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Optimization of first-order impulsive differential inclusions and duality

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Abstract. The paper studies optimization problem described by first order evolution impulsive differential inclusions (DFIs); in terms of locally adjoint mappings in framework of convex and nonsmooth analysis we formulate sufficient conditions of optimality. Then we construct the dual problems for impulsive DFIs and prove duality results. It turns out that the Euler-Lagrange inclusions are "duality relations" for both the primal and dual problems, that is, if some pair of functions satisfies this relation, then one of them is a solution to the primal problem, and the other is a solution to a dual problem. At the end of the paper duality in optimal control problems with first order linear and polyhedral DFIs are considered, where the supremum is taken over the class of non-negative absolutely continuous functions.

1. The first section

The paper deal with the Bolza problem of the evolution impulsive DFIs:

$$\text{minimize} \int_0^T g(x(t),t)dt + f(x(T)), \tag{1}$$

$$(\mathrm{P_C}) \qquad x'(t) \in F(x(t),t), \text{ a.e. } t \in J \backslash \{t_1, \dots, t_m\}, \tag{2}$$

$$x\left(t_k^+\right) - x\left(t_k^-\right) = I_k\left(x\left(t_k^-\right)\right), k = 1, \dots, m, \tag{3}$$

$$x(0) = x_0. \tag{4}$$

Here $F(\cdot,t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a time dependent set-valued mapping, $g(\cdot,t), f$ are continuous cost functionals $g(\cdot,t), f : \mathbb{R}^n \to \mathbb{R}^1$, $J = [0,T]$, where T is an arbitrary positive real number. The functions I_k characterize the jump of the solutions at impulse point $t_k (k = 1, \dots, m)$, $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $k = 1, \dots, m$, $t_0 = 0 < t_1 < \dots < t_m < t_{m+1} = T$, $\Delta x|_{t=t_k} = x\left(t_k^+\right) - x\left(t_k^-\right)$, $x\left(t_k^+\right) = \lim_{\varepsilon \to 0^+} x(t_k + \varepsilon)$ and $x\left(t_k^-\right) = \lim_{\varepsilon \to 0^+} x(t_k - \varepsilon)$. In order to define a solution for problem (1)-(4), we consider the space $PC(J, \mathbb{R}^n) = \left\{x(\cdot) : J \to \mathbb{R}^n : x(t) \text{ is continuous everywhere exept for some } t_k \text{ at which } x(t_k^-) \text{ and } x(t_k^+) \text{exist and satisfy } x\left(t_k^-\right) = x(t_k), k = 1, \dots, m\right\}$. Clearly, $PC(J, \mathbb{R}^n)$ is a Banach space endowed with the norm $\|x\|_{PC} = \sup\{|x(t)| : t \in J\}$. A function $x(\cdot) \in PC \cap AC(t_k, t_{k+1}), k = 0, \dots, m,$

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is said to be a solution of problem (1)-(4) if $x(\cdot)$ satisfies the differential inclusion $x'(t) \in F(x(t), t)$ a.e. on $J \backslash \{t_1, \ldots, t_m\}$, and the conditions $x\left(t_k^+\right) - x\left(t_k^-\right) = I_k\left(x\left(t_k^-\right), k = 1, \ldots, m\right.$ and the initial condition $x(0) = x_0$. Here, $AC(t_k, t_{k+1})$ is a space of absolutely continuous functions on (t_k, t_{k+1}), that is, $x(\cdot)$ is differentiable almost everywhere on $(t_k, t_{k+1}), x'(\cdot) \in L_1(t_k, t_{k+1})$ and $x(t) = x(t_k) + \int_{t_k}^{t} x'(\tau)d\tau$ and $\|x\|_{AC} = |x(t_k)| + \int_{t_k}^{t_{k+1}} |x'(\tau)| d\tau$. The problem is to find an arc $\tilde{x}(\cdot)$ of the problem (1)-(4) satisfying (2) almost everywhere (a.e.) on $[0, T]$, the impulsive constraint (3) on $[0, T]$ and initial-value condition (4) that minimizes the Bolza functional (1). We label this problem as (P_C).

Many evolutionary processes are subject to short-term perturbations, so-called impulses, the duration of which is negligible in comparison with the duration of the process. It is known, for example, that many biological phenomena, including threshold values, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics, and frequency modulation systems, do indeed exhibit impulse effects. Thus, impulsive differential equations / inclusions containing impulse effects appear as a natural description of the observed evolutionary phenomena of many real-life problems. In environmental sciences, impulses correspond to seasonal changes in water levels in artificial reservoirs. Their models are described by impulsive differential equations/inclusions. Important contributions to the study of the mathematical aspects of such equations/inclusions can be found in the works by Aubin [2], Lakshmikantham, Bainov, and Simeonov [12], and Samoilenko and Perestyuk [26]. During the last couple of years, impulsive ordinary differential inclusions and functional differential inclusions with different conditions have been intensely studied; see, for example, the monographs by Aubin [2] and Benchohra and Quahab [4], Liu and Wu [13], Samoilenko and Perestyuk [26], Bahaa [3] and the references therein. Some existence results are investigated in [9] for solutions of a boundary value problem for impulsive fractional differential inclusions supplemented with fractional flux boundary conditions by applying Bohnenblust-Karlin's fixed point theorem for set-valued maps.

