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Approximating multiple integrals over non-rectangular compact set using α-dense curves

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Abstract. In this paper, we develop a method for approximating multiple integrals. The domain of integration Ω is assumed to be a non-rectangular compact of $\mathbb{R}^n$. The main idea is the dimensionality reduction procedure based on the use of parametric α-dense curves $\ell_\alpha(t)$. First, the region whose measure represents the value of the integral, is densified using new results, by a certain α-dense curve of finite length. The multiple integral of a positive continuous function f over Ω is approximated by a unique single integral corresponding to $\ell_\alpha(t)$. Some numerical examples are given.

1. Introduction

Let us consider the following multiple integral:

$$I_n(f) = \underset{\Omega}{\int \cdots \int} f(x_1, \ldots, x_n) dx_n \ldots dx_1 \tag{1.1}$$

where the function f is of class $C^1(\Omega)$ (or Lipschitzean) and Ω is the n-dimensional region in the Euclidean n-space $\mathbb{R}^n$ defined as follows:

$$\Omega = \{x \in \mathbb{R}^n : g_i(x) \leq 0, \ i \in I\},$$

where $g_i(x)$ are given measurable functions and $I = \{i : 1 \leq i \leq m, \ m \leq n\}$ is a finite index set.

Several specific methods for numerical evaluation of integrals over higher dimensional regions have been proposed (see [8, 11]). All of these, including the theory of integration itself, are based on the geometric principles of area and volume. These intuitive concepts are difficult to formalize, but will be crucial for us. The literature on integration formulas is really extensive but mostly deal with functions of a single variable [12]. However, for the development of modern modelling techniques [10, 17, 23–25] it is necessary to

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consider systems that are described by functions with several variables. Therefore, finding the extrema, calculating their integrals etc. turns out to be a fundamental task and this topic presents many challenges. In order to classically integrate (1.1) explicitly, which is not always easy, we can proceed with a change of variables or a succession of simple integrations (Fubini's theorem [8, 12]) and this is only possible if the region of integration Ω, can be formulated as follows:

$$\Omega = \left\{ x \in \mathbb{R}^n, \begin{array}{c} a \leq x_1 \leq b \\ \vdots \\ \varphi_i(x_1, \ldots, x_{i-1}) \leq x_i \leq \psi_i(x_1, \ldots, x_{i-1}), \;\; 2 \leq i \leq n \end{array} \right\} \tag{1.2}$$

where φ_i and ψ_i $(2 \leq i \leq n)$ are continuous and bounded functions.

Even the case of a rectangular area, is not more practical because it involves too much calculus. The situation will be very complicated in the case of a non-rectangular compact area. Here, we develop an approximate method of the multiple integral (1.1) when the region Ω is non-rectangular of the form (1.2). The main idea is based on the reduction of the domain $\Omega_f = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x = (x_1, \ldots, x_n) \in \Omega \text{ and } 0 \leq x_{n+1} \leq f(x)\}$ by means of a transformation which allows us to express the $(n + 1)$ variables x_i, $1 \leq i \leq (n + 1)$ as a single variable $t \in \mathbb{R}$, given by $x_i = \ell_i(t)$, $1 \leq i \leq (n + 1)$, (see below). Because calculating the integral (1.1) over Ω numerically amounts to calculating the volume of Ω_f. The function f is approximated by a univariate function defined by $f_\alpha(t) = f(\ell_1(t), \ldots, \ell_n(t))$. The curve $\ell_\alpha(t) = (\ell_1(t), \ldots, \ell_{n+1}(t))$ is a computable parametric α-dense curve in the non-rectangular domain Ω_f. Then, the multiple integral (1.1) is reduced to the unique simple integral I_α obtained as the limit, when α tends to 0, of the product of α^n by $\mathcal{L}_{\alpha,\ell_\alpha}$ the length of the curve $\ell_\alpha(t)$:

$$I_\alpha = \lim_{\alpha \to 0} \alpha^n \mathcal{L}_{\alpha,\ell_\alpha} = \int_{\mathbb{I}} g_\alpha(t)\, dt \tag{1.3}$$

where $\mathbb{I}$ is an interval and $g_\alpha(t)$ is a function which depends on $f_\alpha(t)$. More precisely $\mathbb{I} = [0, U]$ with U is the upper bound of the function $\ell_\alpha(t)$.

Till now, some authors looked at numerical approximation method of multiple integrals, based on α-dense curves, only over rectangular areas [2, 3, 13, 14]. The goal of this paper is, still using α-dense curves, to give a new formula to approximate the multiple integral (1.1) over some non-rectangular compact Ω of the type (1.2).
It should be also mentioned that, the error analysis for numerical integration methods dealing with multivariate function are not abundant in the literature. And that, because of the approximate integration formula which is generally complicated. For the procedure of reduction consisting of approaching the multiple integrals $I_n(f)$ using α-dense curves, by the simple integral I_α given by (1.3), we define the associated error $E_\alpha(f)$ as:

$$E_\alpha(f) = |I_n(f) - I_\alpha|.$$

In this contest, there is an attempt, but just for a very special type of the α-dense curves and for a very few special cases of function f integrated over rectangular region [4, 12, 15]. Because, in general error analysis in higher dimension is much more difficult than for the function of one variable.

2. Generating α-dense curves in a non-rectangular compact regions

Here, we present some ways for constructing α-dense curves in a compact region. The construction of the curves filling a rectangular region of $\mathbb{R}^n$, was studied and applied by numerous authors. The first type of curves are known as space-filling curves [1, 19] which are the approximation of Peano type curves (1980), then Hilbert curves (1981), see e.g. Butz [5]. The second type are the α-dense curves which have been developed by several authors [7, 9, 20], were introduced mainly to solve multidimensional global optimization with and without constraints [10, 18]. Also, the approximation of multiple integrals on hyper-rectangles of $\mathbb{R}^n$ $(n > 1)$ [3, 6, 14, 16], as being areas and volumes. The introduction of these α-dense curves has allowed analytical argumentation of certain dimensionality reduction procedure [3, 6, 13, 14, 16]. In what follows they will be used as the support for a numerical integration approach. Below, some definitions and properties of these curves are given.

Definition 2.1. *Let Ω be a subset of finite diameter of the metric space $\mathbb{R}^n$ and $\alpha > 0$. We say that a continuous curve ℓ_α of $\mathbb{R}^n$ is α-dense in Ω, if $\ell_\alpha \subset \Omega$ and for all $x \in \Omega$, $\mathbf{d}(x, \ell_\alpha) \leq \alpha$, where: $\mathbf{d}(x, \ell_\alpha) = \inf_{y \in \ell_\alpha} \mathbf{d}(x, y)$ ($\mathbf{d}$ is the Euclidean distance).*

Remark 2.1. *Recall that the curve ℓ_α is rectifiable if its length $\mathcal{L}_{\alpha,\ell_\alpha}$ (which is independent of the chosen parametrization) is finite.*

Some interesting results concerning the existence of α-dense curves with minimal length were given by Ziadi et al. [21]. In the case where the set Ω is a hyper-rectangle of $\mathbb{R}^n$, different ways for constructing the α-dense curves in Ω are presented in [7, 20, 22, 23]. We shall use the recent new results [10, 18] concerning the generation of α-dense curves in non-rectangular compacts (which have been given to solve some constrained global optimization problems). Before giving the new formula for the computation of the single integral (1.3), we first begin with densification results for compacts of the following form:

$$\Omega_f = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x \in \Omega \text{ and } 0 \leq x_{n+1} \leq f(x)\}.$$

We determine the relationship existing between the components $\ell_i, (i = 1, \ldots, n)$ of the parametrization in order to generate a new class of α-dense curves in a compact type (1.2) whose boundary is defined by a continuous functions. Because the integral of a non-negative continuous function f on Ω is the measure of the area or volume of the domain Ω_f. If the domain Ω_f is densified by a certain α-dense curve ℓ_α, we shall prove that the integral (1.1) can be approximated, when α approaches 0, by the expression $\alpha^n.\mathcal{L}_{\alpha,\ell_\alpha}$. The rest of this section is to present densification results of Ω_f in which Ω is not rectangular of the form (1.2).

