\alpha_1, \alpha_2, .., \alpha_m$ between $m(A^*)$ and $m_e(A^*)$ by [10]. Let $\tilde{H_k}$ be the eigenspaces and $\tilde{A}_k^*(1 \le k \le m)$ be the corresponding unitaries associated to γ_k respectively. Since the eigenvectors of $|A^*|$ span a dense space of H, we've necessarily $H = \bigoplus_{k\ge 1} H_k \bigoplus_{1\le j\le m} \tilde{H}_j$. Also, $\sigma(|A^*|) = \{\beta_n\}_{n=1}^\infty \cup \{\beta\} \cup \{\alpha_k\}_{k=1}^m$. Therefore, A^* can be written as

$$A^* = \left(\bigoplus_{k\ge 1} \beta_n V_n \right) \oplus \left(\bigoplus_{1\le j\le m} \alpha_j \tilde{V}_j \right) = \bigoplus_{\alpha\in\Lambda} \alpha V_\alpha \tag{5}$$

Thus,

$$\sigma(A^*) = \bigsqcup_{n=1}^{\infty} \sigma(\beta_n V_n) \oplus \bigsqcup_{k=1}^{n} \sigma(\alpha_k V_k) \subseteq \bigsqcup_{n=1}^{\infty} \beta_n \mathbb{T} \oplus \bigsqcup_{k=1}^{n} \alpha_k \mathbb{T} = \bigoplus_{\alpha\in\Lambda} \alpha \mathbb{T} \tag{6}$$

where $\bigsqcup$ denotes the the disjoint union. If $\alpha \in \sigma_p(|A^*|$ with infinite multiplicity, then $\beta = \alpha$ by [15, Theorem 3.8]. Thus, $\beta \in \sigma_p(|A^*|$ with infinite multiplicity, and also the unique limit point of $\sigma(|A^*|$. Therefore, we get

$$A^* = \left(\bigoplus_{\alpha\in\Lambda, \alpha\neq\beta} \alpha V_\alpha \right) \oplus \beta V_\beta \tag{7}$$

□

The next results give sufficient conditions that ensure the normality of ∗-paranormal operators. We have then

Theorem 1.16. *Let $A \in B(H)$ such that A and A^* are ∗-paranormal, and $A^* \in \mathcal{AN}(H)$. If $|A^*|$ has no eigenvalue with infinite multiplicity, then A is normal.*

Proof. According to the hypotheses, A^* can be written as in Equation (3) or Equation (5) in the proof of Theorem 1.15. Since V_j is a finite dimensional isometry in the two cases, V_j is unitary. Thus, A is necessarily normal. □

Corollary 1.17. *Let $A \in B(H)$ be a compact operator. If both of A and A^* are ∗-paranormal, then A is normal.*

Proof. $A \in \mathcal{AN}(H)$ by [15, Proposition 2.1]. The result follows then by Theorem 1.16.
□

Competing interests. The authors declare that they have no competing interests
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