The existence results and some other qualitative properties of impulsive DFIs were studied in [2, 4, 9, 12, 13, 26, 30]. In [4], the existence of solutions for impulsive neutral functionaldifferential inclusions of the first and second order with variable times is studied. The main tool is the Martelli fixed point theorem for condensing set-valued mappings. In the paper [13], the authors study the existence of solutions to impulsive boundary value problems for SturmLiouville-type differential inclusions that admit nonconvex set-valued mappings on the righthand side. Paper [30] presents several results on the finite-time stability of pulsed DFI.

The results without impulse phenomena, but in terms of set-valued mappings include papers [1, 5, 8, 10, 11, 22, 24, 27, 29]. In [1], a class of boundary value problems for nonlinear nth order differential equations and inclusions with nonlocal and integral boundary conditions is studied. New existence results are obtained with the help of some fixed point theorems. In [11] it is proved a strong convergence theorem for finding a common solution of a combination of equilibrium problems and the set of fixed points of a k-nonspreading set-valued mapping by using shrinking projection hybrid method. Further, it is given a numerical example to justify the main result and compare the shrinking areas of solution set after randomization. The results obtained in [22] complement some well-known results concerning the set-valued Cauchy problem and fuzzy DFIs. The paper [27] presents some results on fixed points for set-valued mappings of contraction type using the notion of w-distance. As an application, the existence of a solution to a nonlinear fractional DFIs is proved.

The necessary and sufficient conditions of optimality given in [14, 15, 18–21] are more precise, since they include more excellent forms of the transversality condition and the Euler-Lagrange inclusion. Moreover, the simplicity of the LAM approach and the method of the "cone of tangent directions" instead of the normal cone [7, 23] simplifies the derivation and formulation of optimality conditions. The paper [20] studies a new class of problems of optimal control theory with Sturm-Liouville-type DFIs involving second-order linear self-adjoint differential operators. Necessary and sufficient conditions, containing both the Euler-Lagrange and Hamiltonian-type inclusions and "transversality" conditions are derived. The paper [21] considers the Mayer problem with higher order evolution DFIs and functional constraints of optimal control theory. Are proved necessary and sufficient conditions incorporating the EulerLagrange inclusion, the Hamiltonian inclusion, the transversality and complementary slackness conditions.

Although a significant part of the present paper is devoted to the derivation of optimal conditions for problem (P_C), its main goal is to construct and study the duality theory for them. On the one hand, duality theory provides a powerful theoretical tool for the analysis of optimization and variational problems, and on the other hand, it opens the way to the development of new algorithms to solve them. In general, duality is associated with convex problems, it turns out that the theory of duality also has a fundamental impact even on the analysis of non-convex problems. The reader can refer to [16, 19, 25, 28] and their references for more details on this topic. In the works of Mahmudov [14–16, 19–21] on the basis of the apparatus of LAM mapping, sufficient conditions of optimality for ordinary and partial DFIs are derived and duality theorems are proved. Also, the duality of optimal control problems with DFIs of the second and higher orders are studied. The duality theorems proved allow one to conclude that a sufficient condition for an extremum is an extremal relation for the primal and dual problems. Use of infimal convolution plays a key role in proofs of duality results for problems with discrete and DFIs. The paper [23] on the whole using Fenchel conjugates develops a geometric approach of variational analysis for the case of convex objects considered in locally convex topological spaces and also in Banach space settings.

In the present work, the optimality conditions for a first order impulsive DFIs and duality approach based on the dual operations of addition and infimal convolution of convex functions [19] is considered for the first time. The construction of the duality of the considered problem with DFIs is primarily accompanied by the duality of the discrete-approximate first order problem. But it turns out that this approach requires a careful and lengthy computations of the construction of optimality conditions and dual problems. Therefore, these cumbersome calculations are omitted in the paper.

Based on the above, we can conclude that various applications of impulsive first-order DFIs and their duality play an important role in modern mathematics and, thus, the problem is of great interest to the scientific community. As far as we know, there are no works in the mathematical literature in which sufficient conditions of optimality for these problems would be considered, and we are trying to fill this gap. Thus, the novelty of our problem statement is justified.

The paper is organized in the following order:

In Section 2, the necessary facts and additional results from the book of Mahmudov [16] are presented; Hamiltonian function and argmaximum sets of a set-valued mapping, the LAM, infimal convolution of proper convex functions, conjugate function.

In Section 3, sufficient condition of optimality for a continuous impulsive problem (P_C) is given. Then we consider the application of Theorem 3.1 on optimality to an interesting linear problem, where the Euler - Lagrange inclusion is simply a well-known adjoint equation of optimal control theory.

In Section 4 formulates the dual problem (P_C^*) to the impulsive DFI problem, which is of interest even without impulse conditions (3). Here, in the formulation of the dual problem to Bolza problem, in contrast to Mayer's problems, an integral term appears and its construction becomes much more complicated. We prove that if ρ and ρ^* are the values of primal and dual problems, respectively, then $\rho \geq \rho^*$ for all feasible solutions. Then a pair of functions $(\tilde{x}^*(\cdot), \tilde{v}^*(\cdot))$ is an optimal solution of the dual problem (P_C^*) if and only if the sufficient conditions of optimality are satisfied. In addition, the optimal values in the primal (P_C) and dual (P_C^*) problems are equal, i.e. $\rho = \rho^*$. At the end of this section, it is shown that maximization in the "linear" dual problem is implemented only on one dual variable $x^*(\cdot)$ (not on a pair $(x^*(\cdot), v^*(\cdot))$).

Section 5 of the paper demonstrates a model with a polyhedral impulsive DFI. From an applied point of view, this example shows that the considered approach to constructing duality turns out to be justified. Here, the calculation of the support function for the graph of a polyhedral set-valued mapping $M_F(x^*, y^*)$, taken with a minus sign, is carried out by reducing it to a linear programming problem. Moreover, in the dual problem the supremum is taken over the class of absolutely continuous non-negative functions.