2.1. The α-densifiable hyper-rectangles and multiple integrals

We first give here some results concerning the densification of rectangular regions by α-dense curve ℓ_α and the approximate calculation of a multiple integral. If the functions $\varphi_2, \ldots, \varphi_n$, $\psi_2, \ldots, \psi_n$ in (1.2) are all constants $\varphi_i(x_1, \ldots, x_{i-1}) = a_i$ and $\psi_i(x_1, \ldots, x_{i-1}) = b_i$ for all x_j $(j = 1, \ldots, (i-1))$, $i = 2, \ldots, n$ and if we put $a = a_1$ and $b = b_1$, then Ω becomes the hyper-rectangle $\prod_{i=1}^{n} [a_i, b_i]$. In this case we have the following result (see [10]).

Theorem 2.1. *Let $\ell = (\ell_1, \ldots, \ell_n) : \mathbb{I} = \left[0, \frac{\pi}{\alpha_1}\right] \to \Omega$ be a function defined by:*

$$\ell_i(t) = (a_i - b_i)\cos^2(\frac{\alpha_i t}{2}) + b_i, \;\; i = 1, \ldots, n,$$

where the constants $\alpha_1, \ldots, \alpha_n, \alpha$ satisfy the relations:

$$\alpha_1 \in \mathbb{R}_+^*, \; \alpha_i = \alpha_1 \left(\frac{1}{\alpha}\right)^{i-1} \prod_{k=1}^{i-1} (b_k - a_k), \quad i = 2, \ldots, n,$$

then, the parameterized curve $\ell(t) = (\ell_1(t), \ldots, \ell_n(t))$ for $t \in \mathbb{I}$, is $\pi\alpha\sqrt{n-1}$-dense in Ω.

Example 2.1. *For illustrating the concept of α-density, we propose two examples for densifying two hyper-rectangles (for $n = 2$ and $n = 3$), by the supports of different α-dense curves ℓ_α (defined on $[0, \pi]$ with $\alpha_1 = 1$) respectively in the hyper-rectangles $\Omega_2 = [-2, 2] \times [1, 2]$ and $\Omega_3 = [-2, 2] \times [1, 2] \times [0, 1]$, see Figure 1 and Figure 2 given below,*

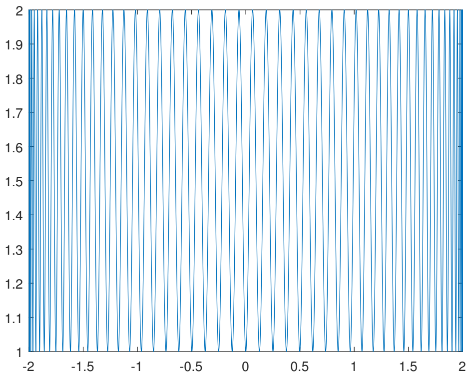

Figure 1: 2D, α-dense curve ℓ_α (in blue) in Ω_2 with $\alpha = 0.2$.

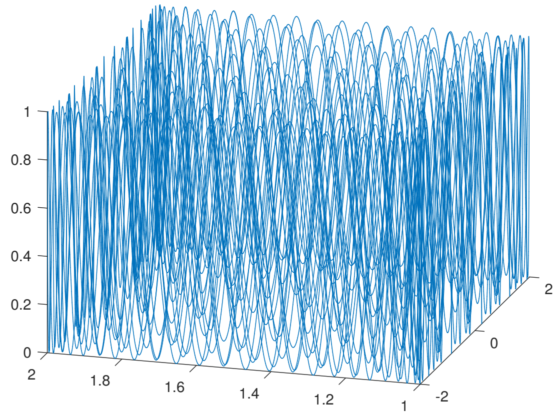

Figure 2: 3D, α-dense curve ℓ_α (in blue) in Ω_3 with $\alpha = 0.3$.

Theorem 2.2. *Let $f(x) = C$, where C is a positive number, be a constant function defined on the hyper-rectangle Ω. Consider the parameterized curve $\ell_\alpha = (\ell_1, \ldots, \ell_{n+1}) : \left[0, \frac{\pi}{\alpha_1}\right] \to \Omega_f = \Omega \times [0, C]$ defined by:*

$$\begin{cases} \ell_i(t) = (a_i - b_i)\cos^2(\frac{\alpha_i t}{2}) + b_i\,, & \textit{for } i = 1, \ldots, n, \\ \ell_{n+1}(t) = -C\cos^2(\frac{\alpha_{n+1} t}{2}) + C, & \end{cases}$$

with

$$\begin{cases} \alpha_1 \in \mathbb{R}_+^*,\ \alpha_i = \alpha_1 \left(\frac{1}{\alpha}\right)^{i-1} \prod\limits_{k=1}^{i-1}(b_k - a_k), \quad \textit{for } i = 2, \ldots, n+1. \\ \textit{and } \alpha = \frac{1}{m}\left(\prod\limits_{k=1}^{n}(b_k - a_k)\right)^{1/n}, \quad m \in \mathbb{N}^*, \quad \alpha \to 0 \Leftrightarrow m \to +\infty. \end{cases}$$

Then the curve ℓ_α is $\pi\alpha\sqrt{n}$-dense in Ω_f and we have:

$$I_n(f) = \lim_{\alpha \to 0} \alpha^n \mathcal{L}_{\alpha, \ell_\alpha}.$$

Proof. Using the theorem 2.1 and by inspiring by [14], we can proof that the curve $\ell_\alpha = (\ell_1, \ldots, \ell_{n+1})$ defined on $\left[0, \frac{\pi}{\alpha_1}\right]$, in the last theorem, is $\pi\alpha\sqrt{n}$-dense in Ω_f. Next set $L_\alpha = \alpha^n \mathcal{L}_{\alpha,\ell_\alpha}$ and $\beta = \left(\prod\limits_{k=1}^{n}(b_k - a_k)\right)^{1/n}$, then

$$L_\alpha = \alpha^n \int_0^{\frac{\pi}{\alpha_1}} \left[\left(\ell_1'(t)\right)^2 + \left(\ell_2'(t)\right)^2 + \cdots + \left(\ell_{n+1}'(t)\right)^2\right]^{\frac{1}{2}} dt,$$

where $\ell_i'(t)$, $(1 \leq i \leq n+1)$ are the derivatives of the components of $\ell_\alpha(t)$ and the constants α_i are given by

$$\begin{cases} \alpha_1 \in \mathbb{R}_+^*,\ \alpha_i = \alpha_1 \left(\frac{m}{\beta}\right)^{i-1} \prod\limits_{k=1}^{i-1}(b_k - a_k)\,, \quad \text{for } i = 2, \ldots, n, \\ \text{and } \alpha_{n+1} = \alpha_1 m^n \ \text{ with } \alpha = \frac{\beta}{m}. \end{cases}$$

