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An effective operational matrix method based on shifted sixth-kind Chebyshev polynomials for solving fractional integro-differential equations with a weakly singular kernel

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Abstract. The principal aim of this study is to present a new algorithm for solving some kinds of weakly singular fractional integro-differential equations. The suggested algorithm uses the shifted sixth-kind Chebyshev polynomials together with the collocation method. Using the suggested algorithm and resultant operational matrices, the main equation converts into a system of algebraic equations which can be efficiently solved. Some theorems are proved and used to deduce an upper error bound for this method. Also, several examples are presented to illustrate the efficiency of the suggested algorithm compared to other methods in the literature. The suggested algorithm provides accurate results, even using a few terms of the proposed expansion.

1. Introduction

In recent years, many phenomena in engineering, physics, chemistry, and other sciences have been modeled successfully by mathematical tools of fractional calculus (.i.e. theory of derivatives and integrals of arbitrary orders)[1–7]. Therefore, with the development of fractional calculus in modeling natural phenomena, theoretical and numerical analysis of these equations has been paid much attention by some researchers. (see previous works[8–14]).

Many natural phenomena in physical science have been formulated as fractional order integro-differential equations with weakly singular kernels. These kinds of equations arise in the field of elasticity and fracture mechanics [15], radiative equilibrium [16], heat conduction problems [17], and other applications of these equations can be found in [18–20]. So far, there are a few achievements in the numerical methods for solving fractional order weakly singular integro-differential equations, for example second kind Chebyshev polynomials [21], Euler functions (FEFs)[22], Legendre wavelet method [23], modified hat functions (MHFs) [24],

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spline collocation method [25], piecewise polynomial [26], the second kind Chebyshev wavelets (SCW) method [27], least residue method [28]. Since solving these equations analytically is hard or not even possible, thus the approximation of the solutions of these equations leads to a new subject.

In this paper, we study three type of fractional order weakly singular equations as follows

$$\mathcal{D}^{\eta}u(x) = q(x)u(x) + \theta_1 \int_0^x \frac{\mathcal{H}(x,z)u(z)}{(x-z)^{\kappa}}\,dz + \theta_2 \int_0^1 \bar{\mathcal{H}}(x,z)u(z)\,dz + g(x), \tag{1}$$

$$\mathcal{D}^{\eta}u(x) = u(x) + \theta_1 \int_0^x \frac{\mathcal{H}(x,z)\mathcal{D}^{\vartheta}u(z)}{(x-z)^{\kappa}}\,dz + \theta_2 \int_0^1 \bar{\mathcal{H}}(x,z)u(z)\,dz + g(x), \qquad 0 < \vartheta \le \eta, \tag{2}$$

$$\mathcal{D}^{\eta}u(x) = u^2(x) + \theta_1 \int_0^x \frac{\mathcal{H}(x,z)\mathcal{D}^{\vartheta}u(z)}{(x-z)^{\kappa}}\,dz + \theta_2 \int_0^1 \bar{\mathcal{H}}(x,z)u^2(z)\,dz + g(x), \qquad 0 < \vartheta \le \eta. \tag{3}$$

The general form of these equations is as below

$$\begin{aligned} &\mathcal{D}^{\eta}u(x) = F(x,u(x)) + \theta_1 \int_0^x \frac{\mathcal{F}(x,z,u(z),\mathcal{D}^{\vartheta}u(z))}{(x-z)^{\kappa}}\,dz + \theta_2 \int_0^1 \bar{\mathcal{F}}(x,z,u(z))\,dz + g(x), \quad x \in [0,1], \\ &u^j(0) = u_0^j, \qquad j = 0,1,\dots,m-1, \quad m-1 < \eta \le m, \end{aligned} \tag{4}$$

where $m = \lceil \eta \rceil$ is the ceiling function of $\eta, F(x,u(x))$ and $g(x)$ are known and continuous functions on $[0,1]$. $\mathcal{F}(x,z,u(z),\mathcal{D}^{\vartheta}u(z)), \bar{\mathcal{F}}(x,z,u(z))$ are linear or nonlinear and sufficiently smooth functions, and $u(x) \in C^m[0,1]$ is an unknown function. $\mathcal{D}^{\eta}$ and $\mathcal{D}^{\vartheta}$ are Caputo fractional derivative operators where $0 \le \vartheta \le \eta$. The parameters $\theta_i, \kappa \in \mathbb{R}$, such that $-1 \le \theta_i \le 1, i = 1,2$, and $0 < \kappa < 1$.

Existence and uniqueness of the solutions of Eq. (4) provided in Das et al. [29] and also Biazar and sadri in [30]. A popular way to solve functional equations is to express the solution as a linear combination of the basis functions. Among the family of polynomials, the Chebyshev polynomials usually give the near-best approximation. Due to their orthogonality, they are proper to obtain the operational matrices. So far, the first kind and second kind of Chebyshev polynomials have been used for weakly singular fractional Volterra integro-differential equations and fractional integro-differential equations with weakly singular kernels (see [21, 31]). But the sixth-kind Chebyshev polynomials, due to the complicated analytical form and their weight function, have been used rarely as a basis function. The basic formula and properties of this class of polynomials can be seen in [32–34] . The main aim of this paper is to consider the sixth-kind Chebyshev polynomials as less used basis functions for solving fractional weakly singular integro-differential equations. Also, we attempt to increase calculation speed by providing algorithms to approximate the singular integral or nonlinear functions without any integration, as Kumar et al. in [40] presented another solution for cost reduction. We demonstrate the obtained approximations of these polynomials are more accurate than the obtained approximations of the second kind. Despite the long and complicated computations, the numerical approach based on sixth-kind Chebyshev polynomials is fast. Moreover, we already knew the distribution of roots of different kinds of Chebyshev polynomials on the interval $[0,1]$, we intend to consider the influence of the distribution of roots of the sixth-kind Chebyshev polynomials on $[0,1]$. Therefore, using sixth-kind Chebyshev polynomials we derive the fractional operational matrices of fractional and integer orders and the product operational matrix by presenting some useful algorithms. Also, we introduce an operational matrix to approximate the integral part with the singular kernel in Eq. (4). By substitution proper approximations in Eq. (4), the original equation converts into an algebraic equation that is collocated at $N+1$ roots of the $(N+1)$th shifted sixth-kind Chebyshev polynomial (SSKCP). By solving this algebraic system, the approximate solution of the original equation is obtained. The nonlinear system can be solved by the well-known Newton iterative method in [35, 36]. In this paper, we use shifted sixth-kind Chebyshev polynomials and acquire a numerical solution of the fractional integro-differential with weakly singular kernels without any integration and multiplication operations on the basic vector. This method is accurate, advantageous, and easy to implement. We arrange the rest of this paper as follows. In Section 2, we present some basic definitions and properties of fractional calculus and shifted sixth-kind Chebyshev polynomials.

Section 3 is focused on constructing operational matrices of SSKCPs. In Section 4, a collocation method based on shifted sixth-kind Chebyshev polynomials is implemented. The error analysis of the proposed method is discussed in Section 5 and convergence of the presented method is investigated in section 6. Some numerical test examples are provided in Section 7. Finally, in Section 8 the main conclusions are presented.

2. Definitions and Preliminaries

In this section, we recall some definitions and properties of fractional integral and derivative, which will be used later. After that, some necessary definitions and fundamental properties of the shifted sixth-kind Chebyshev polynomials are reviewed briefly.

2.1. Fractional calculus

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f(x)$ is defined as[1, 12]*

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-z)^{\alpha-1} f(z)\, dz, \qquad \alpha > 0,\ x > 0.$$

The above integral exist almost everywhere for any absolutely integrable function $f(x)$.

Definition 2.2. *Let $\alpha \in \mathbb{R}$, $n-1 < \alpha \leq n$, $n \in \mathbb{N}$ and $f(x)$ is an absolutely continuous function on the interval $[0,\infty)$, then the Caputo fractional derivative of order $\alpha > 0$ is defined by[1, 21]*

$$\begin{cases} {}_0\mathcal{D}_x^{\alpha} f(x) = \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^x \frac{f^n(z)}{(x-z)^{\alpha+1-n}}\, dz, \\ f^{(n)}(z), \qquad \alpha = n, \end{cases}$$

where $\Gamma(x)$ is the Gamma function given as follows

$$\Gamma(x) = \int_0^{\infty} e^{-z} z^{x-1}\, dz, \qquad Rez > 0,$$

with the property $\Gamma(x+1) = x\Gamma(x)$. Also, the Beta integral can be computed using the Gamma function as

$$B(u,v) = \int_0^1 z^{u-1}(1-z)^{v-1}\, dz = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \qquad Reu > 0, Rev > 0.$$

The Riemann-Liouville integral, J^{α}, and the Caputo fractional derivative, $\mathcal{D}^{\alpha}$, operators satisfy the following properties

1. $J^{\alpha_1}(J^{\alpha_2} f(x)) = J^{\alpha_2}(J^{\alpha_1} f(x)) = J^{\alpha_1+\alpha_2} f(x)$,
2. $J^{\alpha}(\lambda_1 f(x) + \lambda_2 g(x)) = \lambda_1 J^{\alpha} f(x) + \lambda_2 J^{\alpha} g(x)$,
3. $J^{\alpha}(\mathcal{D}^{\alpha} f(x)) = f(x) - \sum_{i=0}^{n-1} f^i(0)\dfrac{x^i}{i!}, \qquad n-1 < \alpha \leq n,\ x > 0,$
4. $\mathcal{D}^{\alpha} x^{\gamma} = \begin{cases} 0 & \alpha > \gamma, \\ \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha} & otherwise. \end{cases},$
5. $J^{\alpha} x^{v} = \frac{\Gamma(v+1)}{\Gamma(v+\alpha+1)} x^{v+\alpha}, \qquad v > -1.$

where $\alpha, \alpha_1, \alpha_2, \gamma \in \mathbb{R}^+$ and $\nu, \lambda_1, \lambda_2 \in \mathbb{R}$.

Theorem 2.3. *If $f(x)$ is any continuous function on the interval $[0, X]$ for some $X > 0$, then*

$$\mathcal{D}^{\alpha_2}\mathcal{D}^{\alpha_1} f(x) = \mathcal{D}^{\alpha_1}\mathcal{D}^{\alpha_2} f(x) = \mathcal{D}^{\alpha_1+\alpha_2} f(x), \qquad x \in [0, X],$$

where $\mathcal{D}^{\alpha_1}, \mathcal{D}^{\alpha_2}$ are the Caputo derivative, $\alpha_1, \alpha_2 \in \mathbb{R}^+$ and $\alpha_1 + \alpha_2 \leq 1$.

Proof. See[38]. □

2.2. Shifted sixth-kind Chebyshev polynomials

Definition 2.4. *The sixth-kind Chebyshev polynomials are orthogonal functions on the interval* $[-1, 1]$ *and can be determined by the following recursive formula[34]*

$$S_j(x) = xS_{j-1}(x) - \frac{j(j+1) + (-1)^j(2j+1) + 1}{4j(j+1)} S_{j-2}(x), \qquad j \geq 2,$$
$$S_0(x) = 1, \quad S_1(x) = x.$$

Definition 2.5. *The shifted sixth-kind Chebyshev polynomials on* $[0, 1]$ *defined by[33, 39]*

$$S_j^*(x) = S_j(2x - 1), \qquad j = 0, 1, 2, \cdots .$$

These polynomials have the following explicit analytic form

$$S_j^*(x) = \sum_{k=0}^{j} \rho_{kj} x^k, \tag{5}$$

where

$$\rho_{kj} = \begin{cases} \frac{2^{2k-j}}{(2k+1)!} \sum\limits_{i=\lfloor \frac{k+1}{2} \rfloor}^{\frac{j}{2}} \frac{(-1)^{\frac{j}{2}+i+k}\,(2i+k+1)!}{(2i-k)!}, & j \text{ even}, \\ \frac{2^{2k-j+1}}{(2k+1)!(j+1)} \sum\limits_{i=\lfloor \frac{k}{2} \rfloor}^{\frac{j-1}{2}} \frac{(-1)^{\frac{j+1}{2}+i+k}(i+1)(2i+k+2)!}{(2i-k+1)!}, & j \text{ odd}. \end{cases} \tag{6}$$

Moreover, the shifted polynomials $S_j^*(x)$ *are orthogonal on* $[0, 1]$ *with respect to the weight function* $V(x) = (2x - 1)^2\sqrt{x - x^2}$ *in the sense that*

$$\int_0^1 S_i^*(x)S_j^*(x)V(x)\,dx = \lambda_i\delta_{ij}, \tag{7}$$

$$\lambda_i = \begin{cases} \frac{\pi}{2^{2i+5}}, & i \text{ even}, \\ \frac{\pi(i+3)}{2^{2i+5}(i+1)}, & i \text{ odd}. \end{cases} \tag{8}$$

Now, let $h(x) \in L^2[0, 1]$ *then* $h(x)$ *can be approximated in terms of* $S_j^*(x)$ *as*

$$h(x) \approx \sum_{j=0}^{N} \varrho_j S_j^*(x) = F^T\mathfrak{S}(x) = \mathfrak{S}^T(x)F,$$

where

$$\mathfrak{S}(x) = \left[S_0^*(x), S_1^*(x), \ldots, S_N^*(x)\right]^T, \qquad F = [\varrho_0, \varrho_1, \ldots, \varrho_N]^T, \tag{9}$$

where the coefficients ϱ_j *are given by*

$$\varrho_j = \frac{1}{\lambda_j}\int_0^1 h(x)S_j^*(x)V(x)\,dx,$$

and λ_j *is defined in Eq.* (8)*. Similarly, any continuous two-variable function,* $G(x, z)$*, defined on* $[0, 1] \times [0, 1]$ *can be approximated by means of the double-shifted sixth-kind Chebyshev polynomials as*

$$G(x, z) \approx \sum_{j=0}^{N}\sum_{i=0}^{N} g_{ij}S_i^*(x)S_j^*(z) = \mathfrak{S}^T(x)G\mathfrak{S}(z), \tag{10}$$

where G *is a* $(N + 1) \times (N + 1)$ *matrix and its entries are given by*

$$g_{ij} = \frac{1}{\lambda_i\lambda_j}\int_0^1\int_0^1 G(x, z)S_i^*(x)S_j^*(z)V(x)V(z)\,dxdz. \tag{11}$$

3. Operational matrices of SSKCPs

In this section, the formulas of operational matrices will be derived from the fractional order for the sixth-kind Chebyshev polynomials. Some lemmas and theorems are proved in the following.