2. Needed Facts and Preliminaries

All the necessary definitions and concepts can be found in Mahmudov's book [16]. Let $\mathbb{R}^n$ be a n-dimensional Euclidean space, $\langle x, u \rangle$ be an inner product of elements $x, y \in \mathbb{R}^n$, and (x, y) be a pair of x, y. Assume that $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping from $\mathbb{R}^n$ into the set of subsets of $\mathbb{R}^n$. Then F is convex

if its gph $F = \{(x, y) : y \in F(x)\}$ is a convex subset of $\mathbb{R}^{2n}$. The set-valued mapping F is convex closed if its graph is a convex closed set in $\mathbb{R}^{2n}$. The domain of F is denoted by $\operatorname{dom} F$ and is defined as follows $\operatorname{dom} F = \{x : F(x) \neq \emptyset\}$.$F$ is convex-valued if $F(x)$ is a convex set for each $x \in domF$.
Let's give important definitions, which we will often see in the paper:

$$H_F(x, y^*) = \sup_y \{\langle y, y^* \rangle : y \in F(x)\}, y^* \in \mathbb{R}^n,$$
$$F_A(x, y^*) = \{y \in F(x) : \langle y, y^* \rangle = H_F(x, y^*)\}.$$

H_F and F_A are called Hamiltonian function and argmaximum set for a set-valued mapping F, respectively. For convex F we put $H_F(x, y^*) = -\infty$ if $F(x) = \emptyset$.

The convex cone $K_Q(z), z = (x, y)$ is called the cone of tangent directions at a point $z \in Q$ to the set Q if from $\bar{z} = (\bar{x}, \bar{y}) \in K_Q(z)$ it follows that $\bar{z}$ is a tangent vector to the set Q at point $z \in Q$, i.e., there exists such function $\eta : \mathbb{R}^1 \to \mathbb{R}^{2n}$ that $z + \gamma\bar{z} + \eta(\gamma) \in Q$ for sufficiently small $\gamma > 0$ and $\gamma^{-1}\eta(\gamma) \to 0$, as $\gamma \downarrow 0$.

A function f is called a proper function if it does not take the value $-\infty$ and is not identically equal to $+\infty$. Clearly, f is proper if and only if $\operatorname{dom} f \neq \emptyset$ and $f(\cdot)$ is finite for $x \in \operatorname{dom} f = \{x : f(x) < +\infty\}$.

In general, for a set-valued mapping F a set-valued mapping $F^* : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$F^*(y^*; (x, y)) := \left\{x^* : (x^*, -y^*) \in K_F^*(x, y)\right\},$$

is called the LAM to a set-valued F at a point $(x, y) \in gphF$, where $K_F^*(x, y)$ is the dual to the cone of tangent directions $K_{gphF}(x, y) \equiv K_F(x, y)$. We provide another definition of LAM to mapping F which is more relevant for further development

$$F^*(y^*; (x, y)) := \{x^* : H_F(x_1, y^*) - H_F(x, y^*) \leq \langle x^*, x_1 - x \rangle,$$
$$\forall (x_1, y_1) \in \mathbb{R}^{2n}\right\}, (x, y) \in gphF, \quad z \in F_A(x, y^*).$$

Obviously, for the convex mapping the Hamiltonian $H(\cdot, y^*)$ is concave and the latter and previous definitions of LAMs coincide. In fact, the given in the paper notion LAM is closely related to the coderivative (Frechet) concept of Mordukhovich [23], $D^*F(x, v)(v^*) = \{x^* : (x^*, -v^*) \in N((x, v); \operatorname{gph} F\}$ which is essentially different for nonconvex mappings. Here $N((x, v); \operatorname{gph} F)$ is a normal cone at $(x, v) \in \operatorname{gph}F$. In the most interesting settings for the theory and applications, coderivatives are nonconvex-valued and hence are not tangentially /derivatively generated. However, for the convex maps the two notions are equivalent.

Definition 2.1. *A function $f(x)$ is said to be a closure if its epigraph epi $f = \left\{\left(x^0, x\right) : x^0 \geq f(x)\right\}$ is a closed set.*

Definition 2.2. *The function $f^*(x^*) = \sup_x \{\langle x, x^* \rangle - f(x)\}$ is said to be the conjugate of f. Clearly, the conjugate function is closed and convex.*

Let us denote

$$M_F(x^*, y^*) = \inf_{x,y} \{\langle x, x^* \rangle - \langle y, y^* \rangle : (x, y) \in \operatorname{gph} F\}.$$

Obviously, the function

$$M_F(x^*, y^*) = \inf_x \{\langle x, x^* \rangle - H_F(x, y^*)\}$$

is a support function taken with a minus sign.

Definition 2.3. *We recall that the operation of infimal convolution $\oplus$ of functions $f_i, i = 1, 2$ is defined as follows*

$$(f_1 \oplus f_2)(u) = \inf\left\{f_1\left(u^1\right) + f_2\left(u^2\right) : u^1 + u^2 = u\right\}, u^i \in \mathbb{R}^n, i = 1, 2.$$

It should be noted that, for a proper convex closed functions $f_i, i = 1, 2$ their infimal convolution $(f_1 \oplus f_2)$ is convex and closed (but not necessarily proper).