Then

$$\begin{aligned} L_\alpha &= \alpha^n \int_0^{\frac{\pi}{\alpha_1}} \left[\left(\alpha_1 \tfrac{b_1-a_1}{2} \sin(\alpha_1 t)\right)^2 + \cdots + \left(\alpha_n \tfrac{b_n-a_n}{2} \sin(\alpha_n t)\right)^2 + \left(\alpha_{n+1} \tfrac{C}{2} \sin(\alpha_{n+1} t)\right)^2 \right]^{\frac{1}{2}} dt \\ &= \alpha^n \int_0^{\frac{\pi}{\alpha_1}} \left[\left(\alpha_1 \tfrac{b_1-a_1}{2} \sin(\alpha_1 t)\right)^2 + \cdots + \left(\alpha_1 \left(\tfrac{m}{\beta}\right)^{n-1} \tfrac{b_n-a_n}{2} (b_1-a_1)\ldots(b_{n-1}-a_{n-1}) \sin(\alpha_n t)\right)^2 \right. \\ &\quad \left. + \left(\alpha_1 m^n \tfrac{C}{2} \sin(\alpha_1 m^n t)\right)^2 \right]^{\frac{1}{2}} dt. \end{aligned}$$

Let us make the following change of variables: $u = m^n \alpha_1 t$,

$$\begin{aligned} L_\alpha &= \alpha^n \int_0^{m^n\pi} \tfrac{1}{m^n\alpha_1} \left[\left(\alpha_1 \tfrac{b_1-a_1}{2} \sin\left(\tfrac{u}{m^n}\right)\right)^2 + \cdots + \left(\alpha_1 \left(\tfrac{m}{\beta}\right)^{n-1} \tfrac{b_n-a_n}{2} (b_1-a_1)\ldots(b_{n-1}-a_{n-1}) \sin\left(\tfrac{\alpha_n u}{m^n\alpha_1}\right)\right)^2 + \right. \\ &\quad \left. \left(\alpha_1 m^n \tfrac{C}{2} \sin(u)\right)^2 \right]^{\frac{1}{2}} du \\ &= \alpha^n \sum_{i=1}^{m^n} \int_{\pi(i-1)}^{\pi i} \left[\left(\tfrac{1}{m^n} \tfrac{b_1-a_1}{2} \sin\left(\tfrac{u}{m^n}\right)\right)^2 + \cdots + \left(\tfrac{C}{2} \sin(u)\right)^2 \right]^{\frac{1}{2}} du \\ &= \alpha^n \sum_{i=1}^{m^n} L_\alpha^{(i)}, \end{aligned}$$

with $L_\alpha^{(i)} = \int_{\pi(i-1)}^{\pi i} \left[\left(\frac{1}{m^n} \frac{b_1-a_1}{2} \sin\left(\frac{u}{m^n}\right)\right)^2 + \cdots + \left(\frac{C}{2} \sin(u)\right)^2 \right]^{\frac{1}{2}} du, \quad i = 1, \ldots, m^n.$

There exist $p \geq 1$ and $q \leq m^n$ such that:

$$L_\alpha^{(p)} = \min_{1 \leq i \leq m^n} L_\alpha^{(i)} \quad \text{and} \quad L_\alpha^{(q)} = \max_{1 \leq i \leq m^n} L_\alpha^{(i)}.$$

Then:

$$\alpha^n \sum_{i=1}^{m^n} L_\alpha^{(p)} \leq L_\alpha \leq \alpha^n \sum_{i=1}^{m^n} L_\alpha^{(q)},$$

that is

$$\beta^n L_\alpha^{(p)} \leq L_\alpha \leq \beta^n L_\alpha^{(q)}. \tag{2.1}$$

On the other hand we have:

$$L_\alpha^{(p)} = \int_{\pi(p-1)}^{\pi p} \left[\left(\frac{1}{m^n} \frac{b_1-a_1}{2} \sin\left(\frac{u}{m^n}\right)\right)^2 + \cdots + \left(\frac{C}{2} \sin(u)\right)^2 \right]^{\frac{1}{2}} du,$$

$$L_\alpha^{(q)} = \int_{\pi(q-1)}^{\pi q} \left[\left(\frac{1}{m^n} \frac{b_1-a_1}{2} \sin\left(\frac{u}{m^n}\right)\right)^2 + \cdots + \left(\frac{C}{2} \sin(u)\right)^2 \right]^{\frac{1}{2}} du.$$

Again, the next change of variables:

$$v = u - (p-1)\pi, \quad w = u - (q-1)\pi,$$

gives:

$$L_\alpha^{(p)} = \int_0^{\pi} \left[\left(\tfrac{1}{m^n} \tfrac{b_1-a_1}{2} \sin\left(\tfrac{v+(p-1)\pi}{m^n}\right)\right)^2 + \cdots + \left(\tfrac{C}{2} \sin(v)\right)^2 \right]^{\frac{1}{2}} dv$$

$$L_\alpha^{(q)} = \int_0^\pi \left[\left(\frac{1}{m^n} \frac{b_1 - a_1}{2} \sin\left(\frac{w + (q-1)\pi}{m^n} \right) \right)^2 + \cdots + \left(\frac{C}{2} \sin(w) \right)^2 \right]^{\frac{1}{2}} dw.$$

According to (2.1), by setting m tends to $+\infty$, and by virtue of uniform convergence, we deduce:

$$\beta^n \int_0^\pi \frac{C}{2} \sin(v)\, dv \le L_\alpha \le \beta^n \int_0^\pi \frac{C}{2} \sin(w)\, dw,$$

which means that:

$$\lim_{\alpha \to 0} L_\alpha = C\beta^n = C \prod_{k=1}^n (b_k - a_k).$$

So we have:

$$I_n(f) = \lim_{\alpha \to 0} \alpha^n \mathcal{L}_{\alpha, \ell_\alpha} = C \prod_{k=1}^n (b_k - a_k).$$

□

Proposition 2.1. *Let* $f(x) = \sum_{j=1}^{p} C_j \chi_j$ *be a non-negative step function defined on the hyper-rectangle* Ω *with* $C_j \ge 0$ *and* χ_j $(1 \le j \le p)$*, are the indicator functions of the sets* P_j *of a partition* P *of* Ω*. Then there exists a parameterized curve* ℓ_α *with density* $\alpha\pi\sqrt{n}$ *and length* $\mathcal{L}_{\alpha,\ell}$ *in the domain:*

$$K_{P,f} = \bigcup_{j=1}^{p} \left\{ \left(x^{(j)}, x_{n+1}^{(j)} \right) \,/\, x^{(j)} \in P_j\,,\ 0 \le x_{n+1}^{(j)} \le C_j \right\},$$

such that the following equality is satisfied:

$$I_n(f) = \lim_{\alpha \to 0} \alpha^n \mathcal{L}_{\alpha, \ell_\alpha}.$$

Proof. The partition $P = \left\{ P_j : j = 1, \ldots, p \right\}$ is obtained by dividing the hyper-rectangle Ω into sub-rectangles P_j, without common interior points, we then have:

$$I_n(f) = \sum_{j=1}^{p} \int \cdots \int_{P_j} C_j dx_n \ldots dx_1.$$

But, we have:

$$\int \cdots \int_{P_j} C_j dx_n \ldots dx_1 = \lim_{\alpha \to 0} \alpha^n \mathcal{L}_{\alpha, \ell_\alpha^{(j)}},$$

for curves $\ell_\alpha^{(j)}$ with density $\alpha\pi\sqrt{n}$ and length $\mathcal{L}_{\alpha,\ell_\alpha^{(j)}}$ in $P_j \times [0, C_j]$. So we can write:

$$I_n(f) = \lim_{\alpha \to 0} \sum_{j=1}^{p} \alpha^n \mathcal{L}_{\alpha, \ell_\alpha^{(j)}}.$$