Lemma 3.1. *If $r \geq l$, $l \in \mathbb{N}$, then we have*

$$\int_0^1 x^r S_l^*(x)V(x)\,dx = \sum_{m=0}^{l} \frac{\rho_{ml}\sqrt{\pi}\Gamma(r+m+\frac{3}{2})}{2\Gamma(r+m+5)}(m^2+m+r^2+2rm+3+r).$$

Proof. The lemma can be easily proved by the integration of analytic form SSKCPs Eq. (5). □

Theorem 3.2. *Let $\mathfrak{S}(x)$ be the SSKCPs vector given by Eq. (9), $\mu \in \mathbb{R}$ then*

$$J^{\mu}\mathfrak{S}(x) \approx \mathscr{P}^{(\mu)}\mathfrak{S}(x),$$

where $\mathscr{P}^{(\mu)}$ is the $(N+1)\times(N+1)$ operational matrix of fractional integration of order μ in the Riemann-Liouville sense, which is defined by

$$\mathscr{P}^{(\mu)} = \begin{bmatrix} \tilde{a}_{00} & \tilde{a}_{01} & \dots & \tilde{a}_{0N} \\ \tilde{a}_{10} & \tilde{a}_{11} & \dots & \tilde{a}_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{N0} & \tilde{a}_{N1} & \dots & \tilde{a}_{NN} \end{bmatrix},$$

$$\tilde{a}_{ij} = \sum_{l=0}^{i} \varpi_{ijl}, \quad i=0,\dots,N,\ j=0,\dots,N. \tag{12}$$

and ϖ_{ijl} are given by

$$\varpi_{ijl} = \rho_{li}\frac{\Gamma(l+1)\Gamma(\frac{3}{2})}{\lambda_j\Gamma(l+\mu+1)}\sum_{k=0}^{j}\rho_{kj}\Big[\frac{4\Gamma(l+\mu+k+\frac{7}{2})}{\Gamma(l+\mu+k+5)} - \frac{4\Gamma(l+\mu+k+\frac{5}{2})}{\Gamma(l+\mu+k+4)} + \frac{\Gamma(l+\mu+k+\frac{3}{2})}{\Gamma(l+\mu+k+3)}\Big], \quad i,j=0,\dots,N.$$

Proof. By applying the Riemann-Liouville integral operator to the SSKCPs analytic form, we have

$$J^{\mu}S_i^*(x) = \sum_{l=0}^{i} \rho_{li}\frac{\Gamma(l+1)}{\Gamma(l+\mu+1)}x^{\mu+l}, \tag{13}$$

now, we can express $x^{\mu+l}$ in terms of the shifted sixth-kind Chebyshev polynomials as the following

$$x^{\mu+l} \approx \sum_{j=0}^{N} \tilde{C}_{lj}S_j^*(x),$$

where the coefficients $\tilde{C}_{lj}$, are given by

$$\tilde{C}_{lj} = \frac{1}{\lambda_j}\int_0^1 x^{\mu+l}S_j^*(x)V(x)\,dx,$$

By substitution of the analytic form of the sixth-kind Chebyshev polynomials into the above integral and using Beta function definition, we calculate the integral and we can rewrite Eq. (13) as follows

$$J^{\mu}S_j^{(*)}(x) \approx \sum_{j=0}^{N}\left\{\sum_{l=0}^{i}\rho_{li}\frac{\Gamma(l+1)\Gamma(\frac{3}{2})}{\lambda_j\Gamma(l+\mu+1)}\right.$$
$$\left.\times\sum_{k=0}^{j}\rho_{kj}\Big[\frac{4\Gamma(l+\mu+k+\frac{7}{2})}{\Gamma(l+\mu+k+5)}-\frac{4\Gamma(l+\mu+k+\frac{5}{2})}{\Gamma(l+\mu+k+4)}+\frac{\Gamma(l+\mu+k+\frac{3}{2})}{\Gamma(l+\mu+k+3)}\Big]\right\}S_j^{*}(x)$$
$$=\sum_{j=0}^{N}\tilde{a}_{ij}S_j^{*}(x).$$

where $\tilde{a}_{ij}$ is given in Eq. (12). The last relation can be rewritten in the vector form as follows

$$J^{\mu}S_j^{*}(x) \approx [\tilde{a}_{i0},\tilde{a}_{i1},\dots,\tilde{a}_{iN}]\,S_j^{*}(x), \qquad i=0,1,...,N.$$

This leads to the desired result. □

In the following, some useful and applicable lemmas are presented to get the Chebyshev operational matrix of product.

Lemma 3.3. *If $S_j^*(x)$ and $S_i^*(x)$ are jth and ith shifthed sixth-kind Chebyshev polynomials, then we can write the product of $S_j^*(x)$ and $S_i^*(x)$ as*

$$\eth_{i+j}(x) = S_i^*(x)S_j^*(x) = \sum_{k=0}^{i+j}\chi_k^{(i,j)}x^k.$$

The coefficients $\chi_k^{(i,j)}$ are computed by Algorithm 1 and the quantities ρ_{rj} are determined using Eq. (6).

Proof. See[30]. □

To clarify the performance of this algorithm consider two following Chebyshev polynomials of the sixth kind

$$S_4^*(x) = \frac{3}{16} - 4x + 20x^2 - 32x^3 + 16x^4, \quad S_2^*(x) = \frac{1}{2} - 4x + 4x^2.$$

Then, the direct product is

$$\eth_{4+2}^{(4,2)}(x) = S_4^*(x)S_2^*(x) = \frac{3}{32} - \frac{11}{4}x + \frac{107}{4}x - 112x^3 + 216x^4 - 192x^5 + 64x^6.$$

Algorithm 1 Computation of the quantity $\chi_k^{(i,j)}$

Input : i, j, ρ_{mn}
If $i \geq j$:
$k = 0, 1, ..., i + j,$
if $k > i$ **then**
  $\chi_k^{(i,j)} = \sum_{r=k-i}^{j} \rho_{(k-r)i}\rho_{rj},$
else
  $\hat{r} = min\{k, j\}$
  $\chi_k^{(i,j)} = \sum_{r=0}^{\hat{r}} \rho_{(k-j)i}\rho_{rj},$
end if

If $i < j$:
$k = 0, 1, ..., i + j,$
if $k \leq i$ **then**
  $\hat{r} = min\{k, i\},$
  $\chi_k^{(i,j)} = \sum_{r=0}^{\hat{r}} \rho_{(k-r)i}\rho_{rj},$
else
  $\tilde{r} = min\{k, j\}$
  $\chi_k^{(i,j)} = \sum_{r=k-i}^{\tilde{r}} \rho_{(k-r)i}\rho_{rj},$
end if
Output : $\chi_k^{(i,j)}$

The coefficients in the above equation are computed using Algorithm 1 as follows

$$i = 4, \quad j = 2, \quad i + j = 6, \quad i \geq j :$$
$$k = 0, 1, \ldots, 6 :$$
$$k = 0 < i, \quad \hat{r} = min\{0, 2\} = 0, \quad \chi_0^{(4,2)} = \sum_{r=0}^{0} \rho_{0-r4}\rho_{r2} = \rho_{04}\rho_{02} = (\frac{3}{16})(\frac{1}{2}) = \frac{3}{32},$$
$$k = 1 < i, \quad \hat{r} = min\{1, 2\} = 1, \quad \chi_1^{(4,2)} = \sum_{r=0}^{1} \rho_{1-r4}\rho_{r2} = \rho_{14}\rho_{02} + \rho_{04}\rho_{12} = (-4)(\frac{1}{2}) + (\frac{3}{16})(-4) = \frac{-11}{4},$$
$$k = 2 < i, \quad \hat{r} = min\{2, 2\} = 2, \quad \chi_2^{(4,2)} = \sum_{r=0}^{2} \rho_{2-r4}\rho_{r2} = \rho_{24}\rho_{02} + \rho_{14}\rho_{12} + \rho_{24}\rho_{02}$$
$$= (20)(\frac{1}{2}) + (-4)(-4) + \frac{3}{16}(4) = \frac{107}{4},$$
$$k = 3 < i, \quad \hat{r} = min\{3, 2\} = 2, \quad \chi_3^{(4,2)} = \sum_{r=0}^{2} \rho_{3-r4}\rho_{r2} = \rho_{34}\rho_{02} + \rho_{24}\rho_{12} + \rho_{14}\rho_{22}$$
$$= (-32)(\frac{1}{2}) + (20)(-4) + (-4)(4) = -112,$$
$$k = 4 = i, \quad \hat{r} = min\{4, 2\} = 2, \quad \chi_4^{(4,2)} = \sum_{r=0}^{2} \rho_{4-r4}\rho_{r2} = \rho_{44}\rho_{02} + \rho_{34}\rho_{12} + \rho_{24}\rho_{22}$$
$$= (16)(\frac{1}{2}) + (-32)(-4) + (20)(4) = 216,$$
$$k = 5 > i, \quad \hat{r} = min\{0, 2\} = 0, \quad \chi_5^{(4,2)} = \sum_{r=1}^{2} \rho_{5-r4}\rho_{r2} = \rho_{44}\rho_{12} + \rho_{34}\rho_{22} = (16)(-4) + (-32)(4) = -192,$$
$$k = 6 > i, \quad \hat{r} = min\{0, 2\} = 0, \quad \chi_6^{(4,2)} = \sum_{r=2}^{2} \rho_{44}\rho_{22} = 64,$$

so we have

$$\eth_{i+j}(x) = S_4^*(x)S_2^*(x) = \sum_{k=0}^{6} \chi_k^{(4,2)} x^k = \frac{3}{32} - \frac{11}{4}x + \frac{107}{4}x - 112x^3 + 216x^4 - 192x^5 + 64x^6.$$

Lemma 3.4. *If $S_i^*(x)$, $S_j^*(x)$ and $S_k^*(x)$ are ith, jth and kth shifted sixth-kind Chebyshev polynomials then*

$$d_{ijk} = \int_0^1 S_i^*(x)S_j^*(x)S_k^*(x)V(x)\,dx = \sum_{r=0}^{j+k}\sum_{l=0}^{i} \frac{\sqrt{\pi}\rho_{li}\chi_k^{(i,j)}\Gamma(r+l+\frac{3}{2})}{2\Gamma(r+l+5)}((l+r)(l+r+1)+3),$$

where $\chi_k^{(i,j)}$ is obtained by Lemma 3.3.

Proof. According to Lemma 3.3, we can write

$$\eth_{j+k}(x) = S_j^*(x)S_k^*(x) = \sum_{r=0}^{j+k} \chi_r^{(j,k)} x^r,$$

then

$$d_{ijk} = \int_0^1 S_i^*(x) \sum_{r=0}^{j+k} \chi_r^{(j,k)} x^r V(x)\,dx = \sum_{r=0}^{j+k} \chi_r^{(j,k)} \int_0^1 S_i^*(x) x^r V(x)\,dx.$$

The value of the integral is obtained by Lemma 3.1. □

Assuming that $\mathbb{E}$ is a $(N+1)\times 1$ vector, we have

$$\mathfrak{S}(x)\mathfrak{S}^T(x)\mathbb{E} \approx \tilde{\mathbb{E}}\mathfrak{S}(x). \tag{14}$$

where $\tilde{\mathbb{E}}$ is a $(N+1)\times(N+1)$ matrix, called the product operational matrix. The next theorem presents a general form for entries of matrix $\tilde{\mathbb{E}}$.

Theorem 3.5. *The entries of matrix $\tilde{\mathbb{E}}$ in Eq.* (14) *are as follows*

$$\tilde{\mathbb{E}}_{jk} = \frac{1}{\lambda_k}\sum_{i=0}^{N} \mathbb{E}_i d_{ijk} \qquad j,k = 0,1,\cdots,N,$$

where d_{ijk} is obtained using Lemma 3.4, and $\mathbb{E}_i$ is the element of the vector $\mathbb{E}$.

Proof. See[30]. □

Remark 3.6. *Let $\mathfrak{S}(x)$ be SSKCPs vector in Eq.* (9) *then*

$$Q = \int_0^1 \mathfrak{S}(x)\mathfrak{S}^T(x)\,dx$$

is a $(N+1)\times(N+1)$ matrix and its entries are determined as follows

$$Q_{ij} = \sum_{r=0}^{i+j} \frac{\chi_r^{(i,j)}}{r+1},$$

the quantities $\chi_r^{(i,j)}$ are introduced in Lemma 3.3.

In the following, we get an approximation for the integral part with singular kernel in Eq. (4). Before that, we present a theorem.

Theorem 3.7. *The following relation is determined for* $0 < \kappa < 1$

$$\int_0^x \frac{z^r}{(x-z)^\kappa}\,dz = \frac{\Gamma(r+1)\Gamma(1-\kappa)}{\Gamma(r-\kappa+2)} x^{r-\kappa+1}, \qquad r = 0,1,2,\dots. \tag{15}$$

Proof. By substituting $z = \xi x$ into Eq.(15) and then using the definition of Beta function, we obtain

$$x^{r-\kappa+1}\int_0^1 (1-\xi)^{-\kappa}\xi^r\,d\xi = \frac{\Gamma(1-\kappa)\Gamma(r+1)}{\Gamma(r-\kappa+2)} x^{r-\kappa+1}, \qquad r = 0,1,2,\dots.$$

□

Now, we present an approximation of integral with a weakly singular kernel. For this purpose, see the following theorem.