3. Sufficient Condition of Optimality for Impulsive DFIs (Pc)

In this section using the limit procedure for a discrete-approximate problem (because of the long and tedious computation, the formulation of the necessary conditions for the discretized problem is omitted), we formulate the following Euler-Lagrange inclusion and transversality condition for the impulsive problem (P_C) :

(a) $-x^{*\prime}(t) \in F^*\left(x^*(t);(\tilde{x}(t),\tilde{x}'(t)),t\right)-\partial g(\tilde{x}(t),t)$, a.e. $t\in J\setminus\{t_1,\ldots,t_m\}$

(b) $\tilde{x}'(t)\in F_A\left(\tilde{x}(t);x^*(t),t\right)$,

(c) $-x^*(T)\in\partial f(\tilde{x}(T)),\left(\Delta x^*|_{t=t_k}=x^*\left(t_k^+\right)-x^*\left(t_k^-\right)=0,k=1,\ldots,m\right)$,

where $\partial g(x,t)$ is a classical subdifferential of $g(\cdot,t)$ at a point $x\in \operatorname{dom} g(\cdot,t)$. Here and below, as a solution to the adjoint DFI (a), we use a function $x(t)$ that is absolutely continuous on the set $t\in J\setminus\{t_1,\ldots,t_m\}$, which, due to $\Delta x^*|_{t=t_k}=0$, are continuous at the points $t_k, k=1,\ldots,m$. Obviously, due to the "homogeneous" adjoint condition, are also absolutely continuous over the entire interval $[0,T]$.

Theorem 3.1. *Suppose that $F(\cdot,t):\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ is a convex set-valued mapping, $g(\cdot,t),f(\cdot):\mathbb{R}^n\to\mathbb{R}^1\cup\{+\infty\}$ are continuous proper convex functions. Then for optimality of the trajectory $\tilde{x}(\cdot)$ in the convex problem (P_C) with impulsive DFI it is sufficient that there exists an absolutely-continuous function $x^*(\cdot),t\in[0,T]$ satisfying a.e. the Euler-Lagrange inclusions (a), (b) and transversality condition (c).*

Proof. Using Theorem 2.1 [16, p.62] and the Euler-Lagrange type inclusion (a) we have

$$-x^{*\prime}(t)\in\partial_x H_F\left(\tilde{x}(t),x^*(t)\right)-\partial g(\tilde{x}(t),t),t\in\left(t_k,t_{k+1}\right],k=0,\ldots,m$$

which implies that

$$\begin{gathered}H_F\left(x(t),x^*(t)\right)-H_F\left(\tilde{x}(t),x^*(t)\right)\le-\left\langle x^{*\prime}(t),x(t)-\tilde{x}(t)\right\rangle\\+g(x(t),t)-g(\tilde{x}(t),t),t\in\left(t_k,t_{k+1}\right],k=0,\ldots,m.\end{gathered}$$

In turn, using the definition of the Hamilton function, the argmaximum set and the last inequality we have

$$\begin{gathered}g(x(t),t)-g(\tilde{x}(t),t)\ge\left\langle x^*(t),x'(t)-\tilde{x}'(t)\right\rangle+\left\langle x^{*\prime}(t),x(t)-\tilde{x}(t)\right\rangle,\\t\in\left(t_k,t_{k+1}\right],k=0,\ldots,m\end{gathered}$$

Integrating this inequality over the time interval $(t_k,t_{k+1}]$ implies

$$\int_{t_k}^{t_{k+1}}[g(x(t),t)-g(\tilde{x}(t),t)]dt\ge\int_{t_k}^{t_{k+1}}d\left\langle x^*(t),x(t)-\tilde{x}(t)\right\rangle,k=0,\ldots,m$$

or

$$\begin{gathered}\int_{t_k}^{t_{k+1}}[g(x(t),t)-g(\tilde{x}(t),t)]dt\ge\left\langle x^*\left(t_{k+1}\right),x\left(t_{k+1}\right)-\tilde{x}\left(t_{k+1}\right)\right\rangle\\-\left\langle x^*\left(t_k\right),x\left(t_k^+\right)-\tilde{x}\left(t_k^+\right)\right\rangle,k=0,\ldots,m.\end{gathered}$$

Then summing these inequalities over $k=0,\ldots,m$ and taking into account that $t_0=0$, $t_{m+1}=T$, $x\left(t_k^+\right)=x\left(t_k^-\right)+I_k\left(x\left(t_k^-\right)\right)=x\left(t_k\right)+I_k\left(x\left(t_k\right)\right),k=1,\ldots,m$, we obtain

$$\begin{gathered}\int_0^T[g(x(t),t)-g(\tilde{x}(t),t)]dt\ge\sum_{k=0}^m\left[\left\langle x^*\left(t_{k+1}\right),x\left(t_{k+1}\right)-\tilde{x}\left(t_{k+1}\right)\right\rangle\right.\\\left.-\left\langle x^*\left(t_k\right),x\left(t_k^+\right)-\tilde{x}\left(t_k^+\right)\right\rangle\right]=\left\langle x^*(T),x(T)-\tilde{x}(T)\right\rangle-\left\langle x^*(0),x(0)-\tilde{x}(0)\right\rangle\\-\sum_{k=0}^m\left\langle x^*\left(t_k\right),I_k\left(x\left(t_k\right)-I_k\left(\tilde{x}\left(t_k\right)\right)\right)\right\rangle.\end{gathered}$$

Now recall that both $x(\cdot)$ and $\tilde{x}(\cdot)$ are feasible trajectories and $x(0) = \tilde{x}(0) = x_0, I_k(x(t_k) = I_k(\tilde{x}(t_k)), k = 1, \ldots, m$. Thus, from the last inequality we have

$$\int_0^T [g(x(t), t) - g(\tilde{x}(t), t)]dt \geq \langle x^*(T), x(T) - \tilde{x}(T)\rangle . \tag{5}$$