Now, consider the curves $\ell_\alpha^{(j)}$ for $j = 1, \ldots, p$ and define a curve ℓ_α in $K_{P,f}$ such that:

$$\ell_\alpha(t) = \ell_\alpha^{(j)}(t) \quad \text{if } t \in int\left(P_j\right) \quad (\text{ interior of } P_j).$$

The curve ℓ_α can be defined in different ways on the boundary planes of the sub-rectangles P_j. Suppose that ℓ_α was constructed with the density $\alpha\pi\sqrt{n}$. Then its length is given by:

$$\mathcal{L}_{\alpha,\ell} = \sum_{j=1}^{p} \mathcal{L}_{\alpha,\ell_\alpha^{(j)}} \pm O,$$

where O denotes the total quantity corresponding to the elements added or removed from the curve linking the $\ell_\alpha^{(j)}$. By substitution, we get:

$$\begin{aligned} I_n(f) &= \lim_{\alpha\to 0} \alpha^n \sum_{j=1}^{p} \mathcal{L}_{\alpha,\ell_\alpha^{(j)}} \\ &= \lim_{\alpha\to 0} \alpha^n \left[\mathcal{L}_{\alpha,\ell_\alpha} \pm O\right] = \lim_{\alpha\to 0} \alpha^n \mathcal{L}_{\alpha,\ell_\alpha}. \end{aligned}$$

□

Theorem 2.3. *Let f be a non-negative continuous function defined on the hyper-rectangle Ω of $\mathbb{R}^n$. Then, for any $\varepsilon > 0$, there exists a parameterized curve ℓ_α with density $\left(\alpha\pi\sqrt{n} + \varepsilon\right)$ in the domain*

$$\Omega_f = \left\{(x, x_{n+1}) \in \mathbb{R}^{n+1} : \ x \in \Omega \ \text{ and } 0 \le x_{n+1} \le f(x)\right\},$$

for which the following inequality is satisfied

$$\left|I_n(f) - \alpha^n \mathcal{L}_{\alpha,\ell_\alpha}\right| < \varepsilon.$$

Proof. According to Riemann's integral theory, there exists a partition $P = \left\{P_j : j = 1, \ldots, p\right\}$ of Ω and a step function:

$$f_P = \sum_{j=1}^{p} m_j \chi_j \,,$$

where χ_j are the indicator functions of the sets P_j and $m_j = \min\left\{f(x) : x \in P_j\right\}$, such that:

$$|I_n(f) - I^n_{f_P}| < \frac{\varepsilon}{2}.$$

Moreover, because of the continuity of f, the function f_P can be chosen so that:

$$\max_j \left\{M_j - m_j : j = 1, \ldots, p\right\} < \varepsilon,$$

with $M_j = \max\left\{f(x) : x \in P_j\right\}$. On the other hand, we can find a curve ℓ_α densifying the region:

$$K_{f_P} = \bigcup_{j=1}^{p} \left\{(x, x_{n+1}) : x \in P_j \,, 0 \le x_{n+1} \le m_j\right\},$$

with the density $\alpha\pi\sqrt{n}$ and such that we have the inequality:

$$|I_n(f_P) - \alpha^n \mathcal{L}_{\alpha,\ell_\alpha}| < \frac{\varepsilon}{2}.$$

The two previous inequalities allow us to write:

$$|I_n(f) - \alpha^n \mathcal{L}_{\alpha,\ell_\alpha}| < \varepsilon.$$

Since $K_{f_P} \subset \Omega_f$, the curve ℓ_α densifies Ω_f with density $\left(\alpha\pi\sqrt{n} + \varepsilon\right)$ and the proof of the theorem is finished. □

3. The main result

In what follows, based on recent results [10, 18], we will extend the results given in the previous section.

3.1. The α-densifiable non-rectangular compacts and multiple integrals

As the functions φ_i and ψ_i $(2 \leq i \leq n)$ are supposed to be continuous and bounded then there exist x_i^l, x_i^u such that Ω is contained in the hyper-rectangle $\mathbb{H} = \prod_{i=1}^{n}[x_i^l, x_i^u]$. In the sequel, the functions φ_i and ψ_i $(2 \leq i \leq n)$ in (1.2) are supposed to be lipschitzian [10] (or hölderian see [18]) with constants respectively $l_i > 0$, $L_i > 0$ $(2 \leq i \leq n)$ over the hyper-rectangles $\mathbb{H}_i = \prod_{k=1}^{i-1}[x_k^l, x_k^u]$ $(2 \leq i \leq n)$. Denote by $\mathbf{m}$ the Lebesgue measure, the arbitrary number α is supposed to be positive. Next, keeping the same notations introduced above, we give a general main result for generating α-dense curves in Ω_f.

Theorem 3.1. *Let f be a real non-negative lipschitzian function of constant $L > 0$ over $\mathbb{H}$. Then, for all $\varepsilon > 0$, there exists a curve ℓ_α densifying the domain:*

$$\Omega_f = \left\{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x \in \Omega \ \textit{and} \ 0 \leq x_{n+1} \leq f(x)\right\},$$

with density $\left(\pi\alpha\sqrt{n} + \varepsilon\right)$, such that:

$$\left|I_n(f) - \alpha^n \mathcal{L}_{\alpha,\ell_\alpha}\right| < \varepsilon.$$

Proof. Let $P = \left\{P_j : j = 1, \ldots, p\right\}$ be a set of hyper-rectangles defined by

$$P_j = \prod_{i=1}^{n}\left[a_i^{(j)}, b_i^{(j)}\right] \subset \Omega,$$

such that:

$$\begin{cases} a_1^{(j)} = a^{(j)}, \ \ a_1^{(1)} = a \\ a_2^{(j)} = \max\limits_{\left[a_1^{(j)}, b_1^{(j)}\right]} \varphi_2(x_1) \\ \vdots \\ a_n^{(j)} = \max\limits_{\prod_{i=1}^{n-1}\left[a_i^{(j)}, b_i^{(j)}\right]} \varphi_n(x_1, \ldots, x_{n-1}) \end{cases}, \qquad \begin{cases} b_1^{(j)} = b^{(j)}, \ \ b_1^{(1)} = b \\ b_2^{(j)} = \min\limits_{\left[a_1^{(j)}, b_1^{(j)}\right]} \psi_2(x_1) \\ \vdots \\ b_n^{(j)} = \min\limits_{\prod_{i=1}^{n-1}\left[a_i^{(j)}, b_i^{(j)}\right]} \psi_n(x_1, \ldots, x_{n-1}) \end{cases}$$

and

$$P = \bigcup_{j=1}^{p} P_j \simeq \Omega \text{ and } \bigcap_{j=1}^{p} Int(P_j) = \emptyset.$$