Theorem 3.8. *Suppose that* $u(x) \in C[0,1]$ *and* $\kappa \in (0,1)$ *and* $u(x) \approx \mathfrak{S}^T(x)F = F^T\mathfrak{S}(x)$ *where* $\mathfrak{S}(x)$ *and* F *are defined by Eq.* (9), *then we have*

$$\int_0^x \frac{u(z)}{(x-z)^\kappa}\,dz \approx F^T\mathfrak{I}^{(\kappa)}\mathfrak{S}(x),$$

where $\mathfrak{I}^{(\kappa)}$ *is a* $(N+1)\times(N+1)$ *matrix as follows*

$$\mathfrak{I}^{(\kappa)} = \begin{bmatrix} \sigma_{00} & \sigma_{01} & \dots & \sigma_{0N} \\ \sigma_{10} & \sigma_{11} & \dots & \sigma_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N0} & \tilde{a}_{N1} & \dots & \sigma_{NN} \end{bmatrix},$$

and its entries are determined as follows

$$\mathfrak{I}^{(\kappa)}_{ij} = \sigma_{ij} = \sum_{l=0}^{i} \rho_{li}\frac{\Gamma(l+1)\Gamma(1-\kappa)}{\Gamma(l-\kappa+2)}\tilde{C}_{j(l-\kappa+1)}, \qquad i,j = 0,1,\dots,N.$$

where the quantities ρ_{li} *and* $\tilde{C}_{j(l-\kappa+1)}$ *are introduced in relation* (6) *and Theorem 3.2, respectively.*

Proof. By definition of vector $\mathfrak{S}(x)$ and Eq. (5), we can write

$$\mathfrak{S}^T(x) = \left[S_0^*(x), S_1^*(x), \dots, S_N^*(x)\right] = \left[\sum_{l=0}^{0}\rho_{l0}x^l, \dots, \sum_{l=0}^{N}\rho_{lN}x^l\right],$$

applying Theorem 3.7, we have

$$\begin{aligned} \int_0^x \frac{\mathfrak{S}^T(z)}{(x-z)^\kappa}\,dz &= \left[\sum_{l=0}^{0}\int_0^x \rho_{l0}\frac{z^l}{(x-z)^\kappa}\,dz, \dots, \sum_{l=0}^{N}\rho_{lN}\int_{l=0}^x \frac{z^l}{(x-z)^\kappa}\,dz\right] \\ &= \left[\sum_{l=0}^{0}\rho_{l0}\frac{\Gamma(l+1)\Gamma(1-\kappa)}{\Gamma(l-\kappa+2)}x^{l-\kappa+1}, \dots, \sum_{l=0}^{N}\rho_{lN}\frac{\Gamma(l+1)\Gamma(1-\kappa)}{\Gamma(l-\kappa+2)}x^{l-\kappa+1}\right]. \end{aligned} \tag{16}$$

Now, we approximate $x^{l-\kappa+1}$ in terms of SSKCPs

$$x^{l-\kappa+1} \approx \sum_{j=0}^{N} \tilde{C}_{j(l-\kappa+1)} S_j^*(x),$$

and

$$\tilde{C}_{j(l-\kappa+1)} = \frac{1}{\lambda_j} \int_0^1 x^{l-\kappa+1} S_j^*(x) V(x)\, dx, \qquad j = 0, 1, \dots, N$$

where $\tilde{C}_{j(l-\kappa+1)}$ are obtained by applying Theorem 3.2. Thus

$$\begin{aligned}\sum_{l=0}^{i} \rho_{li} \frac{\Gamma(l+1)\Gamma(1-\kappa)}{\Gamma(l-\kappa+2)} x^{l-\kappa+1} &\approx \sum_{j=0}^{N} \left\{ \sum_{l=0}^{i} \frac{\rho_{li}\Gamma(l+1)\Gamma(1-\kappa)\tilde{C}_{j(l-\kappa+1)}}{\Gamma(l-\kappa+2)} \right\} S_j^*(x) \\ &= \sum_{j=0}^{N} \sigma_{ij} S_j^*(x), \qquad i = 0, 1, \dots, N.\end{aligned}$$

Eq. (16) is obtained as follows

$$\int_0^x \frac{\mathfrak{S}^T(z)}{(z-x)^{\kappa}}\, dz \approx \begin{bmatrix} \sigma_{00} & \sigma_{01} & \dots & \sigma_{0N} \\ \sigma_{10} & \sigma_{11} & \dots & \sigma_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N0} & \tilde{a}_{N1} & \dots & \sigma_{NN} \end{bmatrix} \begin{bmatrix} S_0^*(x) \\ S_1^*(x) \\ \vdots \\ S_N^*(x) \end{bmatrix} = \mathfrak{I}^{(\kappa)} \mathfrak{S}(x).$$

□

4. Solution method

In this section, we apply the operational Chebyshev method to solve a class of the integro-differential equations with the weakly singular kernel. For this aim, we consider Eqs. (1), (2), (3) under the condition $u(0) = 0$. Where $0 < \eta < 1$, $q(x)$, $\mathcal{H}(x,z)$ and $\bar{\mathcal{H}}(x,z)$ are continuous functions on intervals $[0,1]$ and $[0,1] \times [0,1]$, respectively.

Now, we approximate these equations using matrices introduced in the previous section. We determine the highest order of derivatives in Eqs. (1)-(3) and we approximate the function $\mathcal{D}^{\eta} u$ in a matrix form.

$$\mathcal{D}^{\eta} u(x) = \mathcal{D}^{\eta-\vartheta} \mathcal{D}^{\vartheta} u(x) \approx \mathfrak{S}^T(x) F. \tag{17}$$

From Theorem 3.2, we obtain

$$u(x) \approx \mathfrak{S}^T(x) \mathscr{P}^{(\eta)T} F = \mathfrak{S}^T(x) \mathbb{F}_1, \qquad \mathbb{F}_1 = \mathscr{P}^{(\eta)T} F, \tag{18}$$

$$\mathcal{D}^{\vartheta} u(x) \approx \mathfrak{S}^T(x) \mathscr{P}^{(\eta-\vartheta)T} F = \mathfrak{S}^T(x) \mathbb{F}_2, \qquad \mathbb{F}_2 = \mathscr{P}^{(\eta-\vartheta)T} F, \tag{19}$$

using Eq. (14), we have

$$\mathfrak{S}(x) \mathfrak{S}^T(x) \mathbb{F}_1 = \tilde{\mathbb{F}}_1 \mathfrak{S}(x), \tag{20}$$

Then, the nonlinear function is approximated as follows

$$u^2(x) \approx (\mathfrak{S}^T(x)\mathbb{F}_1)^T (\mathfrak{S}^T(x)\mathbb{F}_1) = \mathbb{F}_1^T \mathfrak{S}(x) \mathfrak{S}^T(x) \mathbb{F}_1 \approx \mathbb{F}_1^T \tilde{\mathbb{F}}_1 \mathfrak{S}(x) = \mathfrak{S}^T(x) \mathbb{U}, \qquad \mathbb{U} = \tilde{\mathbb{F}}_1^{\,T} \mathbb{F}_1. \tag{21}$$

$\tilde{\mathbb{F}}_1$ is the operational matrix of the product corresponding to the vector $\mathbb{F}_1$, $\mathscr{P}^{(\eta)}$, $\mathscr{P}^{(\eta-\vartheta)}$ are the operational matrix of integration of the order η, $\eta - \vartheta$, respectively. Using Eq. (10), we approximate the kernels $\mathcal{H}(x,z)$ and $\bar{\mathcal{H}}(x,z)$ by

$$\mathcal{H}(x,z) \approx \mathfrak{S}^T(x)\mathcal{H}\mathfrak{S}(z), \qquad \bar{\mathcal{H}}(x,z) \approx \mathfrak{S}^T(x)\bar{\mathcal{H}}\mathfrak{S}(z). \tag{22}$$

From Eqs. (20), (22) and Theorem 3.8 we have

$$\begin{aligned}\int_0^x \frac{\mathcal{H}(x,z)u(z)dz}{(x-z)^\kappa} &\approx \int_0^x \frac{\mathfrak{S}^T(x)\mathcal{H}\mathfrak{S}(z)\mathfrak{S}^T(z)\mathbb{F}_1}{(x-z)^\kappa}\,dz\\ &= \mathfrak{S}^T(x)\mathcal{H}\int_0^x \frac{\mathfrak{S}(z)\mathfrak{S}^T(z)\mathbb{F}_1}{(x-z)^\kappa}\,dz\\ &\approx \mathfrak{S}^T(x)\mathcal{H}\tilde{\mathbb{F}}_1\int_0^x \frac{\mathfrak{S}(z)}{(x-z)^\kappa}\,dz\\ &\approx \mathfrak{S}^T(x)\mathcal{H}\tilde{\mathbb{F}}_1\mathfrak{J}^{(\kappa)}\mathfrak{S}(x).\end{aligned}$$

Similarly, by substituting Eq. (19) into integral $\int_0^x \frac{\mathcal{H}(x,z)\mathcal{D}^\vartheta u(z)}{(x-z)^\kappa}\,dz$, we get

$$\int_0^x \frac{\mathcal{H}(x,z)\mathcal{D}^\vartheta u(z)}{(x-z)^\kappa}\,dz \approx \mathfrak{S}^T(x)\mathcal{H}\tilde{\mathbb{F}}_2\mathfrak{J}^{(\kappa)}\mathfrak{S}(x).$$

$\tilde{\mathbb{F}}_2$ is the operational matrix of the product corresponding to the vector $\mathbb{F}_2$ and matrix $\mathfrak{J}^{(\kappa)}$ is introduced by Theorem 3.8. From Eqs. (18) and (21), we approximate the second integral in Eqs. (1)-(3) as follows

$$\begin{aligned}\int_0^1 \bar{\mathcal{H}}(x,z)u(z)\,dz &\approx \int_0^1 \mathfrak{S}^T(x)\bar{\mathcal{H}}\mathfrak{S}(z)\mathfrak{S}^T(z)\mathbb{F}_1\,dz\\ &= \mathfrak{S}^T(x)\bar{\mathcal{H}}\int_0^1 \mathfrak{S}(z)\mathfrak{S}^T(z)\,dz\mathbb{F}_1\\ &= \mathfrak{S}^T(x)\bar{\mathcal{H}}Q\mathbb{F}_1.\end{aligned}$$

Where matrix Q is computed by Remark 3.6. Also, by substituting Eq. (21) into the integral in Eq. (3), we have

$$\begin{aligned}\int_0^1 \bar{\mathcal{H}}(x,z)u^2(z)\,dz &\approx \int_0^1 \mathfrak{S}^T(x)\bar{\mathcal{H}}\mathfrak{S}(z)\mathfrak{S}^T(z)\mathbb{U}\,dz\\ &= \mathfrak{S}^T(x)\bar{\mathcal{H}}\int_0^1 \mathfrak{S}(z)\mathfrak{S}^T(z)\,dz\mathbb{U}\\ &= \mathfrak{S}^T(x)\bar{\mathcal{H}}Q\mathbb{U}.\end{aligned}$$

By substituting the computed matrices and approximations into Eqs. (1)-(3), we get

$$\mathfrak{S}^T(x)F - q(x)\mathfrak{S}^T(x)\mathbb{F}_1 - \theta_1\mathfrak{S}^T(x)\mathcal{H}\tilde{\mathbb{F}}_1\mathfrak{J}^{(\kappa)}\mathfrak{S}(x) - \theta_2\mathfrak{S}^T(x)\bar{\mathcal{H}}Q\mathbb{F}_1 - g(x) \approx 0,$$

$$\mathfrak{S}^T(x)F - \mathfrak{S}^T(x)\mathbb{F}_1 - \theta_1\mathfrak{S}^T(x)\mathcal{H}\tilde{\mathbb{F}}_2\mathfrak{J}^{(\kappa)}\mathfrak{S}(x) - \theta_2\mathfrak{S}^T(x)\bar{\mathcal{H}}Q\mathbb{F}_1 - g(x) \approx 0,$$

$$\mathfrak{S}^T(x)F - \mathfrak{S}^T(x)\mathbb{U} - \theta_1 \mathfrak{S}^T(x)\mathcal{H}\tilde{\mathbb{F}}_2\mathfrak{I}^{(\kappa)}\mathfrak{S}(x) - \theta_2 \mathfrak{S}^T(x)\bar{\mathcal{H}}\mathcal{Q}\mathbb{U} - g(x) \approx 0.$$

The resultant algebraic equations are collocated at $N+1$ roots of the $(N+1)th$ shifted sixth-kind Chebyshev polynomials. By evaluating equations at collocation points, we obtain linear and nonlinear system of algebraic equations. We can solve a nonlinear system by the Newton iterative method. By solving the resultant algebraic system, we can obtain an approximation for vector F, then we can get the approximate solutions of Eqs. (1)-(3) using Eq. (18).

5. Error analysis

In this section, we present some theorems, then we get an upper error bound for approximation errors. For this aim, we consider Eq. (1) and theorems for Eqs. (2) and (3) can be proved similarly.

Theorem 5.1. *Let $\Upsilon(x) \in C[0,1]$ and $\Upsilon_N(x) = \sum_{i=0}^{N} \mathcal{E}_i S_i^*(x)$, be SSKCPs approximate function to $\Upsilon(x)$ on the interval* $[0,1]$. *Then, the coefficients $\mathcal{E}_i$, for $i = 0,1,\ldots,N$ are bounded as follows*

$$|\mathcal{E}_i| \leq \frac{\mathcal{M}_\Upsilon}{\lambda_i} \sum_{m=0}^{i} \rho_{mi} \frac{\sqrt{\pi}\Gamma(m+\frac{3}{2})}{2\Gamma(m+5)}(m^2+m+3), \tag{23}$$

where $\mathcal{M}_\Upsilon$ indicates the maximum value of $\Upsilon(x)$ on the interval $[0,1]$.