On the other hand, by the first condition of (c) we can write

$$f(x(T)) - f(\tilde{x}(T)) \geq -\langle x^*(T), x(T) - \tilde{x}(T)\rangle . \tag{6}$$

Then from (5) and (6) we conclude that for all feasible trajectory $x(\cdot) \in PC \cap AC(t_k, t_{k+1}), k = 0, \ldots, m$

$$\int_0^T g(x(t), t)dt + f(x(T)) \geq \int_0^T g(\tilde{x}(t), t)dt + f(\tilde{x}(T)),$$

which completes the proof of theorem. □

Corollary 3.2. *For a Cauchy problem (P_C) with the Mayer functional the Euler-Lagrange inclusion (a) of Theorem 3.1 is simplified as follows*

$$-x^{*\prime}(t) \in F^*\left(x^*(t); (\tilde{x}(t), \tilde{x}'(t)), t\right) \text{ a.e. } t \in [0, T].$$

Proof. In fact, $g(x(t), t) \equiv 0$ and so $\partial g(x(t), t) \equiv 0$. On the other hand, there is no jump of the solutions at impulse point $t_k (k = 1, \ldots, m)$, and an adjoint inclusion (a) should be satisfied for almost every $t \in [0, T]$. □

Now, we try to investigate the problem (P_{SL}) with linear optimal control problem

$$(P_{SL}) \qquad \begin{aligned} &\text{minimize} \int_0^T g(x(t), t)dt + f(x(T)),\\ &x'(t) = Ax(t) + Bu(t), u(t) \in U, \quad t \in J \setminus \{t_1, \ldots, t_m\},\\ &x\left(t_k^+\right) - x\left(t_k^-\right) = I_k\left(x\left(t_k^-\right)\right), k = 1, \ldots, m,\\ &x(0) = x_0, \end{aligned}$$

where $g(\cdot, t)$ is a continuously differentiable function, A and B are $n \times n$ and $n \times r$ real matrices, respectively, U is a convex closed control domain. Obviously, for the problem (P_{SL}) $F(x, t) \equiv F(x) = Ax + BU$. According to Theorem 3.1 we should calculate the LAM $F^*(y^*; (\tilde{x}, \tilde{y}))$.
It is not hard to see that

$$F^*(y^*; (\tilde{x}, \tilde{y})) = \begin{cases} A^*y^*, & -B^*y^* \in K_U^*(\tilde{u}),\\ \varnothing, & -B^*y^* \notin K_U^*(\tilde{u}), \end{cases} \tag{7}$$

where $\tilde{y} = A\tilde{x} + B\tilde{u}, \tilde{u} \in U$. $\tilde{x}'(t) \in F_A(\tilde{x}(t); x^*(t), t)$.
On the other hand, by the definition of the argmaximum set $F_A(x, y^*)$, it is easy calculate that $F_A(\tilde{x}, y^*) = \{\tilde{u} \in U : \langle B\tilde{u}, y^*\rangle = \max_{u \in U}\langle Bu, y^*\rangle\}$. Note that the same formula can be obtained using the tangent direction cone definition. Indeed, in the formula (7) $K_U(\tilde{u}) = \{\bar{u} : \bar{u} = \lambda(u - \tilde{u}), \lambda > 0, u \in U\}$ and $-B^*y^* \in K_U^*(\tilde{u})$ means that $\langle -B^*y^*, u - \tilde{u}\rangle \geq 0$, or $\langle B\tilde{u}, y^*\rangle = \max_{u \in U}\langle Bu, y^*\rangle$.
Then, based on conditions $(a), (b)$ of Theorem 3.1, we obtain the following conditions

$$-x^{*\prime}(t) = A^*x^*(t) - g'(\tilde{x}(t)), \text{ a.e. } t \in [0, T] \tag{8}$$

$$\langle B\tilde{u}(t), x^*(t)\rangle = \max_{u \in U}\langle Bu, x^*(t)\rangle . \tag{9}$$

Theorem 3.3. *Suppose that $F(x) = Ax + BU : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a convex set-valued mapping, $g(\cdot, t), f : \mathbb{R}^n \to \mathbb{R}^1 \cup \{+\infty\}$ are continuous proper convex functions. Moreover, g is a continuously differentiable function. Then for optimality of the trajectory $\tilde{x}(\cdot)$ corresponding to the controlling parameter $\tilde{u}(t)$ to the convex problem (P_{SL}) it is sufficient that there is a function $x^*(\cdot), t \in [0, T]$ satisfying a.e. adjoint equation (8), maximum principle (9), and transversality condition (c) of Theorem 3.1.*

4. The Dual Problem for Convex Impulsive DFIs

To establish the dual problem $\left(P_C^*\right)$ for the main problem (P_C), we have used a discretizied method [16], which is omitted for the sake of shortening the paper. In this way after passing to the limit when the discrete step tends to zero, we finally obtain the following result:

$$\left(P_C^*\right) \qquad \sup_{x^*(\cdot),v^*(\cdot)} \left\{ -\int_0^T g^*\left(v^*(t),t\right)dt - f^*\left(-x^*(T)\right) + \int_0^T M_F\left(-x^{*\prime}(t)+v^*(t),x^*(t)\right)dt \right.$$
$$\left. -\sum_{k=1}^m \left\langle x^*\left(t_k\right), I_k\left(x\left(t_k\right)\right)\right\rangle - \left\langle x^*(0),x_0\right\rangle \right\}.$$

Further, we assume that $x^*(t), v^*(t), t \in [0,T]$ are absolutely-continuous functions. To prove the duality theorem, we use the results of Theorem 3.1. In fact, the next theorem proves that Euler-Lagrange inclusion (a) of Theorem 3.1 is the dual relation for pairs of primal (P_C) and dual $(P_C{}^*)$ problems.