According to Riemann's integral theory, there exists a step function:

$$f_P = \sum_{j=1}^{p} m_j \chi_j,$$

where χ_j is indicator function of the set P_j and $m_j = \min\left\{f(x) : x \in P_j\right\}$, such that:

$$\left| I_n(f) - \int \cdots \int_P f_P(x_1, \ldots, x_n)\, dx_n \ldots dx_1 \right| < \frac{\varepsilon}{2}. \tag{3.1}$$

The function f_P and the set P can be chosen so that:

$$\max_j \left\{M_j - m_j : j = 1, \ldots, p\right\} < \varepsilon,$$

with

$$M_j = \max\left\{f(x) : x \in P_j\right\}.$$

And

$$\max_{1\le j\le p} \left\{ \max_{\prod_{k=1}^{i-1}\left[a_k^{(j)}, b_k^{(j)}\right]} \varphi_i(x) - \min_{\prod_{k=1}^{i-1}\left[a_k^{(j)}, b_k^{(j)}\right]} \varphi_i(x) \ \ / \ \ i = 2, \ldots, n \right\} < \varepsilon,$$

$$\max_{1\le j\le p} \left\{ \max_{\prod_{k=1}^{i-1}\left[a_k^{(j)}, b_k^{(j)}\right]} \psi_i(x) - \min_{\prod_{k=1}^{i-1}\left[a_k^{(j)}, b_k^{(j)}\right]} \psi_i(x) \ / \ i = 2, \ldots, n \right\} < \varepsilon.$$

(That is to say: $\mathbf{m}\left(\bigcup_{j=1}^{p} P_j - \Omega\right) < \varepsilon$).

By applying the proposition 2.1, to f_P on $P = \bigcup_{j=1}^{p} P_j$, there exists a curve ℓ_α densifying the domain:

$$K_{f_P} = \bigcup_{j=1}^{p} \left\{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x \in P_j, 0 \le x_{n+1} \le m_j\right\},$$

with the density $\left(\pi\alpha\sqrt{n}\right)$ and such that we have the inequality:

$$\left| \int \cdots \int_P f_P(x_1, \ldots, x_n)\, dx_n \ldots dx_1 - \alpha^n \mathcal{L}_{\alpha,\ell_\alpha} \right| < \frac{\varepsilon}{2}. \tag{3.2}$$

By the two previous inequalities (3.1) and (3.2), we deduce:

$$\left| I_n(f) - \alpha^n \mathcal{L}_{\alpha,\ell_\alpha} \right| < \varepsilon.$$

Since $K_{f_P} \subset \Omega_f$, the curve ℓ_α densifies Ω_f with the density $\left(\pi\alpha\sqrt{n} + \varepsilon\right)$. □

Theorem 3.2. *Let f be a real non-negative lipschitzian function with constant $L > 0$ over $\mathbb{H}$ and*

$$\Omega_f = \left\{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x \in \Omega \text{ and } 0 \le x_{n+1} \le f(x)\right\}.$$

Let $\ell_\alpha = (\ell_1, \ldots, \ell_{n+1}) : \left[0, \frac{\pi}{\alpha_1}\right] \to \Omega_f$ be the curve defined by:

$$\begin{aligned}\ell_1(t) &= (a-b)\cos^2\left(\frac{\alpha_1 t}{2}\right)+b\\ &\vdots\\ \ell_n(t) &= \left[\varphi_n(\ell_1(t),\ldots,\ell_{n-1}(t))-\psi_n(\ell_1(t),\ldots,\ell_{n-1}(t))\right]\cos^2\left(\frac{\alpha_n t}{2}\right)+\psi_n(\ell_1(t),\ldots,\ell_{n-1}(t))\\ \ell_{n+1}(t) &= -f(\ell_1(t),\ldots,\ell_n(t))\cos^2\left(\frac{\alpha_{n+1}t}{2}\right)+f(\ell_1(t),\ldots,\ell_n(t)),\end{aligned}$$

where $\alpha_1,\ldots,\alpha_{n+1},\alpha$ are strictly positive real numbers satisfying:

$$\begin{cases}\alpha,\ \alpha_1\in\mathbb{R}_+^*,\ \alpha_2=\frac{1}{\alpha}(M_1-m_1)\alpha_1\\ \alpha_i=\frac{1}{\alpha}\left[(M_{i-1}-m_{i-1})+(l_{i-1}+L_{i-1})\alpha\right]\alpha_{i-1},\quad i=3,\ldots,(n+1)\end{cases}$$

with $M_i=\max\limits_{\mathbb{H}_i}\psi_i$, $m_i=\min\limits_{\mathbb{H}_i}\varphi_i$, and $M_1=b$, $m_1=a$.
Then, the parameterized curve $\ell_\alpha(t)=(\ell_1(t),\ldots,\ell_{n+1}(t))$ is $\pi n\alpha$-dense in Ω_f.

Proof. This result is a consequence of the theorem 2.1. proved in [10] for the Lipschitz case, for the Hölder case see [18]. □

Example 3.1. *Let Ω be the non-rectangular region of $\mathbb{R}^2$ defined by*

$$\Omega=\left\{(x_1,x_2)\in\mathbb{R}^2/\quad \begin{array}{c}-\pi\le x_1\le\pi\\ \varphi_2(x_1)\le x_2\le\psi_2(x_1)\end{array}\right\},$$

where $\varphi_2(x_1)=-\sin x_1+\cos x_1+2$ and $\psi_2(x_1)=\sin x_1+\cos x_1+6$.
The parameterized curve $\ell_\alpha(t)=(\ell_1(t),\ell_2(t))$ defined by

$$\begin{cases}\ell_1(t)=-\pi\cos\alpha_1 t\ \ for\ t\in[0,\pi/\alpha_1]\\ \ell_2(t)=-(\sin(\ell_1(t)+2)\cos(\alpha_2 t)+\cos(\ell_1(t))+4,\end{cases}$$

with $\alpha_2=\frac{\pi\alpha_1}{\alpha}$ is $\pi\alpha$-dense in Ω, see Figure 3.

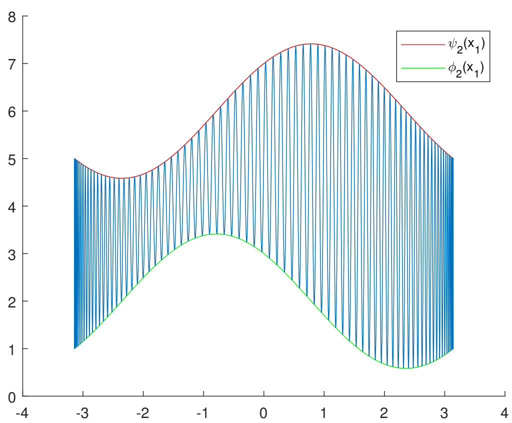

Figure 3: The 2D, α-dense curve ℓ_α (in blue) in the non-rectangular region Ω with α=0.02.