Proof. Using Eqs. (5) and (7) for $i = 0,1,\ldots,N.$ we have

$$\begin{aligned} \mathcal{E}_i &= \frac{1}{\lambda_i}\int_0^1 \Upsilon(x) S_i^*(x) V(x)\,dx = \frac{1}{\lambda_i}\int_0^1 \Upsilon(x)\sum_{m=0}^{i}\rho_{mi}x^m V(x)\,dx \\ &= \frac{1}{\lambda_i}\sum_{m=0}^{i}\rho_{mi}\int_0^1 \Upsilon(x)x^m V(x)\,dx. \end{aligned} \tag{24}$$

Since $\Upsilon(x)$ is a continuous function on the interval $[0,1]$, so it is bounded and there is a constant $\mathcal{M}_\Upsilon$ such that

$$|\Upsilon(x)| \leq \mathcal{M}_\Upsilon. \quad \forall x \in [0,1]. \tag{25}$$

Using Eqs. (24) and (25), inequality (23) is deduced. □

Theorem 5.2. *Suppose that $\Upsilon(x)$ is a continuous function and $\Upsilon_N(x)$ is an approximation to $\Upsilon(x)$ in terms of SSKCPs. Then, a bound for the approximation error can be achieved as follows*

$$\|\Upsilon(x) - \Upsilon_N(x)\|_2 \leq \Big(\sum_{i=N+1}^{\infty} \Omega_i\Big)^{\frac{1}{2}} = \boldsymbol{\Omega}_N,$$

where $\|.\|_2$ denotes L^2-norm and

$$\Omega_i = \frac{\mathcal{M}_\Upsilon^2}{\lambda_i}\sum_{m=0}^{i}\Big(\rho_{mi}\frac{\sqrt{\pi}\Gamma(m+\frac{3}{2})}{2\Gamma(m+5)}(m^2+m+3)\Big)^2.$$

Proof. Assume that $\Upsilon(x)$ is an arbitrary function. So, $\Upsilon(x)$ and $\Upsilon_N(x)$ using SSKCPs have following forms

$$\Upsilon(x) = \sum_{i=0}^{\infty} \mathcal{E}_i S_i^*(x), \qquad \Upsilon_N(x) = \sum_{i=0}^{N} \mathcal{E}_i S_i^*(x),$$

so,

$$\Upsilon(x) - \Upsilon_N(x) = \sum_{i=N+1}^{\infty} \mathcal{E}_i S_i^*(x). \tag{26}$$

Using Eqs. (7), (26) and Theorem 5.1, we have

$$\begin{aligned} \|\Upsilon(x) - \Upsilon_N(x)\|_2^2 &= \int_0^1 |\Upsilon(x) - \Upsilon_N(x)|^2 V(x)\,dx = \int_0^1 \Big(\sum_{i=N+1}^{\infty} \mathcal{E}_i S_i^*(x) \Big)^2 V(x)\,dx \\ &= \int_0^1 \sum_{j=N+1}^{\infty} \sum_{i=N+1}^{\infty} \mathcal{E}_i \mathcal{E}_j S_i^*(x) S_j^*(x) V(x)\,dx = \sum_{i=N+1}^{\infty} \mathcal{E}_i^2 \lambda_i \le \sum_{i=N+1}^{\infty} \Omega_i. \end{aligned}$$

□

Remark 5.3. *In the proof of Theorem 5.2, If m (or N) is sufficiently large, we have from Stirling formula*

$$\frac{\Gamma(m + \frac{3}{2})}{\Gamma(m + 5)} \le c^* m^{\frac{3}{2} - 5} = c^* m^{\frac{-7}{2}},$$

where c^* is a positive constant.

Theorem 5.4. *If $\mathcal{H}(x, y)$ is any continuous two-variable function and approximated on the interval $[0, 1] \times [0, 1]$ by SSKCPs as $\mathcal{H}_N(x, y) = \sum_{i=0}^{N} \sum_{j=0}^{N} \mathcal{H}_{ij} S_i^*(x) S_j^*(y)$, then coefficients $\mathcal{H}_{ij}$ can be bounded as follows*

$$|\mathcal{H}_{ij}| \le \frac{\mathcal{M}_{\mathcal{H}} \pi}{4 \lambda_i \lambda_j} \sum_{m=0}^{i} \frac{\rho_{mi} \Gamma(m + \frac{3}{2})}{\Gamma(m + 5)} (m^2 + m + 3) \sum_{r=0}^{j} \frac{\rho_{rj} \Gamma(r + \frac{3}{2})}{\Gamma(r + 5)} (r^2 + r + 3), \qquad i, j = 0, 1, \dots, N$$

where $\mathcal{M}_{\mathcal{H}}$ denotes the maximum value of $\mathcal{H}(x, y)$ on the interval $[0, 1] \times [0, 1]$.

Proof. Using Eq. (5) and (11), we have

$$\begin{aligned} \mathcal{H}_{ij} &= \frac{1}{\lambda_i \lambda_j} \int_0^1 \int_0^1 \mathcal{H}(x, y) S_i^*(x) S_j^*(y) V(x) V(y)\,dxdy \\ &= \frac{1}{\lambda_i \lambda_j} \int_0^1 \sum_{m=0}^{i} \rho_{mi} x^m V(x) \Big(\int_0^1 \mathcal{H}(x, y) \sum_{r=0}^{j} \rho_{rj} y^r V(y) dy \Big)\,dx \\ &= \frac{1}{\lambda_i \lambda_j} \sum_{m=0}^{i} \rho_{mi} \sum_{r=0}^{j} \rho_{rj} \int_0^1 \int_0^1 x^m \mathcal{H}(x, y) y^r V(x) V(y)\,dxdy. \end{aligned} \tag{27}$$

Where λ_i is the normalization factor introduced in Eq. (8). Since $\mathcal{H}(x, y)$ is a continuous function on the interval $[0, 1] \times [0, 1]$ and it is bounded. So, there is a constant $\mathcal{M}_{\mathcal{H}}$ such that

$$|\mathcal{H}(x, y)| \le \mathcal{M}_{\mathcal{H}}, \quad \forall (x, y) \in [0, 1] \times [0, 1]. \tag{28}$$

Using Eqs. (27) and (28), Theorem 5.4 is proved. □

Theorem 5.5. *Assume that $\mathcal{H}(x,y)$ is a continuous two-variable function such that $\mathcal{H}_N(x,y)$ is the SSKCPs approximate function to $\mathcal{H}(x,y)$. Then the error bound can be obtained as follows*

$$\|\mathcal{H}(x,y)(x) - \mathcal{H}_N(x,y)\|_2 \leq \Big(\sum_{i=0}^{N}\sum_{j=N+1}^{\infty} \varsigma_{ij}^2 \lambda_i \lambda_j\Big)^{\frac{1}{2}} + \Big(\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty} \varsigma_{ij}^2 \lambda_i \lambda_j\Big)^{\frac{1}{2}} = \Lambda_{\mathcal{H}},$$

$$\varsigma_{ij} = \frac{\mathcal{M}_{\mathcal{H}}\pi}{4\lambda_i\lambda_j}\sum_{m=0}^{i}\rho_{mi}\frac{\Gamma(m+\frac{3}{2})}{\Gamma(m+5)}(m^2+m+3)\sum_{r=0}^{j}\rho_{rj}\frac{\Gamma(r+\frac{3}{2})}{\Gamma(r+5)}(r^2+r+3).$$

Proof. Suppose that $\mathcal{H}(x,y)$ is an arbitrary function. SSKCPs series of $\mathcal{H}(x,y)$ and its approximation in terms of SSKCPs have the following form

$$\mathcal{H}(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathcal{H}_{ij}S_i^*(x)S_j^*(y), \qquad \mathcal{H}_N(x,y) = \sum_{i=0}^{N}\sum_{j=0}^{N}\mathcal{H}_{ij}S_i^*(x)S_j^*(y),$$

thus

$$\mathcal{H}(x,y) - \mathcal{H}_N(x,y) = \sum_{i=0}^{N}\sum_{j=N+1}^{\infty}\mathcal{H}_{ij}S_i^*(x)S_j^*(y) + \sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty}\mathcal{H}_{ij}S_i^*(x)S_j^*(y), \tag{29}$$

using Eqs. (7), (29) and Theorem 5.4, we conclude that

$$\begin{aligned}
\|\mathcal{H}(x,y)(x) - \mathcal{H}_N(x,y)\|_2 \leq & \Big\|\sum_{i=0}^{N}\sum_{j=N+1}^{\infty}\mathcal{H}_{ij}S_i^*(x)S_j^*(y)\Big\|_2 + \Big\|\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty}\mathcal{H}_{ij}S_i^*(x)S_j^*(y)\Big\|_2 \\
&= \Big(\int_0^1\int_0^1\Big(\sum_{i=0}^{N}\sum_{j=N+1}^{\infty}\mathcal{H}_{ij}S_i^*(x)S_j^*(y)\Big)^2 V(x)V(y)\,dydx\Big)^{\frac{1}{2}} \\
&+ \Big(\int_0^1\int_0^1\Big(\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty}\mathcal{H}_{ij}S_i^*(x)S_j^*(y)\Big)^2 V(x)V(y)\,dydx\Big)^{\frac{1}{2}} \\
&= \Big(\sum_{i=0}^{N}\sum_{j=N+1}^{\infty}\mathcal{H}_{ij}^2\lambda_i\lambda_j\Big)^{\frac{1}{2}} + \Big(\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty}\mathcal{H}_{ij}^2\lambda_i\lambda_j\Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_{i=0}^{N}\sum_{j=N+1}^{\infty}\varsigma_{ij}^2\lambda_i\lambda_j\Big)^{\frac{1}{2}} + \Big(\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty}\varsigma_{ij}^2\lambda_i\lambda_j\Big)^{\frac{1}{2}}.
\end{aligned}$$

□

In Theorem 5.6, we will achieve an upper error bound of the presented method. Assume that the following assumptions are satisfied

1. $\|\mathcal{H}(x,y)\|_2 \leq \mathcal{M}_{\mathcal{H}}, \qquad \|\bar{\mathcal{H}}(x,y)\|_2 \leq \mathcal{M}_{\bar{\mathcal{H}}},$
2. $\|q(x)u(x) - q(x)u_N(x)\|_2 \leq \beta\|u(x) - u_N(x)\|_2, \qquad \|q(x)\|_\infty \leq \beta,$
3. $\|u_N(x)\|_2 \leq \Big(\sum_{i=0}^{N}\frac{\mathcal{M}_{u_N}^2}{\lambda_i}\Big(\sum_{l=0}^{i}\rho_{li}\frac{\sqrt{\pi}\Gamma(l+\frac{3}{2})}{2\Gamma(l+5)}(l^2+l+3)\Big)^2\Big)^{\frac{1}{2}} = \mathbf{\Theta}_N.$

Theorem 5.6. *Suppose that $u(x)$ and $u_N(x)$ are the exact and approximate solution respectively of Eq.* (1). *Moreover, assume that Hypotheses 1-3 are satisfied and also $\frac{\beta}{\Gamma(\eta+1)} + \frac{|\theta_1|\Gamma(1-\kappa)}{\Gamma(\eta-\kappa+2)}\mathcal{M}_{\mathcal{H}} + \frac{3|\theta_2|}{\Gamma(\eta+2)}\mathcal{M}_{\bar{\mathcal{H}}} < 1$, then an error bound for the method can be achieved as follows*

$$\|u(x) - u_N(x)\|_2 \leq \frac{\boldsymbol{\Delta}_N + \left(\frac{|\theta_1|\Gamma(1-\kappa)}{\Gamma(\eta-\kappa+2)}\Lambda_{\mathcal{H}} + \frac{3|\theta_2|}{\Gamma(\eta+2)}\Lambda_{\bar{\mathcal{H}}}\right)\boldsymbol{\Theta}_N}{1 - \left(\frac{\beta}{\Gamma(\eta+1)} + \frac{|\theta_1|\Gamma(1-\kappa)}{\Gamma(\eta-\kappa+2)}\mathcal{M}_{\mathcal{H}} + \frac{3|\theta_2|}{\Gamma(\eta+2)}\mathcal{M}_{\bar{\mathcal{H}}}\right)}.$$

Proof. We apply the Riemann-Liouville integral operator on Eq. (1) and obtain the following equation

$$\begin{aligned} u(x) = G(x) &+ \frac{1}{\Gamma(\eta)}\int_0^x (x-z)^{\eta-1} q(z)u(z)\,dz + \frac{\theta_1\Gamma(1-\kappa)}{\Gamma(\eta-\kappa+1)}\int_0^x (x-z)^{\eta-\kappa}\mathcal{H}(x,z)u(z)\,dz \\ &+ \frac{\theta_2}{\Gamma(\eta)}\int_0^1\int_0^x (x-\tilde{z})^{\eta-1}\bar{\mathcal{H}}(\tilde{z},z)u(z)\,d\tilde{z}dz, \end{aligned} \tag{30}$$

where $G(x) = \frac{1}{\Gamma(\eta)}\int_0^x (x-z)^{\eta-1}g(z)dz + \sum_{j=0}^{m-1}\frac{u_0^j x^j}{\Gamma(j+1)}$. We can write the approximate equation of Eq. (30) as follows

$$\begin{aligned} u_N(x) = G(x) &+ \frac{1}{\Gamma(\eta)}\int_0^x (x-z)^{\eta-1} q(z)u_N(z)\,dz + \frac{\theta_1\Gamma(1-\kappa)}{\Gamma(\eta-\kappa+1)}\int_0^x (x-z)^{\eta-\kappa}\mathcal{H}_N(x,z)u_N(z)\,dz \\ &+ \frac{\theta_2}{\Gamma(\eta)}\int_0^1\int_0^x (x-\tilde{z})^{\eta-1}\bar{\mathcal{H}}_N(\tilde{z},z)u_N(z)\,d\tilde{z}dz + H_N(x), \end{aligned} \tag{31}$$

where $H_N(x)$ is the perturbation term. We subtract Eq. (31) from Eq. (30) and obtain the following result

$$\begin{aligned} u(x) - u_N(x) = -H_N(x) &+ \frac{1}{\Gamma(\eta)}\int_0^x (x-z)^{\eta-1} q(z)(u(z) - u_N(z))\,dz \\ &+ \frac{\theta_1\Gamma(1-\kappa)}{\Gamma(\eta-\kappa+1)}\int_0^x (x-z)^{\eta-\kappa}(\mathcal{H}(x,z)u(z) - \mathcal{H}_N(x,z)u_N(z))\,dz \\ &+ \frac{\theta_2}{\Gamma(\eta)}\int_0^1\int_0^x (x-\tilde{z})^{\eta-1}(\bar{\mathcal{H}}(\tilde{z},z)u(z) - \bar{\mathcal{H}}_N(\tilde{z},z)u_N(z))\,d\tilde{z}dz. \end{aligned} \tag{32}$$

First, we obtain a bound for the perturbation term.