Theorem 4.1. *Suppose that $\tilde{x}(t)$ is an optimal solution of the primal convex problem (P_C) with impulsive DFIs. Then a pair of functions $(\tilde{x}^*(\cdot), \tilde{v}^*(\cdot))$ is an optimal solution of the dual problem (P_C^*) if and only if the conditions (a)-(c) of Theorem 3.1 are satisfied. In addition, the optimal values in the primal (P_C) and dual (P_C^*) problems are equal.*

Proof. First of all, we should prove that for all feasible solutions $x(\cdot)$ and dual variables $(\tilde{x}^*(\cdot), \tilde{v}^*(\cdot))$ of the primal (P_C) and dual (P_C^*) problems, respectively, the following inequality holds:

$$\int_0^T g(x(t),t)dt + f(x(T)) \geq -\int_0^T g^*\left(v^*(t),t\right)dt - f^*\left(-x^*(T)\right)$$
$$+\int_0^T M_F\left(-x^{*\prime}(t)+v^*(t),x^*(t)\right)dt - \sum_{k=1}^m \left\langle x^*\left(t_k\right), I_k\left(x\left(t_k\right)\right)\right\rangle - \left\langle x^*(0),x_0\right\rangle. \tag{10}$$

For this, we use the definitions of the conjugate functions $g^*(\cdot,t), f^*$ and the Hamilton function. Therefore, observing the rule of integral calculus [17] we obtain

$$-\int_{t_k}^{t_{k+1}} g^*\left(v^*(t),t\right)dt \leq -\int_{t_k}^{t_{k+1}} \left\langle x(t),v^*(t)\right\rangle dt + \int_{t_k}^{t_{k+1}} g(x(t),t)dt,$$
$$-f^*\left(-x^*(T)\right) \leq f(x(T)) - \left\langle x(T),-x^*(T)\right\rangle, t \in \left(t_k,t_{k+1}\right], k=0,\ldots,m \tag{11}$$

$$\int_{t_k}^{t_{k+1}} M_F\left(-x^{*\prime}(t)+v^*(t),x^*(t)\right)dt \leq \int_{t_k}^{t_{k+1}} \left[\left\langle -x^{*\prime}(t)+v^*(t),x(t)\right\rangle - \left\langle x^*(t),x'(t)\right\rangle\right]dt$$
$$= -\int_{t_k}^{t_{k+1}} \left[\left\langle x^{*\prime}(t),x(t)\right\rangle + \left\langle x^*(t),x'(t)\right\rangle\right]dt + \int_{t_k}^{t_{k+1}} \left\langle v^*(t),x(t)\right\rangle dt$$
$$= -\int_{t_k}^{t_{k+1}} d\left\langle x^*(t),x(t)\right\rangle + \int_{t_k}^{t_{k+1}} \left\langle v^*(t),x(t)\right\rangle dt, t \in \left(t_k,t_{k+1}\right], k=0,\ldots,m. \tag{12}$$

In view of $x\left(t_k^+\right)=x\left(t_k^-\right)+I_k\left(x\left(t_k^-\right)\right)=x\left(t_k\right)+I_k\left(x\left(t_k\right)\right)$ summing the inequalities (11), (12) we deduce

$$\begin{aligned}
&-\int_{t_k}^{t_{k+1}} g^*\left(v^*(t),t\right)dt-f^*\left(-x^*(T)\right)+\int_{t_k}^{t_{k+1}} M_F\left(-x^{*\prime}(t)+v^*(t),x^*(t)\right)dt\\
&\le \int_{t_k}^{t_{k+1}} g(x(t),t)dt+f(x(T))+\langle x(T),x^*(T)\rangle-\int_{t_k}^{t_{k+1}} d\left\langle x^*(t),x(t)\right\rangle dt\\
&=\int_{t_k}^{t_{k+1}} g(x(t),t)dt+f(x(T))+\langle x(T),x^*(T)\rangle-\left\langle x^*\left(t_{k+1}\right),x\left(t_{k+1}\right)\right\rangle\\
&+\left\langle x^*\left(t_k\right),x\left(t_k^+\right)\right\rangle=\int_{t_k}^{t_{k+1}} g(x(t),t)dt+f(x(T))+\langle x(T),x^*(T)\rangle\\
&-\left\langle x^*\left(t_{k+1}\right),x\left(t_{k+1}\right)\right\rangle+\left\langle x^*\left(t_k\right),x\left(t_k\right)\right\rangle+\left\langle x^*\left(t_k\right),I_k\left(x\left(t_k\right)\right)\right\rangle,t\in\left(t_k,t_{k+1}\right],k=0,\dots,m,
\end{aligned}$$

whence since $I_0\left(x\left(t_0\right)=I_0(x(0))=0\right.$ we have

$$\begin{aligned}
&-\int_0^T g^*\left(v^*(t),t\right)dt-f^*\left(-x^*(T)\right)+\int_0^T M_F\left(-x^{*\prime}(t)+v^*(t),x^*(t)\right)dt\\
&\le\int_0^T g(x(t),t)dt+f(x(T))+\langle x(T),x^*(T)\rangle\\
&-\langle x^*(T),x(T)\rangle+\langle x^*(0),x(0)\rangle+\sum_{k=1}^m\left\langle x^*\left(t_k\right),I_k\left(x\left(t_k\right)\right)\right\rangle.
\end{aligned}$$