Example 3.2. *Let Ω be the non-rectangular region of $\mathbb{R}^3$ defined by*

$$\Omega = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 / \begin{array}{c} -1 \le x_1 \le 1 \\ \varphi_2(x_1) \le x_2 \le \psi_2(x_1) \\ \varphi_3(x_1, x_2) \le x_3 \le \psi_3(x_1, x_2) \end{array} \right\}$$

where $\varphi_2(x_1) = \frac{-1}{4}x_1^2 + \frac{x_1}{2} - 1$, $\psi_2(x_1) = \frac{1}{4}x_1^2 + \frac{x_1}{2} + 1$ *and* $\varphi_3(x_1, x_2) = 0$, $\psi_3(x_1, x_2) = x_1 + 1$.

The parameterized curve $\ell_\alpha(t) = (\ell_1(t), \ell_2(t), \ell_3(t))$ *defined by*

$$\begin{cases} \ell_1(t) = -\cos \alpha_1 t \ \text{ for } t \in [0, \pi\alpha_1] \\ \ell_2(t) = -(\frac{1}{4}\cos^2(\alpha_1 t) + 1)\cos(\alpha_2 t) - \frac{1}{2}\cos(\alpha_1 t) \\ \ell_3(t) = \left(\left(\frac{1}{8}\cos^2(\alpha_1 t) + \frac{1}{2}\right)\cos(\alpha_2 t) + \frac{3}{4}\cos(\alpha_1 t) - \frac{11}{8}\right)(\cos(\alpha_3 t) - 1), \end{cases}$$

with $\left(\alpha_2 = \frac{\alpha_1}{\alpha}, \alpha_3 = \frac{(7+4\alpha)\alpha_1}{2\alpha^2}\right)$ *is* $\pi\sqrt{2}\alpha$*-dense in* Ω*, see Figure 4.*

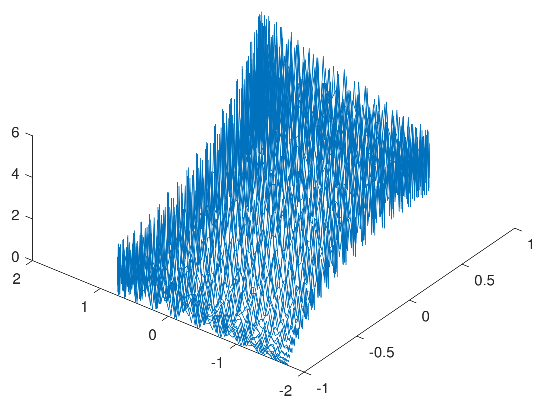

Figure 4: The 3D, α-dense curve ℓ_α (in blue) in the non-rectangular region Ω with α=0.07.

Theorem 3.3. *Let f be a non-negative function of class $C^1(\Omega)$. Then there exists a parameterized curve* $\ell_\alpha(t) = (\ell_1(t), \ldots, \ell_{n+1}(t))$ *densifying* Ω_f *such that the multiple integral* (1.1) *can be approached by the single integral:*

$$I_\alpha = \frac{\alpha_1}{2}\prod_{i=1}^{n}(M_i - m_i)\int_0^{\frac{\pi}{\alpha_1}} f_\alpha(t)\,|\sin(\alpha_{n+1}t)|\,dt,$$

where $f_\alpha(t) = f(\ell(t)) = f(\ell_1(t), \ldots, \ell_n(t))$.

Proof. By applying the theorem 3.1, if $\varepsilon \to 0$, then we have:

$$I_n(f) = \lim_{\alpha\to 0}\alpha^n \mathcal{L}_{\alpha,\ell_\alpha},$$

where the curve $\ell_\alpha(t)$ is given in theorem 3.2, which densify the domain Ω_f with density $\pi n\alpha$. According to the previous theorem, the multiple integral (1.1) can be approached by $\alpha^n\mathcal{L}_{\alpha,\ell}$.
For $L_\alpha = \alpha^n\mathcal{L}_{\alpha,\ell}$ we have:

$$
\begin{aligned}
L_\alpha &= \alpha^n \int_0^{\frac{\pi}{\alpha_1}} \left[\left(\ell_1'(t)\right)^2 + \cdots + \left(\ell_{n+1}'(t)\right)^2\right]^{\frac{1}{2}} dt \\
&= \tfrac{\alpha^n}{2} \int_0^{\frac{\pi}{\alpha_1}} \left[((b_1-a_1)\alpha_1 \sin(\alpha_1 t))^2 + \ldots + \left(f(\ell(t))\,\alpha_{n+1}\sin(\alpha_{n+1}t) - [f(\ell(t))]' \cos(\alpha_{n+1}t) + [f(\ell(t))]'\right)^2\right]^{\frac{1}{2}} dt.
\end{aligned}
$$

But we have:

$$\alpha_2 = \tfrac{1}{\alpha}(M_1 - m_1)\,\alpha_1, \;\; \alpha, \alpha_1 \in \mathbb{R}_+^*$$

$$\alpha_i = \tfrac{1}{\alpha}\left[(M_{i-1} - m_{i-1}) + (l_{i-1} + L_{i-1})\,\alpha\right]\alpha_{i-1}\,, \quad i = 3, \ldots, n+1,$$

so

$$
\begin{aligned}
\alpha_{n+1} &= \frac{\alpha_n}{\alpha}\left[(M_n - m_n) + (l_n + L_n)\,\alpha\right] \\
&= \frac{\alpha_1}{\alpha^n}\left[(M_n - m_n) + (l_n + L_n)\,\alpha\right] \ldots \left[(M_2 - m_2) + (l_2 + L_2)\,\alpha\right].
\end{aligned}
$$

As $f(\ell(t))$ is non-negative, by making $\alpha \to 0$ we find:

$$\lim_{\alpha \to 0} L_\alpha = \frac{1}{2} \int_0^{\frac{\pi}{\alpha_1}} f(\ell(t))\,(M_n - m_n)\ldots(M_1 - m_1)\,\alpha_1\,|\sin(\alpha_{n+1}t)|\,dt.$$

We conclude that

$$I_n(f) = \lim_{\alpha \to 0} \alpha^n \mathcal{L}_{\alpha,\ell_\alpha} = \frac{\alpha_1}{2} \prod_{i=0}^{n} (M_i - m_i) \int_0^{\frac{\pi}{\alpha_1}} f_\alpha(t)\,|\sin(\alpha_{n+1}t)|\,dt,$$

which is what we wanted. □

4. Numerical examples

Numerical results of four examples of double and triple integrals in the tables 1, 2, 3, 4, to illustrate the usefulness of the above procedure are reported. To compute I_α, we use the Simpson's integration method to approximate I_α by $\tilde{I}_\alpha$ and the numerical tests have been implemented in Matlab R2017a runtine environment and the experiments have been executed at a PC with Intel(R) Core(TM) i5-7200U CPU 2.50 GHz and 8.00 RAM. We tested these examples just because the exact value is known and to compare the approximate values obtained by using the α-dense curves. We have chosen different values of the densification parameter α in a decreasing way and different values of the number k of the subintervals used in the Simpson's algorithm to calculate I_α. The absolute and relative errors ε_α^A, ε_α^R are evaluated and we show how these errors decrease with respect to α and k.

Some notations are necessary in tables 1–4.

Notations.