$$\begin{aligned} \|H_N(x)\|_2 \leq \|u(x) - u_N(x)\|_2 &+ \frac{1}{\Gamma(\eta)}\int_0^x |x-z|^{\eta-1}\|q(z)\|_\infty\|u(z) - u_N(z)\|_2\,dz \\ &+ \frac{|\theta_1|\Gamma(1-\kappa)}{\Gamma(\eta-\kappa+1)}\int_0^x |x-z|^{\eta-\kappa}\|\mathcal{H}(x,z)u(z) - \mathcal{H}_N(x,z)u_N(z)\|_2\,dz \\ &+ \frac{|\theta_2|}{\Gamma(\eta)}\int_0^1\int_0^x |x-\tilde{z}|^{\eta-1}\|\bar{\mathcal{H}}(\tilde{z},z)u(z) - \bar{\mathcal{H}}_N(\tilde{z},z)u_N(z)\|_2\,d\tilde{z}dz. \end{aligned}$$

Using Hypothesis 2 and Theorem 5.2, we have

$$\frac{1}{\Gamma(\eta)}\int_0^x |x-z|^{\eta-1}\|q(z)\|_\infty\|u(z) - u_N(z)\|_2\,dz \leq \frac{\beta}{\Gamma(\eta+1)}\|u(x) - u_N(x)\|_2 \leq \frac{\beta}{\Gamma(\eta+1)}\boldsymbol{\Omega}_N. \tag{33}$$

From Hypotheses 1 and 3 and Theorems 5.2 and 5.5, the following inequalities are obtained

$$\begin{aligned}\frac{|\theta_1|\Gamma(1-\kappa)}{\Gamma(\eta-\kappa+1)}&\int_0^x |x-z|^{\eta-\kappa}\|\mathcal{H}(x,z)(u(z)-u_N(z))+u_N(z)\big(\mathcal{H}(x,z)-\mathcal{H}_N(x,z)\big)\|_2\,dz\\ &\leq \frac{|\theta_1|\Gamma(1-\kappa)}{\Gamma(\eta-\kappa+2)}\mathcal{M}_{\mathcal{H}}\|u(x)-u_N(x)\|_2+\frac{|\theta_1|\Gamma(1-\eta)}{\Gamma(\eta-\kappa+2)}\Lambda_{\mathcal{H}}\|u_N(x)\|_2\\ &\leq \frac{|\theta_1|\Gamma(1-\kappa)}{\Gamma(\eta-\kappa+2)}\mathcal{M}_{\mathcal{H}}\boldsymbol{\Omega}_N+\frac{|\theta_1|\Gamma(1-\eta)}{\Gamma(\eta-\kappa+2)}\Lambda_{\mathcal{H}}\boldsymbol{\Theta}_N,\end{aligned} \tag{34}$$

$$\begin{aligned}\frac{|\theta_2|}{\Gamma(\eta)}&\int_0^1\int_0^x |x-\tilde{z}|^{\eta-1}\|\bar{\mathcal{H}}(\tilde{z},z)(u(z)-u_N(z))+u_N(z)(\bar{\mathcal{H}}(\tilde{z},z)-\bar{\mathcal{H}}_N(\tilde{z},z))\|_2\,d\tilde{z}dz\\ &\leq \frac{3|\theta_2|}{\Gamma(\eta+2)}\mathcal{M}_{\bar{\mathcal{H}}}\|u(x)-u_N(x)\|_2+\frac{3|\theta_2|\Lambda_{\bar{\mathcal{H}}}}{\Gamma(\eta+2)}\|u_N(x)\|_2\\ &\leq \frac{3|\theta_2|}{\Gamma(\eta+2)}\mathcal{M}_{\bar{\mathcal{H}}}\boldsymbol{\Omega}_N+\frac{3|\theta_2|\Lambda_{\bar{\mathcal{H}}}}{\Gamma(\eta+2)}\boldsymbol{\Theta}_N.\end{aligned} \tag{35}$$

From Theorem 5.2 and Eqs. (33)-(35), we can get the following upper bound for $H_N(x)$

$$\begin{aligned}\|H_N(x)\|_2 &\leq \Big(1+\frac{\beta}{\Gamma(\eta+1)}+\frac{|\theta_1|\Gamma(1-\kappa)}{\Gamma(\eta-\kappa+2)}\mathcal{M}_{\mathcal{H}}+\frac{3|\theta_2|}{\Gamma(\eta+2)}\mathcal{M}_{\bar{\mathcal{H}}}\Big)\boldsymbol{\Omega}_N\\ &\quad+\Big(\frac{|\theta_1|\Gamma(1-\eta)}{\Gamma(\eta-\kappa+2)}\Lambda_{\mathcal{H}}+\frac{3|\theta_2|\Lambda_{\bar{\mathcal{H}}}}{\Gamma(\eta+2)}\Big)\boldsymbol{\Theta}_N\\ &=\boldsymbol{\Delta}_N.\end{aligned}$$

Again, we consider Eq. (32). So, we have

$$\begin{aligned}\|u(x)-u_N(x)\|_2 &\leq \|H_N(x)\|_2+\frac{1}{\Gamma(\eta)}\int_0^x |x-z|^{\eta-1}\|q(z)\|_\infty\|u(z)-u_N(z)\|_2\,dz\\ &\quad+\frac{|\theta_1|\Gamma(1-\kappa)}{\Gamma(\eta-\kappa+1)}\int_0^x |x-z|^{\eta-\kappa}\|\mathcal{H}(x,z)u(z)-\mathcal{H}_N(x,z)u_N(z)\|_2\,dz\\ &\quad+\frac{|\theta_2|}{\Gamma(\eta)}\int_0^1\int_0^x |x-\tilde{z}|^{\eta-1}\|\bar{\mathcal{H}}(\tilde{z},z)u(z)-\bar{\mathcal{H}}_N(\tilde{z},z)u_N(z)\|_2\,d\tilde{z}dz\\ &\leq \|H_N(x)\|_2+\frac{\beta}{\Gamma(\eta+1)}\|u(x)-u_N(x)\|_2\\ &\quad+\frac{|\theta_1|\Gamma(1-\kappa)}{\Gamma(\eta-\kappa+2)}\mathcal{M}_{\mathcal{H}}\|u(x)-u_N(x)\|_2+\frac{|\theta_1|\Gamma(1-\eta)}{\Gamma(\eta-\kappa+2)}\Lambda_{\mathcal{H}}\|u_N(x)\|_2\\ &\quad+\frac{3|\theta_2|}{\Gamma(\eta+2)}\mathcal{M}_{\bar{\mathcal{H}}}\|u(x)-u_N(x)\|_2+\frac{3|\theta_2|\Lambda_{\bar{\mathcal{H}}}}{\Gamma(\eta+2)}\|u_N(x)\|_2.\end{aligned}$$

In a similar way, we obtain the following upper bound for our method error

$$\|u(x)-u_N(x)\|_2 \leq \frac{\boldsymbol{\Delta}_N+\Big(\frac{|\theta_1|\Gamma(1-\kappa)}{\Gamma(\eta-\kappa+2)}\Lambda_{\mathcal{H}}+\frac{3|\theta_2|}{\Gamma(\eta+2)}\Lambda_{\bar{\mathcal{H}}}\Big)\boldsymbol{\Theta}_N}{1-\Big(\frac{\beta}{\Gamma(\eta+1)}+\frac{|\theta_1|\Gamma(1-\kappa)}{\Gamma(\eta-\kappa+2)}\mathcal{M}_{\mathcal{H}}+\frac{3|\theta_2|}{\Gamma(\eta+2)}\mathcal{M}_{\bar{\mathcal{H}}}\Big)}.$$

□

Remark 5.7. *Consider the linearity property of Caputo derivative and Lipschitz condition as follows*

$$\begin{aligned} &\mathcal{D}^{\vartheta}u(x) - \mathcal{D}^{\vartheta}u_N(x) = \mathcal{D}^{\vartheta}(u(x) - u_N(x)),\\ &\|\mathcal{D}^{\vartheta}(u(x) - u_N(x))\|_2 \leq \gamma\|u(x) - u_N(x)\|_2 \end{aligned} \tag{36}$$

and

$$\|\mathcal{D}^{\vartheta}u_N(x)\|_2 \leq \Psi_N. \tag{37}$$

Also, we can rewrite the non-linear expression as

$$u^2(x) - u_N^2(x) = (u(x) - u_N(x))(u(x) - u_N(x) + 2u_N(x)). \tag{38}$$

Now, similar to the linear case to obtain an error estimate in the non-linear case, first, we form the residual function and using Eqs. (36)–(38) *and applying norm-L^2 to both sides of the residual function, we have*

$$\begin{aligned} \|u(x) - u_N(x)\|_2 \leq \|H_N(x)\|_2 &+ \frac{1}{\Gamma(\eta+1)}\|u(x) - u_N(x)\|_2\Big(\|u(x) - u_N(x)\|_2 + 2\|u_N(x)\|_2\Big)\\ &+ \frac{|\theta_1|\gamma\Gamma(1-\kappa)\mathcal{M}_{\mathcal{H}}}{\Gamma(\eta-\kappa+2)}\|u(x) - u_N(x)\|_2 + \Psi_N\Lambda_{\mathcal{H}} + \frac{3|\theta_2|\mathcal{M}_{\bar{\mathcal{H}}}}{\Gamma(\eta+2)}\|u(x) - u_N(x)\|_2\\ &+ \frac{3|\theta_2|\Lambda_{\bar{\mathcal{H}}}}{\Gamma(\eta+2)}\|u_N(x)\|_2 \end{aligned} \tag{39}$$

Substituting the attained upper bounds into the right hand-side of Eq. (39)*, for sufficiently large N, the error bound can be obtained sufficiently small.*

6. Convergence analysis

In this section, we consider the convergence of the presented method in the Chebyshev-weighted sobolev space. A different method for discrete problems can be found in [41]. We will demonstrate convergence for Eq. (1), and the proofs for Eq. (2) and Eq. (3) are similar.

Definition 6.1. *The Chebyshev-weighted sobolev space is indicate as*

$$CH_w^m(I) = \left\{z(x)\middle|\frac{d^kz}{dx^k} \in L_w^2(I), 0 \leq k \leq m, m \in \mathbb{N}\right\},$$

equipped with the following norm and semi-norm and inner product

$$\|z\|_{CH_w^m} = \Big(\sum_{k=0}^{m}\left\|\frac{d^kz}{dx^k}\right\|_w^2\Big)^{\frac{1}{2}}, \quad |z|_{CH_w^m} = \left\|\frac{d^mz}{dx^m}\right\|_w, \quad < z, y >_{CH_w^m} = \sum_{k=0}^{m}\left\langle\frac{d^kz}{dx^k}, \frac{d^ky}{dx^k}\right\rangle_w.$$

Theorem 6.2. *Suppose that $0 \leq m \leq N$ and $z(x) \in CH_w^m(I)$. Also, let $u_N(x)$ be the sixth-kind Chebyshev approximation to the exact solution $z(x)$. Then, we can obtain an error bound of the approximate solution as follows*

$$\|z - z_N\|_{L_w^2} \leq cN^{\frac{-9}{4}m}|z|_{CH_w^m},$$

where c is a positive constant independent of N.

Proof. See[37]. □

Definition 6.3. *The two-dimensional Chebyshev-weighted sobolev space, $CH_w^m(\Omega), m \in \mathbb{N}$, is introduced as*

$$CH_w^m(\Omega) = \Big\{ Z(x,t) \in L_w^2(\Omega) | \frac{\partial^{i+j} Z(x,t)}{\partial x^i \partial t^j} \in L_w^2(\Omega), 0 \le i+j \le m \Big\}.$$

The norm of this space for $\iota = (\iota_1, \iota_2), \iota_i \in \mathbb{Z}^+, |\iota| = \iota_1 + \iota_2$ is defined by

$$\|Z\|_{CH_w^m}(\Omega) = \Big(\sum_{|\iota| \le m} \|D^\iota Z\|^2_{L_w^2(\Omega)} \Big)^{(\frac{1}{2})} = \Big(\Big\| \frac{\partial^{|\iota|} Z}{\partial x^{\iota_1} \partial t^{\iota_2}} \Big\|^2_{L_w^2(\Omega)} \Big)^{\frac{1}{2}}.$$

Theorem 6.4. *Let $0 \le m < N+1$, $Z(x,t) \in CH_w^m(\Omega)$ and $Z_N(x,t)$ be the Chebyshev approximation of $Z(x,t)$. We can obtain an error bound as follows*

$$\|Z - Z_N\|_{L_w^2} \le \sqrt{3} C_0 N^{\frac{-9}{4}m} \Big\| \frac{\partial^m Z}{\partial x^m} \Big\|_{L_m^2}.$$

where C_0 is a positive constant independent of any function.