Hence, from the last inequality, finally, we derive

$$\begin{aligned}
&-\int_0^T g^*\left(v^*(t),t\right)dt-f^*\left(-x^*(T)\right)+\int_0^T M_F\left(-x^{*\prime}(t)+v^*(t),x^*(t)\right)dt\\
&-\sum_{k=1}^m\left\langle x^*\left(t_k\right),I_k\left(x\left(t_k\right)\right)\right\rangle-\langle x^*(0),x_0\rangle\le\int_0^T g(x(t),t)dt+f(x(T))
\end{aligned}$$

and this proves the inequality (10). Further, let $(\tilde{x}^*(\cdot),\tilde{v}^*(\cdot))$ satisfies conditions $(a)-(c)$ of Theorem 3.1. Then the Euler-Lagrange type inclusion (a) and the condition (b) of Theorem 3.1 imply that

$$H_F\left(x(t),x^*(t)\right)-H_F\left(\tilde{x}(t),\tilde{x}^*(t)\right)\le-\left\langle\tilde{x}^{*\prime}(t)+\tilde{v}^*(t),x(t)-\tilde{x}(t)\right\rangle,\tilde{v}^*(t)\in\partial g(\tilde{x}(t),t),$$

whence by the definition of function M_F we deduce that

$$-\left\langle\tilde{x}^{*\prime}(t),\tilde{x}(t)\right\rangle-\left\langle\tilde{x}^*(t),\tilde{x}'(t)\right\rangle-H_F\left(\tilde{x}(t),\tilde{x}^*(t)\right)=M_F\left(-\tilde{x}^{*\prime}(t)+\tilde{v}^*(t),\tilde{x}^*(t)\right).\tag{13}$$

Consequently, by Theorem 1.27 [16] the transversality condition (c) is equivalent to the relation

$$-f^*\left(-\tilde{x}^*(T)\right)=f(\tilde{x}(T))+\left\langle\tilde{x}(T),\tilde{x}^*(T)\right\rangle.\tag{14}$$

Besides, $\tilde{v}^*(t)\in\partial g(\tilde{x}(t))$ means that

$$-\int_0^T g^*\left(\tilde{v}^*(t),t\right)dt=-\int_0^T\left\langle\tilde{x}(t),\tilde{v}^*(t)\right\rangle dt+\int_0^T g(\tilde{x}(t),t)dt.\tag{15}$$

Hence, if we take into account relations (13)-(15) in (10), then we will make sure that it will be satisfied as equality, and thus for $\tilde{x}(\cdot)$ and $(\tilde{x}^*(\cdot),\tilde{v}^*(t))$ the equality of the values of the primal and dual problems is guaranteed. On the other hand, $\tilde{x}(\cdot)$ and $(\tilde{x}^*(\cdot),\tilde{v}^*(t))$ satisfy conditions (a) - (c) of Theorem 3.1, and $(a)-(c)$ is the dual relation for the primal (P_C) and dual (P_C^*) problems. Thus, we have proved that from conditions

$(a)-(c)$ it follows that $(\tilde{x}^*(\cdot), \tilde{v}^*(t))$ is a solution of the dual problem $(\mathrm{P}^*_{\mathrm{C}})$. The converse is proved by analogy. By Lemma 2.6 [16, p.64] x^* is an element of LAM $F^*(\cdot, x, v) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ if and only if $M_F(x^*, v^*) = \langle x, x^* \rangle - H_F(x, v^*)$. The last formula for our problem means that (13) is satisfied, whence we immediately have an inclusion of Euler-Lagrange type (a) of Theorem 3.1. Regarding (c), it suffices to recall that (14) is equivalent to (c) inscribed for a pair of functions $(\tilde{x}^*(\cdot), \tilde{v}^*(t))$. The proof of theorem is completed. □

Corollary 4.2. *For a Cauchy problem (P_C) with the Mayer functional the dual problem is as follows*

$$\sup_{x^*(\cdot)} \left\{ -f^*(-x^*(T)) + \int_0^T M_F(-x^{*\prime}(t), x^*(t))\, dt - \langle x^*(0), x_0 \rangle \right\}.$$

Proof. Indeed, in this case $g(x(t), t) \equiv 0$ and so $v^*(t) \equiv 0$. On the other hand, there is no jump of the solutions at impulse point $t_k (k = 1, \dots, m)$, i.e. $I_k(x(t_k) = 0, k = 1, \dots, m$. □

Now, try to construct the dual problem to the Bolza problem (P_{SL}), considered in Section 3. To this end calculate the M_F function:

$$M_F(x^*, y^*) = \inf_{(x,y)\in \mathrm{gph}F} \{\langle x, x^* \rangle - \langle y, y^* \rangle\} = \inf_x [\langle x, x^* - A^* y^* \rangle]$$
$$- \max_{u\in U} \langle u, B^* y^* \rangle = \begin{cases} -W_U(B^* y^*), & \text{if } x^* = A^* y^* \\ -\infty, & \text{otherwise.} \end{cases} \tag{16}$$