α : the density parameter.
k: the number of subintervals used in the Simpson's algorithm.
I_{exact} : The exact value of the integral $I_n(f)$.
$\tilde{I}_\alpha$: The approximate value of the integral I_α.
$\varepsilon_\alpha^A = |I_{exact} - \tilde{I}_\alpha|$: The absolute error.
$\varepsilon_\alpha^R = \frac{\varepsilon_\alpha^A}{I_{exact}}.100\%$: The relative error.

Example 4.1. *Consider*

$$\Omega = \left\{ (x_1, x_2) \in \mathbb{R}^2 / \quad \begin{array}{c} -1 \leq x_1 \leq 1 \\ \varphi_2(x_1) \leq x_2 \leq \psi_2(x_1) \end{array} \right\}$$

with $\varphi_2(x_1) = \frac{1}{2}x_1 - \frac{1}{2}$, $\psi_2(x_1) = \frac{1}{2}x_1 + \frac{1}{2}$ *and* $f(x) = C$ *a constant positive function. The region*

$$\Omega_f = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 / \quad (x_1, x_2) \in \Omega \text{ and } 0 \leq x_3 \leq C \right\}.$$

We show that the exact value of

$$I_2(f) = \iint\limits_{\Omega} f(x_1, x_2)\, dx_2 dx_1 = Vol\left(\Omega_f\right) = 2C$$

is the same as that given by using the integration formula of Theorem 3.1. By Theorem 3.2, The curve $\ell_\alpha = (\ell_1, \ell_2, \ell_3) : \left[0, \frac{\pi}{\alpha_1}\right] \to \Omega_f$ *defined by:*

$\ell_1(t) = -\cos(\alpha_1 t)$
$\ell_2(t) = -\frac{1}{2}(\cos(\alpha_2 t) + \cos(\alpha_1 t))$
$\ell_3(t) = -\frac{C}{2}\cos(\alpha_3 t) + \frac{C}{2}$,

with $M_2 = 1$, $m_2 = -1$, $l_2 = L_2 = \frac{1}{2}$, $\alpha = \frac{1}{m}$, $\alpha_1 \in \mathbb{R}_+^*$, $\alpha_2 = m\alpha_1$ *and* $\alpha_3 = m(2m+1)\alpha_1$, $m \in \mathbb{N}^*$ *(*$\alpha \to 0 \Leftrightarrow m \to +\infty$*), is* $2\pi\alpha$*-dense in* Ω_f*. From theorem 3.1 we have:*

$$I_2(f) = \iint\limits_{\Omega} f(x_1, x_2)\, dx_2 dx_1 = \lim_{\alpha \to 0} \alpha^2 \mathcal{L}_{\alpha, \ell_\alpha}.$$

Then we have:

$$\lim_{\alpha \to 0} \alpha^2 \mathcal{L}_{\alpha, \ell_\alpha} = \lim_{\alpha \to 0} L_\alpha = \lim_{\alpha \to 0} \alpha^2 \int_0^{\frac{\pi}{\alpha_1}} \left[\left(\ell_1'(t)\right)^2 + \left(\ell_2'(t)\right)^2 + \left(\ell_3'(t)\right)^2 \right]^{\frac{1}{2}} dt.$$

$\ell_1'(t) = \alpha_1 \sin(\alpha_1 t)$
$\ell_2'(t) = \frac{\alpha_1}{2\alpha} \sin\left(\frac{\alpha_1}{\alpha} t\right) + \frac{\alpha_1}{2} \sin(\alpha_1 t)$
$\ell_3'(t) = \frac{(2+\alpha)\alpha_1 C}{2\alpha^2} \sin\left(\frac{(2+\alpha)\alpha_1}{\alpha^2} t\right)$,

then

$$\begin{aligned} \lim_{\alpha \to 0} L_\alpha &= \lim_{\alpha \to 0} \alpha^2 \int_0^{\frac{\pi}{\alpha_1}} \left[(\alpha_1 \sin(\alpha_1 t))^2 + \left(\frac{\alpha_1}{2\alpha} \sin\left(\frac{\alpha_1}{\alpha} t\right) + \frac{\alpha_1}{2} \sin(\alpha_1 t) \right)^2 + \left(\frac{(2+\alpha)\alpha_1 C}{2\alpha^2} \sin\left(\frac{(2+\alpha)\alpha_1}{\alpha^2} t\right) \right)^2 \right]^{\frac{1}{2}} dt \\ &= \lim_{\alpha \to 0} \int_0^{\frac{\pi}{\alpha_1}} \left[\left(\frac{(2+\alpha)\alpha_1 C}{2} \sin\left(\frac{(2+\alpha)\alpha_1}{\alpha^2} t\right) \right)^2 \right]^{\frac{1}{2}} dt. \end{aligned}$$

Let us make the following change of variables: $u = \frac{(2+\alpha)\alpha_1}{\alpha^2} t$, *hence*

$$\begin{aligned} \lim_{m \to \infty} L_{1/m} &= \lim_{\alpha \to 0} \int_0^{m(2m+1)\pi} \frac{1}{m(2m+1)\alpha_1 \pi} \left[\left(\frac{(2m+1)\alpha_1 C}{2m} \sin(u) \right)^2 \right]^{\frac{1}{2}} du \\ &= \lim_{m \to \infty} \int_0^{m(2m+1)\pi} \frac{C}{2m^2} |\sin(u)|\, du = \lim_{m \to \infty} \frac{C}{2m^2} \sum_{i=1}^{m(2m+1)} L_{1/m}^{(i)}, \end{aligned}$$

where $L_{1/m}^{(i)} = \int\limits_{\pi(i-1)}^{\pi i} |\sin(u)|\, du.$

If we denote by

$$L_{\alpha}^{(p)} = \min\left\{L_{1/m}^{(i)} : i = 1, \ldots, m(2m+1)\right\} \text{ and } L_{\alpha}^{(q)} = \max\left\{L_{1/m}^{(i)} : i = 1, \ldots, m(2m+1)\right\},$$

hence

$$\frac{C}{2m^2} \sum_{i=1}^{m(2m+1)} L_{1/m}^{(p)} \leq L_{1/m} \leq \frac{C}{2m^2} \sum_{i=1}^{m(2m+1)} L_{1/m}^{(q)},$$

with $L_{1/m}^{(p)} = \int\limits_{\pi(p-1)}^{\pi p} |\sin(u)|\, du$ *and* $L_{1/m}^{(q)} = \int\limits_{\pi(q-1)}^{\pi q} |\sin(u)|\, du$*, then we have:*

$$\frac{C(2m+1)}{2m} L_{1/m}^{(p)} \leq L_{1/m} \leq \frac{C(2m+1)}{2m} L_{1/m}^{(q)}.$$

By another change of variables: $u - (p-1)\pi = v$ *and* $u - (q-1)\pi = w$*, it comes:*

$$L_{1/m}^{(p)} = \int_0^{\pi} |\sin(v)|\, dw = 2 \text{ and } L_{1/m}^{(q)} = \int\limits_0^{\pi} |\sin(w)|\, dw = 2.$$

Let m tender to $+\infty$*, we obtain and by virtue of the uniform convergence:*

$$\lim_{m\to\infty} \frac{C(2m+1)}{2m} L_{1/m}^{(p)} \leq \lim_{m\to\infty} L_{1/m} \leq \lim_{m\to\infty} \frac{C(2m+1)}{2m} L_{1/m}^{(q)},$$

hence $\lim\limits_{m\to\infty} L_{1/m} = 2C$*. So we deduce:*

$$I_2(f) = \lim_{\alpha\to 0} L_{\alpha} = \lim_{\alpha\to 0} \alpha^2 \mathcal{L}_{\alpha,\ell_\alpha} = 2C = Vol\left(\Omega_f\right).$$

For $C = 2$*, and* $I_{exact} = 4$*, the integral* $I_2(f)$ *is approximated by the simple integral:* $I_\alpha = 2\alpha_1 \int\limits_0^{\frac{\pi}{\alpha_1}} |\sin(\alpha_3 t)|\, dt.$

Table 1: Numerical results of example 4.1.