Proof. See[37]. □

Theorem 6.5. *Assume that $u_N(x)$ is the approximate solution of Eq. (1) with $q(x) = 1$ obtained from the presented method. If $H_N(x)$ is the perturbation term, then $H_N(x) \longrightarrow 0$ as N is sufficiently large.*

Proof. We first apply the Riemann-Liouville integral operator on Eq. (1), and we achieve the following equation

$$\begin{aligned} u(x) = G(x) &+ \frac{1}{\Gamma(\eta)} \int_0^x (x-z)^{\eta-1} q(z) u(z)\, dz + \frac{\theta_1 \Gamma(1-\kappa)}{\Gamma(\eta-\kappa+1)} \int_0^x (x-z)^{\eta-\kappa} \mathcal{H}(x,z) u(z)\, dz \\ &+ \frac{\theta_2}{\Gamma(\eta)} \int_0^1 \int_0^x (x-\tilde{z})^{\eta-1} \bar{\mathcal{H}}(\tilde{z},z) u(z)\, d\tilde{z} dz, \end{aligned} \tag{40}$$

where $G(x) = \frac{1}{\Gamma(\eta)} \int_0^x (x-z)^{\eta-1} g(z) dz + \sum_{j=0}^{m-1} \frac{u_0^j x^j}{\Gamma(j+1)}$. By substitution $u_N(x)$ (approximate solution of Eq. (1)) into Eq. (40), we have

$$\begin{aligned} u_N(x) = G(x) &+ \frac{1}{\Gamma(\eta)} \int_0^x (x-z)^{\eta-1} q(z) u_N(z)\, dz + \frac{\theta_1 \Gamma(1-\kappa)}{\Gamma(\eta-\kappa+1)} \int_0^x (x-z)^{\eta-\kappa} \mathcal{H}_N(x,z) u_N(z)\, dz \\ &+ \frac{\theta_2}{\Gamma(\eta)} \int_0^1 \int_0^x (x-\tilde{z})^{\eta-1} \bar{\mathcal{H}}_N(\tilde{z},z) u_N(z)\, d\tilde{z} dz + H_N(x), \end{aligned} \tag{41}$$

where $H_N(x)$ is the perturbation term. We subtract Eq. (41) from Eq. (40), and define the error function $e_N = u(x) - u_N(x)$.

$$\begin{aligned} e_N(x) = -H_N(x) &+ \frac{1}{\Gamma(x)} \int_0^x (x-z)^{\eta-1} q(z) e_N(z)\, dz \\ &+ \frac{\theta_1 \Gamma(1-\kappa)}{\Gamma(\eta-\kappa+1)} \int_0^x \Big(\mathcal{H}(x,z) e_N(z) + (\mathcal{H}(x,z) - \mathcal{H}_N(x,z))(u(z) - e_N(z)) \Big) dz \\ &+ \frac{\theta_2}{\Gamma(\eta)} \int_0^1 \int_0^x (x-\tilde{z})^{\eta-1} \Big(\bar{\mathcal{H}}_N(\tilde{z},z) e_N(z) + (\bar{\mathcal{H}}(\tilde{z},z) - \bar{\mathcal{H}}_N(\tilde{z},z))(u(z) - e_N(z)) \Big) d\tilde{z} dz. \end{aligned} \tag{42}$$

Since $q(z)$ is a continuous function on the interval $[0,1]$, therefore it is bounded. ($\|q(x)\|_\infty \le \beta$). We utilize the L^2-norm, Theorem 6.2 and Theorem 6.4, then we have

$$\begin{aligned}\|H_N(x)\|_{L^2_w} \le N^{\frac{-9}{4}m}\|\frac{d^m u}{dx^m}\|_{L^2_w}\Big(c + \frac{c_1\beta}{\Gamma(\eta)} + \frac{c_2\theta_2\Gamma(\kappa)}{\Gamma(\eta-\kappa+1)} + \frac{\theta_2 c_5}{\Gamma(\eta)}\|\bar{\mathcal{H}}\|_{L^2_w}\Big)\\ + \sqrt{3}C_0 N^{\frac{-9}{4}m}\|u\|_{L^2_w}\Big(\frac{\acute{c}_3\theta_1\Gamma(1-\kappa)}{\Gamma(\eta-\kappa+1)}\|\frac{\partial^m\mathcal{H}}{\partial x^m}\|_{L^2_w} + \frac{\theta_2\acute{c}_6}{\Gamma(\eta)}\|\frac{\partial^m\bar{\mathcal{H}}}{\partial x^m}\|_{L^2_w}\Big)\\ + \sqrt{3}C_0 N^{\frac{-9}{2}m}\|\frac{d^m u}{dx^m}\|_{L^2_w}\Big(\frac{c_4\theta_1\Gamma(1-\kappa)}{\Gamma(\eta-\kappa+1)}\|\frac{\partial^m\mathcal{H}}{\partial x^m}\|_{L^2_w} + \frac{\theta_2 c_7}{\Gamma(\eta)}\|\frac{\partial^m\bar{\mathcal{H}}}{\partial x^m}\|_{L^2_w}\Big).\end{aligned}$$

where $c, \acute{c}_i, c_i, 1 \le i \le 7$, are the positive constants. According to the boundedness of $u(x), \mathcal{H}(x,z), \bar{\mathcal{H}}(x,z)$, it can be seen $H_N(x) \longrightarrow 0$ as $N \longrightarrow \infty$. □

Theorem 6.6. *Suppose that $u(x)$ and $u_N(x)$ are the exact and approximate solution of Eq.* (1) *, respectively. If we define $e_N(x) = u(x) - u_N(x)$, then $e_N(x) \longrightarrow 0$ when $N \longrightarrow \infty$.*

Proof. For obtaining the bound for method error, we use Theorem 6.2 and Theorem 6.4, thus we have

$$\|e_N(x)\|_{L^2_w} \le A_1 N^{\frac{-9}{4}m}\|\frac{d^m u}{dx^m}\|_{L^2_w} + \sqrt{3}C_0 A_2 N^{\frac{-9}{4}m}\|u\|_{L^2_w} + \sqrt{3}C_0 A_3 N^{\frac{-9}{2}m}\|\frac{d^m u}{dx^m}\|_{L^2_w},$$

where

$$\begin{aligned}A_1 &= c + \frac{c_1\beta}{\Gamma(\eta)} + \frac{c_2\theta_2\Gamma(1-\kappa)}{\Gamma(\eta-\kappa+1)} + \frac{\theta_2 c_5}{\Gamma(\eta)}\|\bar{\mathcal{H}}\|_{L^2_w},\\ A_2 &= \frac{\acute{c}_3\theta_1\Gamma(1-\kappa)}{\Gamma(\eta-\kappa+1)}\|\frac{\partial^m\mathcal{H}}{\partial x^m}\|_{L^2_w} + \frac{\theta_2\acute{c}_6}{\Gamma(\eta)}\|\frac{\partial^m\bar{\mathcal{H}}}{\partial x^m}\|_{L^2_w},\\ A_3 &= \frac{c_4\theta_1\Gamma(1-\kappa)}{\Gamma(\eta-\kappa+1)}\|\frac{\partial^m\mathcal{H}}{\partial x^m}\|_{L^2_w} + \frac{\theta_2 c_7}{\Gamma(\eta)}\|\frac{\partial^m\bar{\mathcal{H}}}{\partial x^m}\|_{L^2_w}.\end{aligned}$$

where $c, \acute{c}_i, c_i, 1 \le i \le 7$, are the positive constants. When $N \longrightarrow \infty$ then the right-hand side tends to zero. □

Here, we prepare the upper bounds to estimate the errors of the operational matrices in Theorem 3.2 and Theorem 3.8. For this aim, we define the error vectors as follows

$$\begin{aligned}E(x) &= \int_0^x \frac{\mathfrak{S}^T(z)}{(x-z)^\kappa}\,dz - \mathfrak{I}^{(\kappa)}\mathfrak{S}(x) = [E_0(x)E_1(x)\ldots E_N(x)]^T,\\ F(x) &= J^\mu\mathfrak{S}(x) - \mathscr{P}^{(\mu)}\mathfrak{S}(x) = [F_0(x)F_1(x)\ldots F_N(x)]^T.\end{aligned}$$

Where

$$\begin{aligned}E_n(x) &= \int_0^x \frac{S_n^*(z)}{(x-z)^\kappa}\,dz - \mathfrak{I}_n^{(\kappa)}\mathfrak{S}(x),\\ F_n(x) &= J^\mu S_n^*(x) - \mathscr{P}_n^{(\mu)}\mathfrak{S}(x).\end{aligned}$$

For $n = 0, 1, \ldots, N$, $\mathfrak{I}_n^{(\kappa)}$ and $\mathscr{P}_n^{(\mu)}$ are the nth rows of the operational matrices in Theorem 3.2 and Theorem 3.8.

Lemma 6.7. *Consider $E_n(x) = \int_0^x \frac{S_n^*(z)}{(x-z)^\kappa}\,dz - \mathfrak{I}_n^{(\kappa)}\mathfrak{S}(x) \in CH_w^m(I)$. Thus, we can obtain error bound for $E_n(x)$ by*

$$\begin{aligned}\|E_n(x)\|_{L^2_w(I)} \le CN^{\frac{-9}{4}m}\Big(\sum_{l=m}^{N}\sum_{s=m}^{N}\rho_{ln}\rho_{sn}\frac{\Gamma(l+1)\Gamma(l-\kappa)\Gamma(s+1)\Gamma(s-\kappa)}{\Gamma(l-\kappa-m)\Gamma(s-\kappa-m)}\\ \times\Big(4B(l-2\kappa-2m+\frac{11}{3},\frac{3}{2}) - 4B(l-2\kappa-2m+\frac{9}{2},\frac{3}{2}) + B(l-2\kappa-2m+\frac{7}{2},\frac{3}{2})\Big)\Big)^{\frac{1}{2}},\end{aligned}$$

where $B(u,v)$ is the Beta function.

Proof. We set $f_n(x) = \int_0^x \frac{S_n^*(z)}{(x-z)^\kappa}\,dz$. Then, we use definition of the semi-norm in the space $CH_w^m(I)$, we can write

$$|f_n(x)|^2_{CH_w^m(I)} = \|\frac{d^m}{dx}\int_0^x \frac{S_n^*(z)}{(x-z)^\kappa}\,dz\|^2_{L_w^2(I)} = \|\sum_{l=0}^{N}\rho_{ln}\frac{\Gamma(l+1)\Gamma(l-\kappa)}{\Gamma(l-\kappa-m)}x^{l-\kappa-m+1}\|^2_{L_w^2(I)}$$
$$= \sum_{l=m}^{N}\sum_{s=m}^{N}\rho_{ln}\rho_{sn}\frac{\Gamma(l+1)\Gamma(l-\kappa)\Gamma(s+1)\Gamma(s-\kappa)}{\Gamma(l-\kappa-m)\Gamma(s-\kappa-m)}$$
$$\times\int_0^1\Big(4x^{l-2\kappa-2m+\frac{9}{2}}-4x^{l-2\kappa-2m+\frac{7}{2}}+x^{l-2\kappa-2m+\frac{5}{4}}\Big)(1-x)^{\frac{1}{2}}\,dx$$
$$= \sum_{l=m}^{N}\sum_{s=m}^{N}\rho_{ln}\rho_{sn}\frac{\Gamma(l+1)\Gamma(l-k)\Gamma(s+1)\Gamma(s-\kappa)}{\Gamma(l-\kappa-m)\Gamma(s-\kappa-m)}$$
$$\times\Big(4B(l-2\kappa-2m+\frac{11}{3},\frac{3}{2})-4B(l-2\kappa-2m+\frac{9}{2},\frac{3}{2})+B(l-2\kappa-2m+\frac{7}{2},\frac{3}{2})\Big).$$

Using Theorem 6.2, we achieve the desired result. □

Lemma 6.8. *Assume $F_n(x) = J^\mu S_n^*(x) - \mathscr{P}_n^{(\mu)}\mathfrak{S}(x) \in CH^m_{w(I)}$. So, we can obtain an error bound of $F_n(x)$ as follows*

$$\|F_n\|_{L^2_{w(I)}} \le CN^{\frac{-9}{4}m}\Big(\sum_{l=p}^{n}\sum_{s=p}^{n}\rho_{ln}\rho_{sn}\frac{\Gamma(l+1)\Gamma(s+1)}{\Gamma(l-m+\mu+1)\Gamma(s-m+\mu+1)}$$
$$\times\Big(4B(l+s-2m+2\mu+\frac{7}{2},\frac{3}{2})-4B(l+s-2m+2\mu+\frac{5}{2},\frac{3}{2})+B(l+s+2\mu-2m+\frac{3}{2},\frac{3}{2})\Big)\Big)^{\frac{1}{2}}.$$

Where $p = \lceil m-\mu\rceil$.

Proof. We set $f_n(x) = J^\mu S_n^*(x)$, and we use the properties of the Caputo fractional derivative and Riemann-Liouville integral. Thus, we have

$$|f_n(x)|^2_{CH^m_{w(I)}} = \|D^m J^\mu S_n^*(x)\|^2_{L^2_{w(I)}} = \|D^{m-\mu}D^\mu(J^\mu S_n^*(x))\|^2_{L^2_{w(I)}} = \|D^{m-\mu}S_n^*(x)\|^2_{L^2_{w(I)}}$$
$$= \|\sum_{l=p}^{n}\rho_{ln}\frac{\Gamma(l+1)}{\Gamma(l-m+\mu+1)}x^{l-m+\mu}\|^2_{L_w^2(I)}$$
$$= \sum_{l=p}^{n}\sum_{s=p}^{n}\rho_{ln}\rho_{sn}\frac{\Gamma(l+1)\Gamma(s+1)}{\Gamma(l-m+\mu+1)\Gamma(s-m+\mu+1)}$$
$$\times\int_0^1\Big(4x^{l+s+2\mu-2m+\frac{5}{2}}-4x^{l+s+2\mu-2m+\frac{3}{2}}+x^{l+s+2\mu-2m+\frac{1}{2}}\Big)(1-x)^{\frac{1}{2}}\,dx.$$

By using the Beta function definition and Theorem 6.2, we obtain the desired result. □

7. Numerical illustration

In this section, some different fractional integro-differential equations with the weakly singular kernel are solved by applying the suggested method. In order to evaluate the error of this method, we present the absolute error and root-mean-square error

$$e_N(x) = |u(x) - u_N(x)|, \quad \xi_N = \Big(\int_0^1 e_N^2(x)V(x)\,dx\Big)^{\frac{1}{2}},$$

where $u(x)$ is the exact solution and $u_N(x)$ is the numerical solution by the presented method. In addition, to measure the computational complexity of the proposed algorithm, the CPU time is computed for all given examples. All algorithms are performed by Maple 18. In addition, we compute the convergence rate for all examples. Hence, the convergence rate, C_r, for all examples is computed by the following formula:

$$C_r = \frac{\left| ln(E_{i+1}/E_i) \right|}{ln(N_{i+1}/N_i)}$$

where E_i is the maximum absolute error for $N = N_i$. Also, You can see the updated method to solve discrete problems in [42].