Then, by virtue of (16) according to the problem $(\mathrm{P}^*_{\mathrm{C}})$ we can write

$$M_F(-x^{*\prime}(t) + v^*(t), x^*(t)) = \begin{cases} -W_U(B^* x^*(t)), & \text{if } -x^{*\prime}(t) + v^*(t) = A^* x^*(t), \\ -\infty, & \text{otherwise,} \end{cases} \tag{17}$$

and consequently, it follows that the Euler-Lagrange type DFI (equation):

$$-x^{*\prime}(t) + v^*(t) = A^* x^*(t). \tag{18}$$

Then taking into account (17) and (18) it is clear that, according to problem $(\mathrm{P}^*_{\mathrm{C}})$, the problem dual to problem (P^*_{SL}) has the form

$$\left(P^*_{SL}\right) \quad \sup_{x^*(\cdot)} \left\{ -\int_0^T g^*(A^* x^*(t) + x^{*\prime}(t), t)\, dt - f^*(-x^*(T)) - \int_0^T W_U(B^* x^*(t))\, dt \right.$$
$$\left. - \sum_{k=1}^{T} \langle x^*(t_k), I_k(x(t_k)) \rangle - \langle x^*(0), x_0 \rangle \right\},$$

where $x^*(\cdot)$ is a solution of the adjoint Euler-Lagrange type inclusion/equation (18).

Theorem 4.3. *Let $\tilde{x}(\cdot)$ be an optimal solution of the primal Bolza problem (P_{SL}) with a linear impulsive DFI. Then $\tilde{x}^*(t), t \in [0, T]$ is an optimal solution of the dual problem (P^*_{SL}) if and only if the conditions of Theorem 4.1 are satisfied. In addition, the optimal values in the primal (P_{SL}) and dual (P^*_{SL}) problems are equal.*

5. Model of control problem with first order polyhedral impulsive DFIs

Here we establish a dual problem $(\mathrm{P}^*_{\mathrm{L}})$ to a Mayer ($g \equiv 0$) problem with a first-order polyhedral impulsive DFI:

$$(\mathrm{P_L}) \qquad \begin{aligned} &\text{minimize} \quad f(x(T)), \\ &x'(t) \in F(x(t)), \text{ a.e. } t \in [0, T],\ t \in J \setminus \{t_1, \dots, t_m\}, \\ &x\left(t_k^+\right) - x\left(t_k^-\right) = I_k\left(x\left(t_k^-\right)\right),\ k = 1, \dots, m,\ x(0) = x_0 \\ &F(x) = \{y : Ax - By \le d\}, \end{aligned} \tag{19}$$

where F is first order polyhedral set-valued mapping, A, B are $s \times n$ matrices, d is a s dimensional column-vector, $f : \mathbb{R}^n \to \mathbb{R}^1 \cup \{+\infty\}$ is a convex function. We label this problem by ($\mathrm{P_L}$). According to the dual problem ($\mathrm{P_C^*}$), we calculate the function $M_F(x^*, y^*)$:

$$M_F(x^*, y^*) = \inf\{\langle x, x^* \rangle - \langle y, y^* \rangle : (x, y) \in \operatorname{gph} F\}. \tag{20}$$

Denoting $w = (x, y) \in \mathbb{R}^{2n}, w^* = (x^*, -y^*) \in \mathbb{R}^{2n}$, we compose the linear programming problem:

$$\inf\{\langle w, w^* \rangle : Dw \leq d\}, \tag{21}$$

where $D = [A : -B]$ is $s \times 2n$ block matrix. Then, according to the theory of linear programming, in order for $\tilde{w} = (\tilde{x}, \tilde{y})$ to be a solution to problem (21), it is necessary and sufficient to satisfy the following conditions

$$w^* = -D^*\lambda, \langle A\tilde{x} - B\tilde{y} - d, \lambda \rangle = 0, \lambda \geq 0.$$

Hence, $w^* = -D^*\lambda$ means that $x^* = -A^*\lambda, y^* = B^*\lambda, \lambda \geq 0$. Thus, for a function (20) with (19) we find that

$$M_F(x^*, y^*) = \langle \tilde{x}, -A^*\lambda \rangle + \langle \tilde{y}, B^*\lambda \rangle = -\langle A\tilde{x}, \lambda \rangle + \langle B\tilde{y}, \lambda \rangle = -\langle d, \lambda \rangle. \tag{22}$$

On the other hand, from the form of $M_F(-x^{*\prime}(t), x^*(t))$ by Theorem 3.1 we derive that

$$-x^{*\prime}(t) = -A^*\lambda(t), \quad x^*(t) = B^*\lambda(t), \quad \lambda(t) \geq 0 \tag{23}$$

or

$$A^*\lambda(t) - B^*\lambda'(t) = 0, \quad \lambda(t) \geq 0 \tag{24}$$

. Therefore, taking into account (22)-(24) we have the following dual problem

$$\left(P_L^*\right) \qquad \sup_{\lambda(t)\geq 0} \left\{ -f^*\left(-B^*\lambda(T)\right) - \int_0^T \langle d, \lambda(t) \rangle dt - \sum_{k=1}^T \langle B^*\lambda\left(t_k\right), I_k\left(x\left(t_k\right)\right) \rangle - \langle B^*\lambda(0), x_0 \rangle \right\}.$$

Theorem 5.1. *Let $\tilde{x}(\cdot)$ be an optimal solution of the primal Mayer problem (P_L) with a polyhedral impulsiveDFI. Then $\lambda(t) \geq 0, t \in [0, T]$ is an optimal solution of the dual problem (P_L^*) if and only if the conditions (24) and $-B^*\lambda(T) \in \partial f(\tilde{x}(T))$, $B^*\lambda(0) = 0$ are satisfied. In addition, the optimal values in the primal (P_L) and dual (P_L^*) problems are equal.*

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