α	k	$\tilde{I}_\alpha$	ε_α^A	ε_α^R
0.3	16	3.87652	0.123476	3.1%
	32	3.92822	0.071777	1.8%
	64	3.95902	0.040976	1.0%
0.01	16	4.00912	0.009119	0.2%
	32	4.00054	0.000538	0.01%
	64	4.00003	0.000033	0.0008%
0.004	16	4.00054	0.000538	0.01%
	32	4.00003	0.000033	0.0008%
	64	4.00000	0.000002	0.00005%

Example 4.2. *Consider the following double integral:*

$$I_2(f) = \int_{-1}^{1} \int_{\varphi_2(x_1)}^{\psi_2(x_1)} f(x_1, x_2)\, dx_2 dx_1,$$

with $\varphi_2(x_1) = \frac{-1}{4}x_1^2 + \frac{x_1}{2} - 1$, $\psi_2(x_1) = \frac{1}{4}x_1^2 + \frac{x_1}{2} + 1$ *and* $f(x_1, x_2) = x_1 + x_2 + \frac{11}{4}$.
$I_2(f)$ *is approximated by the simple integral:*

$$I_\alpha = \frac{\alpha_1}{2} \prod_{i=1}^{2} (M_i - m_i) \int_0^{\frac{\pi}{\alpha_1}} f_\alpha(t) \left|\sin(\alpha_3 t)\right| dt,$$

with $f_\alpha(t) = f(\ell_1(t), \ell_2(t))$ *and* $I_{exact} = 11.917$.

Table 2: Numerical results of example 4.2.

α	k	$\tilde{I}_\alpha$	ε_α^A	ε_α^R
	32	13.37120	1.454195	12.2%
0.3	64	13.13939	1.222392	10.3%
	128	12.83984	0.922837	7.7%
	32	12.91740	1.000404	8.4%
0.03	64	11.59811	0.318891	2.7%
	128	11.95262	0.035620	0.3%
	32	12.37103	0.454027	3.8%
0.001	64	11.95120	0.034198	0.3%
	128	11.92798	0.010979	0.1%

Example 4.3. *Let*

$$\Omega = \left\{ (x_1, x_2) \in \mathbb{R}^2 / \begin{array}{c} -\pi \leq x_1 \leq \pi \\ -\sin x_1 + \cos x_1 + 2 \leq x_2 \leq \sin x_1 + \cos x_1 + 6 \end{array} \right\},$$

and consider the following double integral:

$$I_2(f) = \iint_\Omega f(x_1, x_2)\, dx_2 dx_1,$$

with $f(x_1, x_2) = x_2 \exp(x_1)$, *the intgral* $I_2(f)$ *is approximated by the integral* I_α *given by*

$$I_\alpha = 4\pi\alpha_1 \int_0^{\frac{\pi}{\alpha_1}} \ell_2(t) \exp(\ell_1(t)) \left|\sin(\alpha_3 t)\right| dt,$$

where $(\ell_1(t), \ell_2(t))$ *is the* $\pi\alpha$*-dense curve in* Ω *and the parameters* $\alpha, \alpha_1, \alpha_2, \alpha_3$ *verify the relation given in Theorem 3.2 with* $I_{exact} = 406.5156$.

Table 3: Numerical results of example 4.3.

α	k	$\tilde{I}_\alpha$	ε_α^A	ε_α^R
0.1	128	478.78700	72.271400	17.8%
	256	459.00116	52.485562	12.9%
0.04	128	447.27258	40.756979	10.0%
	256	442.21437	35.698770	8.9%
0.001	128	403.91521	2.600394	0.6%
	256	407.65131	1.135707	0.3%

Example 4.4. *In the following example, we show that we can suppose the functions φ_i, ψ_i $(2 \leq i \leq 3)$ to be hölderian instead of lipschitzian thanks to a result given in [18]. Consider the following triple integral:*

$$I_3(f) = \iiint\limits_{\Omega} f(x_1, x_2, x_3) dx_3 dx_2 dx_1,$$

with

$$\Omega = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 / \begin{array}{c} 0 \leq x_1 \leq 1 \\ 0 \leq x_2 \leq \sqrt{1 - x_1^2} \\ 0 \leq x_3 \leq \sqrt{1 - x_1^2 - x_2^2} \end{array} \right\},$$

and $f(x_1, x_2, x_3) = x_1 x_2 x_3$. The integral $I_3(f)$ is approximated by

$$I_\alpha = \frac{\alpha_1}{32} \int_0^{\frac{\pi}{\alpha_1}} \ell_1(t)\ell_2(t)\ell_3(t) \left|\sin(\alpha_4 t)\right| dt,$$

where $(\ell_1(t), \ell_2(t), \ell_3(t))$ is the $2\pi\alpha$-dense curve in Ω and the parameters $\alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ verify the relation given in theorem 3.2 with $I_{exact} = 0.0208$.

Table 4: Numerical results of example 4.4.

α	k	$\tilde{I}_\alpha$	ε_α^A	ε_α^R
	128	0.05162	0.030818	148.2%
0.6	256	0.04930	0.028497	137.0%
	512	0.04922	0.028420	136.6%
	128	0.02265	0.001848	8.9%
0.2	256	0.02150	0.000703	3.4%
	512	0.02135	0.000551	2.6%
	128	0.02048	0.000321	1.5%
0.01	256	0.02093	0.000133	0.6%
	512	0.02083	0.000028	0.1%

5. Concluding remarks

Despite the variety of results and their particular interests, the theory of numerical integration in higher dimensions is in a very crude state of development compared with the numerical integration of functions of a single variable. In this paper, we have developed a new process where by a multiple integral over a non-rectangular region of $\mathbb{R}^n$ is reduced to a unique single integral over an interval of $\mathbb{R}$, by using α-dense curves in the region Ω_f whose measure represents the value of the multiple integral. The integral $I_n(f)$ is equal to the limit, when the coefficient of densification α tends to 0, of the length of the curve which densifies the region Ω_f. But the approximated formula I_α generally takes large values and not necessarily easy. This will involve numerical calculations, created significant errors. It is therefore sometimes necessary to approach this integral by classical integration methods such as Newton-Cotes, Simpson, etc., so we still lose precision. In order to make the calculations less expensive and more accurate, the use of an α-dense curve of minimal length, since it exists theoretically, is recommended.

The numerical results given in Tables 1–4 are satisfactory when comparing with the exact value I_{exact}; we also show that in the reducing process, if one takes numbers k of subintervals used in the algorithm of Simpson and successively larger and if one chooses values of the density α in a decreasing way, the precision of the calculation increases.

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