Example 7.1. *Consider the following fractional integro-differential equation with weakly singular kernel [21, 22]*

$$\mathcal{D}^{\eta}u(x) = 2x - \frac{16}{15}x^{\frac{1}{2}}u(x) + \int_0^x (x-z)^{-\frac{1}{2}}u(z)\,dz, \qquad 0 \le x \le 1, \tag{43}$$

with the initial condition $u(0) = 0$. The exact solution at $\eta = 1$ is $u(x) = x^2$. We apply the computed approximations in Section 3, and we get the following approximations

$$\mathcal{D}^{\eta}u(x) \approx \mathfrak{S}^T(x)F, \quad u(x) \approx \mathfrak{S}^T(x)\mathscr{P}^{(\eta)T}F = \mathfrak{S}^T(x)U, \quad \int_0^x (x-z)^{-\frac{1}{2}}u(z)\,dz \approx U^T\mathfrak{I}^{(\frac{1}{2})}\mathfrak{S}(x).$$

where $\mathscr{P}^{(\eta)}$ is the operational matrix of integration of the order η, $U = \mathscr{P}^{(\eta)T}F$ and $\mathfrak{I}^{(\frac{1}{2})}$ is the matrix introduced by Theorem 3.8. We substitute these approximations into Eq. (43) *and get the following algebraic equation*

$$\mathfrak{S}^T(x)F - 2x + \frac{16}{15}x^{\frac{1}{2}}\mathfrak{S}^T(x)U - U^T\mathfrak{I}^{(\frac{1}{2})}\mathfrak{S}(x) \approx 0.$$

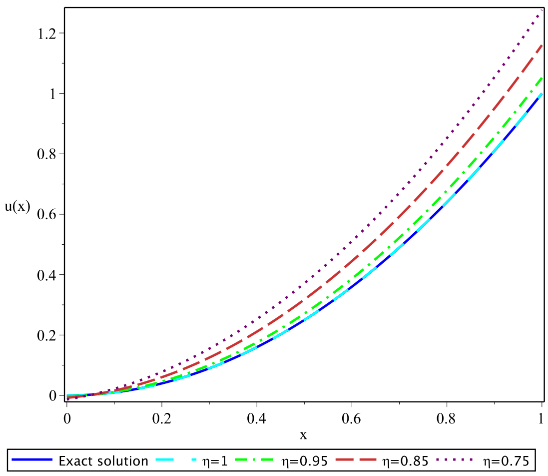

Figure 1: Plot of the function $u(x)$ for $N = 2$ with $\eta = 0.75, 0.85, .095, 1$ for Example 7.1

In Figure 1 as η approached 1, the numerical solutions converge to the exact solution. Table 1 shows the root-mean-square errors for different values of N and the comparison between results obtained from the introduced method and SKCPs method, FEFs method reported by [21, 22] and also shows computed CPU time. From Table 1, we can see that the errors decay as N increases, and the SSKCPs collocation method has smaller root-mean-square errors. Figure 2 and Figure 3 display the numerical result for different values of N. The last column of Table 1 shows that the proposed method has a reasonable computational time.

Table 1: The root-mean-square errors ξ_N for different values of N for Example 7.1

N	SKCPs method[21]	FEFs method[22]	SSKCPs method	CPU time
2	1.0680×10^{-2}	1.7113×10^{-4}	1.1887×10^{-4}	0.296
4	5.1492×10^{-5}	2.7335×10^{-5}	7.5333×10^{-6}	0.375
6	1.7022×10^{-5}	9.5741×10^{-6}	6.8170×10^{-7}	0.546
8	6.5612×10^{-5}	6.2536×10^{-6}	4.1744×10^{-8}	0.749

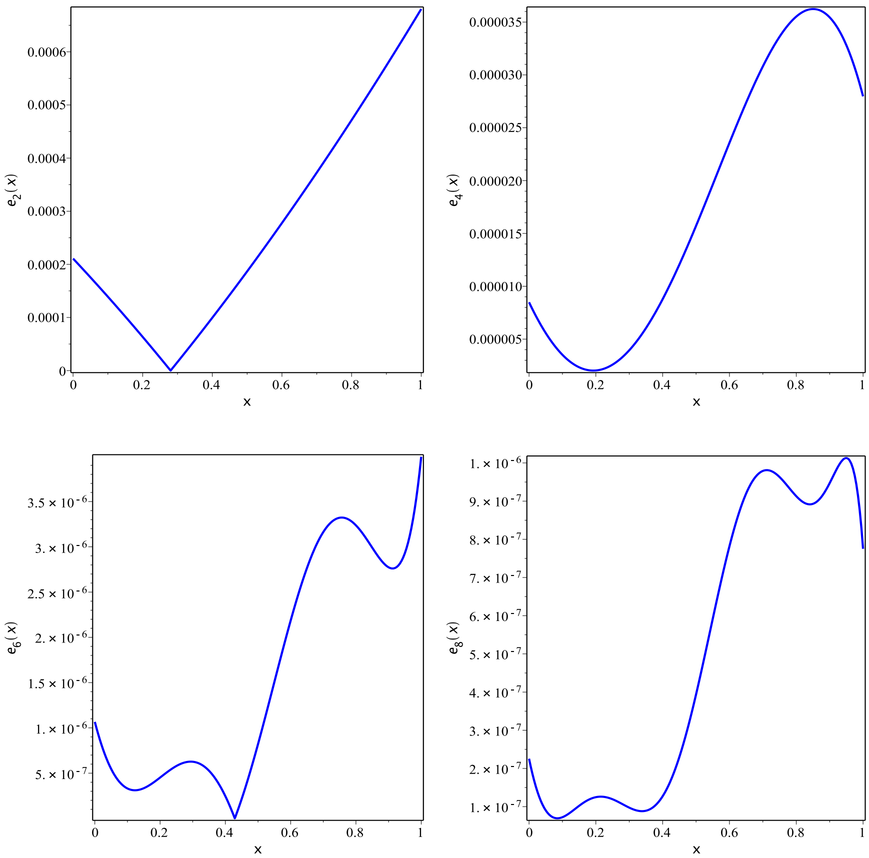

Figure 2: Plot of the absolute error function with $N = 2, 4, 6, 8$ for Example 7.1

Table 2: Convergence rate for different value of N in Example 7.1

N	2, 4	4, 6	6, 8	8, 10	10, 12	12, 14	14, 16
Convergence rate (C_r)	4.2290	5.4384	4.7701	4.3071	4.8080	7.0391	4.6554

Example 7.2. *Consider the following equation [21, 22]*

$$\mathcal{D}^{\frac{1}{3}} u(x) = g(x) - \frac{32}{35} x^{\frac{1}{2}} u(x) + \int_0^x (x-z)^{-\frac{1}{2}} u(z)\, dz, \qquad 0 \le x \le 1, \tag{44}$$

where

$$g(x) = \frac{6x^{\frac{8}{3}}}{\Gamma(\frac{11}{3})} + \Big(\frac{32}{35} - \frac{\Gamma(\frac{1}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{17}{6})}\Big) x^{\frac{11}{6}} + \Gamma(\frac{7}{3}) x,$$

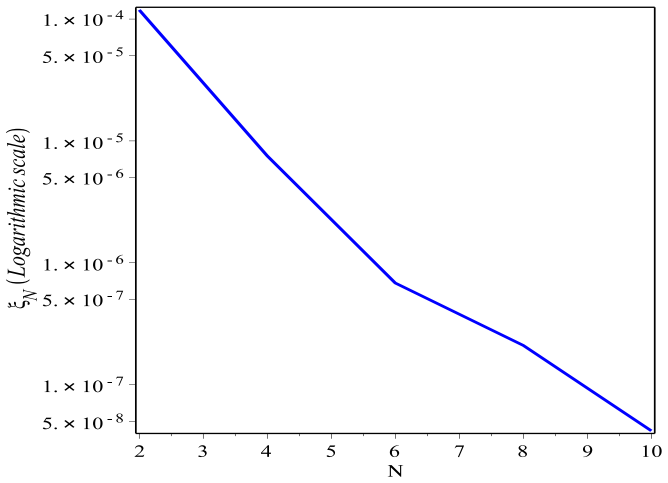

Figure 3: ξ_N on logarithmic scale for Example 7.1

with the initial value $u(0) = 0$. *The exact solution of this problem is* $u(x) = x^3 + x^{\frac{4}{3}}$. *We substitute the approximations given in Section 3 into Eq.* (44), *then we have the following algebraic equation*

$$\mathfrak{S}^T(x)F + \frac{32}{35}x^{\frac{1}{2}}\mathfrak{S}^T(x)\mathscr{P}^{(\frac{1}{3})}F - U^T\mathfrak{I}^{(\frac{1}{2})}\mathfrak{S}(x) - g(x) \approx 0.$$

We apply the presented method to this example and we consider $N = 2, 4, 6, 8$. *Table 3 shows the root-mean-square errors for the SSKCPs collocation method, SKCPs method, and FEFs method with* $\alpha = 1$. *It is seen that the results obtained by the SSKCPs collocation method are more accurate than the other two methods in [21, 22]. Figure 4 displays the absolute error and Figure 5 shows root-mean-square errors for different values of N.*

Table 3: The root-mean-square errors ξ_N for different values of N for Example 7.2

N	SKCPs method[21]	FEFs method[22]	SSKCPs method	CPU time
2	1.3426×10^{-2}	1.0502×10^{-2}	5.1209×10^{-3}	0.577
4	5.1379×10^{-4}	2.6252×10^{-4}	1.8074×10^{-4}	0.702
6	1.5578×10^{-4}	6.6216×10^{-5}	5.1336×10^{-5}	1.061
8	6.4612×10^{-5}	4.8349×10^{-4}	1.8356×10^{-5}	1.435

Table 4: Convergence rate for different value of N in Example 7.2

N	4, 6	6, 8	8, 10	10, 12	12, 14	14, 16	16, 18
Convergence rate (C_r)	2.5358	2.7257	2.8552	2.3939	2.3712	2.3927	2.4395

Example 7.3. *Consider the following linear fractional integro-differential equation with the weakly singular kernel [25, 30]*

$$\mathcal{D}^{\frac{1}{2}}u(x) + x^{\frac{1}{2}}u(x) + \int_0^x (x-z)^{-\frac{1}{4}}u(z)\,dz + \int_0^x (x-z)^{-\frac{1}{3}}\mathcal{D}^{\frac{3}{8}}u(z)\,dz = g(x), \qquad 0 \le x \le 1, \tag{45}$$

where

$$g(x) = \frac{3\Gamma^2(\frac{3}{4})\sqrt{2}}{\pi}x^{\frac{1}{4}} + 2x^{\frac{5}{4}} + \frac{2\Gamma^2(\frac{3}{4})}{\sqrt{\pi}}x^{\frac{3}{2}} + \frac{864\Gamma(\frac{3}{4})\Gamma(\frac{2}{3})\Gamma(\frac{23}{24})}{25\pi Csc(\frac{\pi}{24})}x^{\frac{25}{24}},$$

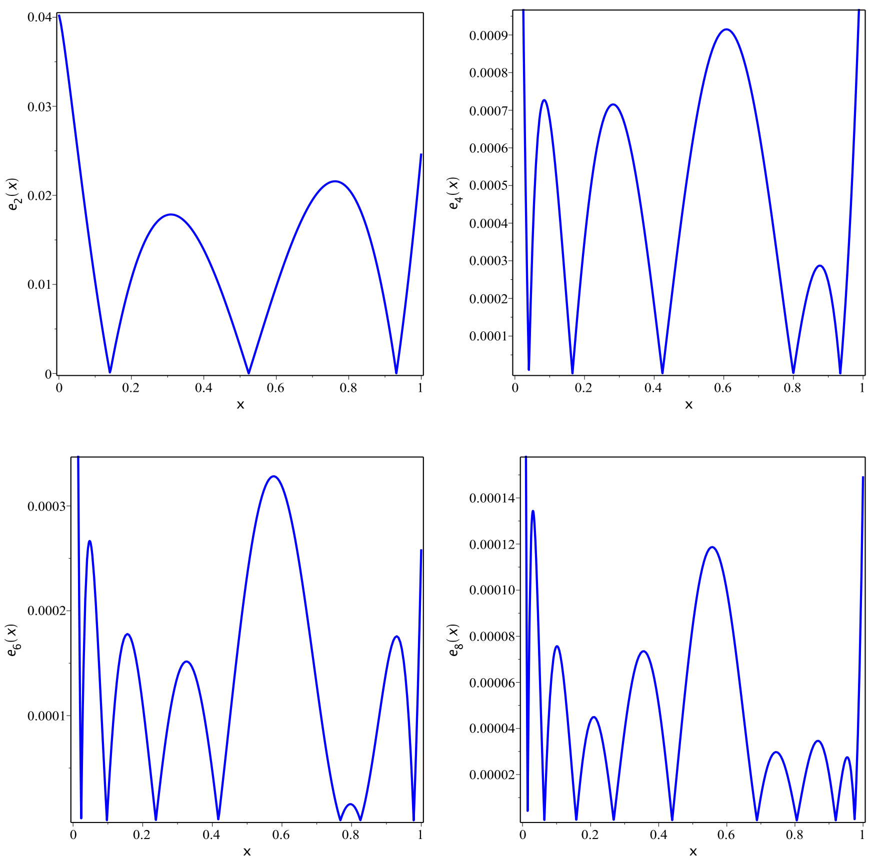

Figure 4: Plot of the absolute error function with $N = 2, 4, 6, 8$ for Example 7.2

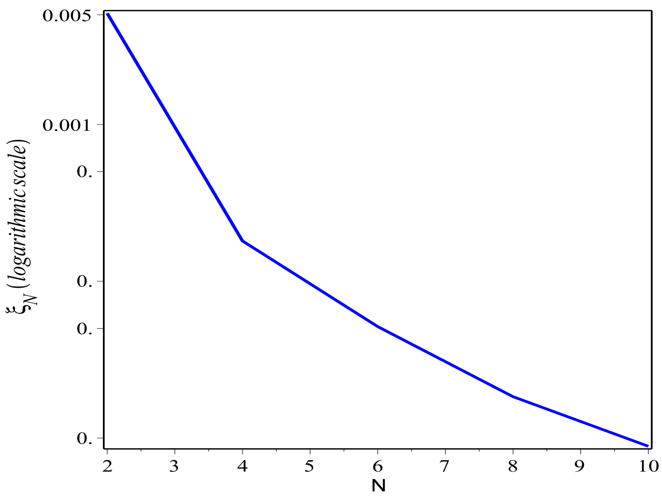

Figure 5: ξ_N on logarithmic scale for Example 7.2

with the initial condition $u(0) = 0$ *and the exact solution is* $u(x) = 2x^{\frac{3}{4}}$. *We substitute the approximations obtained in Section 3 into Eq.* (45) *and get the following algebraic equation*

$$\mathfrak{S}^T(x)F + x^{\frac{1}{2}}\mathfrak{S}^T(x)\mathbb{F}_1 + \mathbb{F}_1^T\mathfrak{I}^{(\frac{1}{4})}\mathfrak{S}(x) + \mathbb{F}_2\mathfrak{I}^{(\frac{1}{3})}\mathfrak{S}(x) - g(x) \approx 0, \tag{46}$$

where $\mathbb{F}_1 = \mathscr{P}^{(\frac{1}{2})}F$ *and* $\mathbb{F}_2 = \mathscr{P}^{(\frac{1}{8})T}F$. *By choosing* $N = 26$, *Eq.* (46) *is collocated at roots of* $S^*_{27}(x)$. *We solve the resultant algebraic system and thus we can determine the unknown vector* F. *We computed the values of the numerical solutions at points* $x_i = 0.2i$ *for* $i = 0, 1, \ldots, 5$ *and* $N = 26$, *where the results are reported in Table 5. Table 6 displays the maximum absolute errors for different values of N and the comparison between the presented method, Spline collocation method and the Jacobi collocation method. This table shows the results obtained from the SSKCPs collocation method are consistent with the other two methods. Table 7 displays the numerical results for different values of N.*

Table 5: Values of absolute errors at equally spaced point for $N = 26$ in Example 7.3

x_i	Exact solution	Approximate solution	$Error_{Abs}$
0.0	0.0000000	0.0075659	7.5659×10^{-3}
0.2	0.5981395	0.5981647	2.5211×10^{-5}
0.4	1.0059467	1.0059427	4.0836×10^{-6}
0.6	1.3634632	1.3634351	2.8045×10^{-5}
0.8	1.6917940	1.6917697	2.4346×10^{-5}
1.0	2.0000000	1.9998943	1.0571×10^{-4}

Table 6: Maximum absolute errors obtained from SSKCPs collocation, Jacobi collocation and Spline collocation methods in Example 7.3

Method	$N = 8$	$N = 16$
SSKCPs collocation method	3.3350×10^{-2}	1.4350×10^{-2}
Jacobi collocation method[30]	2.8900×10^{-2}	1.1579×10^{-2}
Spline collocation method[25]	2.6400×10^{-2}	1.6900×10^{-2}

Table 7: Numerical results for Example 7.3

N	Maximum absolute error	CPU time
8	3.3350×10^{-2}	1.825
16	1.4350×10^{-2}	7.379
26	7.5659×10^{-3}	31.62

Table 8: Convergence rate for different value of N in Example 7.3

N	10, 12	12, 14	14, 16	16, 18	18, 20	20, 22	22, 24
Convergence rate (C_r)	0.7084	0.8443	0.8584	1.0128	1.0109	1.1536	1.0960

Example 7.4. *Consider the following linear fractional integro-differential equation with the weakly singular kernel[30]*

$$\mathcal{D}^{\frac{1}{3}}u(x) = \int_0^x \frac{\mathcal{D}^{\frac{1}{3}}u(x)}{(x-z)^{\frac{1}{2}}}\,dz + \frac{6}{\Gamma(\frac{11}{3})}x^{\frac{8}{3}} - \frac{6\sqrt{\pi}}{\Gamma(\frac{25}{6})}x^{\frac{19}{6}}, \qquad 0 \le x \le 1. \tag{47}$$

The initial condition and exact solution of the problem are $u(0) = 0$ *and* $u(x) = x^3$, *respectively. We apply the approximations introduced in Section 3 in Eq.* (47) *and we achieve the following algebraic equation*

$$\mathfrak{S}^T(x)F - \mathbb{F}^T\mathfrak{I}^{(\frac{1}{2})}\mathfrak{S}(x) - \frac{6}{\Gamma(\frac{11}{3})}x^{\frac{8}{3}} + \frac{6\sqrt{\pi}}{\Gamma(\frac{25}{6})}x^{\frac{19}{6}} \approx 0,$$

where $\mathbb{F} = \mathscr{P}^{(\frac{1}{2})T}F$. We have calculated the numerical solution using the SSKCPs collocation method for $N = 11$. In part (a) of Figure 6, we plotted the numerical and exact solutions and the part (b) displays the absolute error for $N = 11$ that shows the numerical solution coincides with the exact solution. Figure 7 shows the absolute errors for $N = 3, 6, 11$. CPU times for $N = 3, 6$ and 11 are 0.484, 0.749, 1.576s, respectively. From Table 9, we can see that both of SSKCPs collocation method and Jacobi collocation method obtain good approximations to the exact solution and the SSKCPs collocation method has a smaller absolute error than Legendre wavelet method.

Table 9: Maximum absolute errors obtained from SSKCPs collocation, Jacobi collocation and Legendre wavelet methods in Example 7.4

Method	$N = 3$	$N = 6$	$N = 11$
SSKCPs collocation method	1.2395×10^{-2}	8.5374×10^{-5}	3.8206×10^{-6}
Jacobi collocation method[30]	7.8474×10^{-3}	1.3458×10^{-4}	2.3492×10^{-6}
Legendre wavelet method[23]	8.8579×10^{-2}	1.4235×10^{-3}	1.0105×10^{-3}

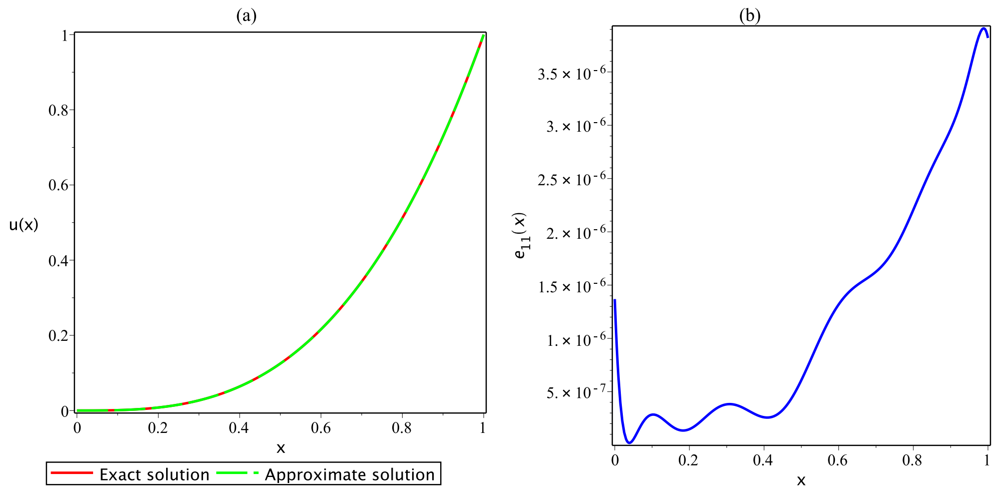

Figure 6: (a) Exact and approximate solutions, (b) Absolute error function for $N = 11$ in Example 7.4

Table 10: Convergence rate for different value of N in Example 7.4

N	2, 4	4, 6	6, 8	8, 10	10, 12	12, 14	14, 16
Convergence rate (C_r)	5.6201	7.4105	4.1242	8.1978	3.9701	8.6586	3.9878

Example 7.5. *As the last example, consider the following nonlinear fractional integro-differential equation with weakly singular kernel [23, 30]*

$$\mathcal{D}^{\eta}u(x) = \int_0^x \frac{u^2(z)}{(x-z)^{\frac{1}{2}}}dz + \int_0^1 xzu^2(z)dz + 3x^2 - \frac{\Gamma(7)\Gamma(\frac{1}{2})}{\Gamma(\frac{15}{2})}x^{\frac{13}{2}} - \frac{x}{8}, \qquad 0 \le x \le 1, \tag{48}$$

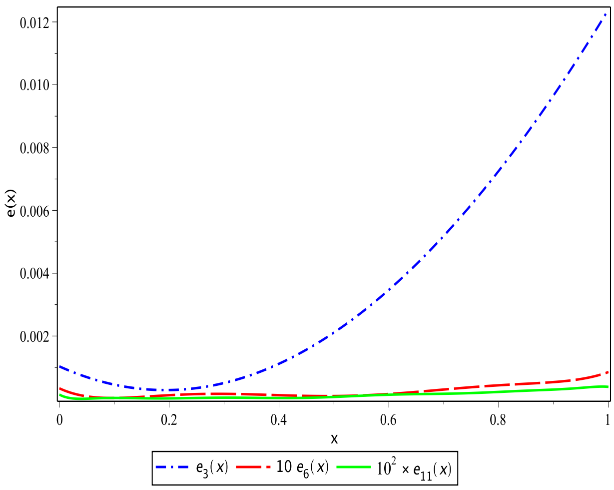

Figure 7: Absolute error functions obtained by SSKCPs collocation method for $N = 3, 6, 11$ in Example 7.4

with $u(0) = 0$. *The exact solution of the equation for* $\eta = 1$ *is* $u(x) = x^3$. *Using Eq.* (21) *and the obtained operational matrices, Eq.* (48) *converts into the following algebraic equation*

$$\mathfrak{S}^T(x)F - \mathbb{U}^T\mathfrak{I}^{(\frac{1}{2})}\mathfrak{S}(x) - \mathfrak{S}^T\mathcal{H}Q\mathbb{U} - 3x^2 + \frac{\Gamma(7)\Gamma(\frac{1}{2})}{\Gamma(\frac{15}{2})}x^{\frac{13}{2}} + \frac{x}{8} \approx 0, \tag{49}$$

where $\mathbb{U} = \tilde{\mathbb{F}}_1{}^T\mathbb{F}_1$ *and* $\mathbb{F}_1 = \mathscr{P}^{(\eta)T}F$ *where* $\tilde{\mathbb{F}}_1$ *is the operational matrix of the product corresponding to the vector* $\mathbb{F}_1$. *Table 11 displays the maximum absolute errors and the computing time that are obtained for different values of* N. *We can see that by increasing* N, *good approximations are obtained to the exact solution. In Figure 8, we show the SSKCPs collocation solutions and the exact solution for various values of* η. *It is evident from Figure 8 that as* η *gets close to* 1, *the numerical solutions obtained from the SSKCPs collocation method converge to the exact solution. Figure 9 shows a graphical comparison between the exact and approximate solutions and the plot of the absolute error function in parts (a) and (b) for* $N = 11$. *In this figure, we can observe excellent agreement between the exact and approximate solution.*

Table 11: Numerical results for Example 7.5

N	Maximum absolute error	CPU time
3	2.6597×10^{-2}	1.513
8	9.1313×10^{-9}	5.148
11	3.0000×10^{-11}	11.123
16	6.7636×10^{-13}	38.438
20	4.8129×10^{-14}	95.784

Table 12: Convergence rate for different value of N in Example 7.5

N	4, 6	6, 8	8, 10	10, 12	12, 14	14, 16	16, 18
Convergence rate (C_r)	17.3580	15.1978	13.5795	13.8716	13.2876	13.8285	13.2458

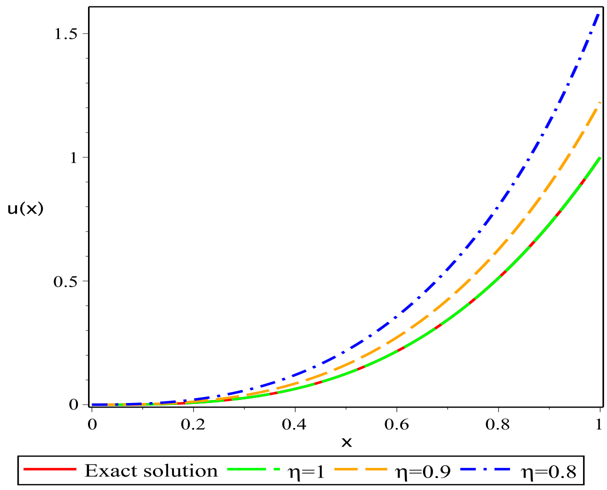

Figure 8: Exact and numerical solutions for values of $\eta = 0.8, 0.9, 1$ and $N = 11$ in Example 7.5

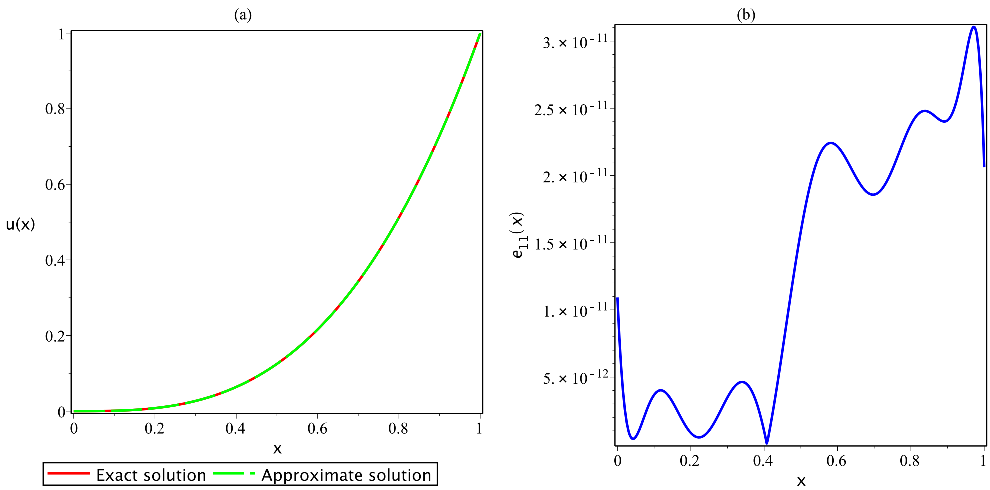

Figure 9: (a) Exact and approximate solutions, (b) Absolute error function for $N = 11$ in Example 7.5

8. Conclusion

In this paper, we presented the numerical solution of a category of fractional integro-differential equations based on shifted sixth-kind Chebyshev polynomials. We applied these polynomials together with the collocation method for solving the weakly singular fractional integro-differential equations. For this aim, we derived the operational matrices of integration and product. We used these approximations, and we converted the main equations into a linear or nonlinear system of algebraic equations that we solved using the Newton iterative method. By solving the linear and nonlinear systems, approximate solutions are achieved. Moreover, the error analysis of the proposed method was investigated. We solved some problems to show the applicability of the presented method. Graphical illustrations and tables of the numerical results show very good consistency between the numerical results and the analytic solutions. The results got by this method in comparison with some existing methods, such as the SKCPs collocation method, FEFs method, and the Legendre wavelet method, are more precise, and the results obtained from the proposed scheme are close to those of the Jacobi collocation method [30]. At the end of this paper, it should be noted that CPU time had been calculated for all of the examples but Example 7.5 had a CPU time longer than the others. This issue can be due to nonlinear nature of Eq. (48). CPU time in this equation may be reduced if a suitable approximation is substituted into the terms of the integral equation. However, our method can be helpful in complicated calculations and faster than many other methods.

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